Deformation Theory after Lurie (DAG X)

§ Preliminaries

Disclaimer: ∞categories work™

Recall: For nice 1category $\mathcal{C}$ (eg. $\mathcal{C} = \mathrm{Set}$) have monadic adjunction with category of abelian group objects of $\mathcal{C}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "{\mathbb{Z}[-]}"] \arrow[r, phantom, "\perp"] & \mathrm{Ab}(\mathcal{C}) \arrow[l, bend left, "u"] \end{tikzcd}$$

For nice ∞category $\mathcal{C}$ (eg. $\mathcal{C} = \mathrm{Ani} :=$ ∞category of anima/spaces/∞groupoids) have adjunction to stabilization of $\mathcal{C}$

pointed obj of $\mathcal{C}$

loop functor
$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\Sigma^\infty_+"] \arrow[r, phantom, "\perp"] & \mathrm{Stab}(\mathcal{C}) \arrow[l, bend left, "\Omega^\infty"] \end{tikzcd} = \lim\left(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*\right) \in \mathrm{Cat}_\infty$$

$$= \left\{ E : \mathrm{Ani}^f_* \to \mathcal{C} \;\middle|\; E(0) = *,\ E\left(\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}\right) = \begin{tikzcd} E(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(Y) \arrow[d] \\ E(Z) \arrow[r] & E(W) \end{tikzcd} \right\}$$

$$=: \mathrm{Exc}(\mathrm{Ani}^f_*, \mathcal{C})$$ strongly excissive functors

$\mathrm{Ani}^f_* \subsetneqq \mathrm{Ani}^\omega_*$
honest inclusion
by Wall-finiteness

Eilenberg Maclane functor

For example monoidal: $\mathrm{Ab}(\mathrm{Set}) = \mathrm{Ab} \xrightarrow{H} \mathrm{Stab}(\mathrm{Ani}) = \mathrm{Sp}$

$\otimes_\mathbb{Z}$, $\mathbb{Z} = \mathbb{Z}[*]$ lax sym monoidal $\otimes_\mathbb{S} = \wedge$, $\mathbb{S} = \Sigma^\infty_+(*)$

think Brown representability
any spectrum $E \in \mathrm{Sp}$ defines
$\mathrm{Ani}^f_* \to \mathrm{Ani}$
$X \mapsto \Omega^\infty(\Sigma^\infty_+(X) \otimes_\mathbb{S} E)$
$0 \mapsto \Omega^\infty(0 \otimes_\mathbb{S} E) = *$

Warning on notation: for monoidal 1cat $\mathcal{C}$ write $\mathrm{CMon}(\mathcal{C}) \ni R$, CAlg_R, Alg_R
for monoidal ∞cat $\mathcal{C}$ write $\mathbb{E}_\infty(\mathcal{C}) \ni R$, $\mathbb{E}_\infty\mathrm{Alg}_R$, $A_\infty\mathrm{Alg}_R = \mathbb{E}_1\mathrm{Alg}_R$

§ Deformation Context and Smallness

accessible localization of some $\mathrm{Fun}(K, \mathrm{Ani})$ for small K
(in particular it is complete and cocomplete)

Def: A deformation context is a pair (Γ, E) with Γ a presentable ∞cat and $E \in \mathrm{Stab}(\Gamma)$

Ex: $R \in \mathbb{E}_\infty(\mathrm{Sp}) \rightsquigarrow \Gamma = \mathbb{E}_\infty\mathrm{Alg}^{aug}_R \rightsquigarrow \mathrm{Stab}(\Gamma) \simeq \mathrm{Mod}_R \rightsquigarrow \Omega^{\infty-n}(E) \simeq R \oplus R[n]$
$E \leftarrow\!\shortmid R$
call this the commutative context
" $= R[\varepsilon]$ with $|\varepsilon| = n$ "

Recall: for stack $X : \mathrm{CAlg}_k \to \mathrm{Grpd}$ considered completion $\hat{X} : \mathrm{Art}_k \to \mathrm{Grpd}$
Q: in the coherent setting, what are Artin algebras/nilpotent extensions?

Def: A morphism $\phi : A \to A'$ in a defo ctxt (Γ, E) is elementary, if $\exists n > 0$:
$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d] \\ * \arrow[r] & \Omega^{\infty-n}E \end{tikzcd}$$
ϕ is small, if it is a finite composite of elem morphisms
A is small, if $A \to *$ is small

Fact: The full subcategory $\Gamma^{sm} \subseteq \Gamma$ on small objects is essentially small
If $A \to A'$ and A' are small, so is A
Γ^{sm} admits pullbacks along small morphisms:
$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\phi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\phi"] \\ B' \arrow[r] & B \end{tikzcd}$$
in Γ with ϕ, A, B, B' small
$\Rightarrow \phi', A'$ small

Ex: (ex 1.1.6, essentially referring to HA 7.4)

In the commutative context for $K \in CMon(Ab) \subseteq \mathbb{E}_\infty(Sp)$ a field

- $\phi : A' \to A \in \mathbb{E}_\infty Alg^{aug}_{K, \geq 0}$ elem $\Leftrightarrow \exists n \geq 0$: $$\begin{tikzcd} K[n] \arrow[r] \arrow[d] & A' \arrow[d, "\phi"] \\ 0 \arrow[r] & A \end{tikzcd}$$ in $Mod_{A'}$ and if $n=0$: $\pi_0 K \otimes_{\pi_0 A'} \pi_0 K \xrightarrow{\cdot} \pi_0 K$ is zero

  connective ie $\pi_* = 0$ for $* < 0$

- $A \in \mathbb{E}_\infty Alg^{aug}_K$ small $\Leftrightarrow A \in \mathbb{E}_\infty Alg^{aug}_{K, \geq 0}$ st. $\pi_* A \in Vect^f_K$ and $\pi_0 A$ local comm ring with max ideal $\mathfrak{m}$ st. $K \xrightarrow{\cong} \pi_0 A / \mathfrak{m}$
- $\phi : A' \to A \in \mathbb{E}_\infty Alg^{aug, sm}_K$ small $\Leftrightarrow \pi_0(\phi) : \pi_0 A' \to \pi_0 A$ surjective
- Somehow the forgetful functor $\mathbb{E}_\infty Alg^{aug, sm}_K \to \mathbb{E}_\infty Alg_K$ is full and faithful $\Rightarrow$ can forget about the augmentations and simply write $\mathbb{E}_\infty Alg^{sm}_K$

§ Formal Moduli Problems

Def: A formal moduli problem in a defo ctxt (Γ, E) is a functor $X : \Gamma^{sm} \to Ani$ with

$$X(*) = * \quad \text{and} \quad X\left(\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "small"] \\ B' \arrow[r] & B \end{tikzcd}\right) = \begin{tikzcd} X(A') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X(A) \arrow[d] \\ X(B') \arrow[r] & X(B) \end{tikzcd}$$

or equivalently (cf. Prop 1.1.15) $$X\left(\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d] \\ * \arrow[r] & \Omega^{\infty-n} E \end{tikzcd}\right) \simeq \begin{tikzcd} X(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X(A') \arrow[d] \\ * \arrow[r] & X(\Omega^{\infty-n} E) \end{tikzcd}$$

Obtain presentable ∞category $Moduli^\Gamma \subseteq Fun(\Gamma^{sm}, Ani)$

Ex: For $A \in \Gamma$ have FMP $Spec(A) : \Gamma^{sm} \to Ani$, $B \longmapsto Map_\Gamma(A, B)$ $\rightsquigarrow$ functor $Spec : \Gamma^{op} \to Moduli^\Gamma$

Ex: In the comm ctxt over a field

$X : \Gamma^{sm} \to Ani$ FMP $\Leftrightarrow X(*) = *$ and $$X\left(\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R_0 \arrow[d, "\pi_0 surj"] \\ R_1 \arrow[r] & R_2 \end{tikzcd}\right) \simeq \begin{tikzcd} X(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X(R_0) \arrow[d] \\ X(R_1) \arrow[r] & X(R_2) \end{tikzcd}$$

§ The Tangent Complex

Prop: Let (Γ, E) defo ctxt, recall $E \in Stab(\Gamma) = Exc(Ani^f_*, \Gamma)$

(i) Have factorization $$\begin{tikzcd} Ani^f_* \arrow[rr, "E"] \arrow[dr, "\tilde{E}"'] & & \Gamma \\ & \Gamma^{sm} \arrow[ur, phantom, "\subseteq"] & \end{tikzcd}$$

(ii) For any FMP $X \in Moduli^\Gamma$: $X(E) := Ani^f_* \xrightarrow{\tilde{E}} \Gamma^{sm} \xrightarrow{X} Ani \in Exc(Ani^f_*, Ani) = Sp$

Def: The tangent spectrum (tangent complex) to a FMP $X \in Moduli^\Gamma$ is $X(E) \in Sp$

Sketch (i) HTS $E(K) \to * = E(0)$ is small $\forall K \in Ani^f_*$

Write $K = K_0 \subseteq \ldots \subseteq K_n \simeq *$ as finite seq of cell attachements

ie. $$\begin{tikzcd} K_{i-1} \arrow[r] \arrow[d] & K_i \arrow[d] \\ * \arrow[r] & \mathbb{S}^{d_i} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$ with $d_i > 0$ $\rightsquigarrow$ applying E yields $$\begin{tikzcd} E(K_{i-1}) \arrow[r, "elem"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(K_i) \arrow[d] \\ * \arrow[r] & E(\mathbb{S}^{d_i}) \overset{!}{=} \Omega^{\infty-d_i} E \end{tikzcd}$$

(ii) Clearly $X\tilde{\hat{E}}(0) = X(*) = *$ thus HTS po $\mapsto$ pb

Problem: X only pres pb along small morphisms

Fact (analogously to (i)): $f: K \to K'$ surj on π_0 $\Rightarrow$ $\tilde{\hat{E}}(f)$ small

For
$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d] & K' \arrow[r, "p"] \arrow[d] \arrow[dl, phantom, "\lrcorner" very near end] & K'/(K' \setminus \mathrm{im} f) \arrow[d] \arrow[dl, phantom, "\lrcorner" very near end] \\
L \arrow[r, "g"'] & L' \arrow[r, "q"'] & L'/(L' \setminus \mathrm{im} g)
\end{tikzcd}$$
have p, pf surj on π_0 $\Rightarrow$ $\tilde{\hat{E}}(p)$, $\tilde{\hat{E}}(pf)$ small

Obtain
$$\begin{tikzcd}
X\tilde{\hat{E}}(K) \arrow[r] \arrow[d] \arrow[rr, bend left, "\text{by FMP}"] \arrow[dr, phantom, "\text{by pb cancellation}"] & X\hat{\tilde{E}}(K') \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{by FMP}"] & X\tilde{\hat{E}}(K'/(K' \setminus \mathrm{im} f)) \arrow[d] \\
X\tilde{\hat{E}}(L) \arrow[r] & X\hat{\tilde{E}}(L') \arrow[r] & X\tilde{\hat{E}}(L'/(L' \setminus \mathrm{im} g))
\end{tikzcd}$$
$\square$

Ex: (ex 1.2.8, again making heavy use of HA 7.4)
In the comm ctxt over $\mathbb{C}$ consider augmented $\mathbb{C}$-alg $R \xrightarrow{\varepsilon} \mathbb{C}$

It defines FMP
$$\hat{X}: \mathbb{E}_\infty \mathrm{Alg}_{\mathbb{C}}^{sm} \longrightarrow \mathrm{Ani}$$
$$B \longmapsto \mathrm{Map}_{\mathbb{E}_\infty \mathrm{Alg}_{\mathbb{C}}^{aug}}(HR, B)$$

mapping spectrum

unwinding definitions $\hat{X}(E) \simeq \mathrm{map}_{\mathrm{Mod}_{HR}}(\mathbb{L}^{top}_{R|\mathbb{C}}, \mathbb{C})$

(topological cotangent cplx, coincides with algebraic one over $\mathbb{C}$ ...)

Have refinement $\hat{X}(E) \in \mathrm{Mod}_{H\mathbb{C}} \simeq D(\mathbb{C})$, so tangent cplx is a chain cplx

Rem: The tangent spectrum is a strong invariant of FMPs!

Prop: Let (Γ, E) defo ctxt, $X, Y \in \mathrm{Moduli}^{\Gamma}$
Then $\alpha: X \to Y$ is equiv $\Leftrightarrow$ $\alpha_*: X(E) \to Y(E)$ is equiv

Prf: "$\Rightarrow$" clear, precomposing with $\tilde{\hat{E}}$ preserves equiv
"$\Leftarrow$" HTS α_A equiv $\forall A \in \Gamma^{sm}$
Write $A = A_0 \to \ldots \to A_n = *$ as composite of elem morphisms
Have $X(*) = * = Y(*)$ ✓

$$\begin{tikzcd}
X(A_i) \arrow[rr] \arrow[dr, "\simeq \text{ as pb}"] \arrow[dd] & & X(A_{i+1}) \arrow[dr, "\simeq \text{ by induction}"] \arrow[dd] & \\
& Y(A_i) \arrow[rr] \arrow[dd] & & Y(A_{i+1}) \arrow[dd] \\
* \arrow[rr] \arrow[dr, equal] & & X(\underline{\Omega}^{\infty-k_i}E) \arrow[dr] & \\
& * \arrow[rr] & & Y(\underline{\Omega}^{\infty-k_i}E)
\end{tikzcd}$$

RTS
$$\begin{tikzcd}
X(\underline{\Omega}^{\infty-k_i}E) \arrow[r] \arrow[d, equal] & Y(\underline{\Omega}^{\infty-k_i}E) \arrow[d, equal] \\
X\tilde{\hat{E}}(\mathbb{S}^{k_i}) \arrow[r, "\simeq", "\alpha_{*,\mathbb{S}^{k_i}}"'] & Y\tilde{\hat{E}}(\mathbb{S}^{k_i})
\end{tikzcd}$$
equiv

$\square$

§ Deformation Theories

Goal: Recognition principle for an equivalence $\mathrm{Moduli}^{\Gamma} \simeq \Xi$

Approach: in terms & conditions of $D: \Gamma^{op} \xrightarrow{\mathrm{Spec}} \mathrm{Moduli}^{\Gamma} \simeq \Xi$

$A \longmapsto \mathrm{Map}_{\Gamma}(A,-)$

Why? Suppose we have such an equivalence Yoneda warning: po ↦ pb ↦ po ↦ pb

For any $K \in \Xi$ we get

$$\begin{tikzcd} \mathrm{Ani}_*^f \arrow[r, "E"] \arrow[rr, bend right, "{:= t}"'] & \Gamma \arrow[r, "D^{op}"] & \Xi^{op} \arrow[r, "\text{よ}(K)"] & \mathrm{Ani} \end{tikzcd}$$

which is strongly excissive

$\Longrightarrow$ have functor $e: \Xi \xrightarrow{\text{よ}} \mathrm{Fun}(\Xi^{op}, \mathrm{Ani}) \xrightarrow{t^*} \mathrm{Exc}(\mathrm{Ani}_*^f, \mathrm{Ani}) = \mathrm{Sp}$, which is limit preserving and conservative (use previous prop)

Assume Ξ presentable & e pres sifted colim $\underset{\substack{\text{Lurie}\\ \text{Barr-Beck}}}{\Longrightarrow}$ $\Xi \xrightarrow{e} \mathrm{Sp}$ monadic

$\Longrightarrow$ $\Xi = T\text{-}\mathrm{Alg}$ for nice algebraic/monadic spectra

Def: A deformation theory for a defo ctxt is a functor $D: \Gamma^{op} \to \Xi$ satisfying

(D1) Ξ is presentable

(D2) D admits a ladj $D': \Xi \to \Gamma^{op}$ (equivly D pres limits)

(D3) (i) $D' \dashv D$ restricts to an equiv $\Xi_0 \simeq (\Gamma^{sm})^{op}$ with $\Xi_0 \subseteq \Xi$ full replete

(ii) $\emptyset \in \Xi_0$, in particular $\emptyset = DD'(\emptyset) = D(*)$ initial in $(\Gamma^{sm})^{op}$

(iii)
$$\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$
small pullback in Γ^{sm} $\Longrightarrow$
$$\begin{tikzcd} D(A') & D(B') \arrow[l] \\ D(A) \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & D(B) \arrow[l] \arrow[u] \end{tikzcd}$$
pushout in Ξ hence Ξ_0

(D4) The functor $e: \Xi \xrightarrow{\text{よ}} \mathrm{Fun}(\Xi^{op}, \mathrm{Ani}) \xrightarrow{D^*} \mathrm{Fun}(\Gamma^{sm}, \mathrm{Ani}) \xrightarrow{\tilde{E}^*} \mathrm{Exc}(\mathrm{Ani}_*^f, \mathrm{Ani}) = \mathrm{Sp}$ is conservative and pres sifted colim

Rem: By (D3) any $K \in \Xi$ gives rise to a FMP $\Gamma^{sm} \xrightarrow{D} \Xi^{op} \xrightarrow{\text{よ}(K)} \mathrm{Ani}$

$\Longrightarrow$ have functor $\Psi: \Xi \xrightarrow{\text{よ}} \mathrm{Fun}(\Xi^{op}, \mathrm{Ani}) \xrightarrow{D^*} \mathrm{Moduli}^{\Gamma}$ and $e = \hat{E}^* \circ \Psi$ is welldef.

Warning - This definition is a mashup of Def 1.3.1 & Prop 1.3.5, made for didactical purposes: this way it is clearer how Ξ_0 relates to Γ^{sm} and why e is welldefined. The conditions of Def 1.3.1 are probably easier to verify in practice...

- There is a notion of a weak deformation theory (essentially D1, D2, D3). This is necessary, since not all defo ctxts admit a deformation theory

Ex: In the comm ctxt over a field of char 0 we will see next time that

$$\begin{tikzcd} (E_\infty \mathrm{Alg}_k^{aug})^{op} \arrow[r, bend right, "D"'] \arrow[r, phantom, "\perp"] & \mathcal{L}ie_k \arrow[l, bend right, "D'"'] \end{tikzcd}$$

$\mathcal{L}ie_k := \mathrm{dgLie}_k[\mathrm{qiso}^{-1}]$ with $D'(\mathfrak{g}_*) \simeq C^*_{CE}(\mathfrak{g}_*)$

Chevalley Eilenberg cplx

is a deformation theory

It remains to demonstrate that we reached our goal:

Thm: Let (Γ, E) be a defo ctxt and $D : \Gamma^{sm} \longrightarrow \Xi^{op}$ be a defo thy
Then $\Psi : \Xi \longrightarrow \mathrm{Moduli}^{\Gamma}$ is an equiv

Sketch (essentially blackboxing 2 subchapters of DAG X)

The functor $\Psi : \Xi \longrightarrow \mathrm{Moduli}^{\Gamma} \subseteq \mathrm{Fun}(\Gamma^{sm}, \mathrm{Ani})$ pres limits and filtered colim
$K \longmapsto (A \longmapsto \mathrm{Map}_{\Xi}(D(A), K))$

Fact: $D(A) \in \Xi^{\omega}$ by Lem 1.5.10

$\Longrightarrow$ has ladj Φ and STS 1. Ψ conservative
2. unit $u : \mathrm{id}_{\mathrm{Moduli}^{\Gamma}} \Rightarrow \Psi\Phi$ is equiv

1. $\Psi(f) : \Psi(K) \simeq \Psi(K') \underset{e = \tilde{\Xi}^{*} \circ \Psi}{\Longrightarrow} e(f) : e(K) \simeq e(K') \underset{e \text{ cons}}{\Longrightarrow} f : K \simeq K'$ ✓

2. For $X \in \mathrm{Moduli}^{\Gamma}$ HTS $u_X : X \longrightarrow \Psi\Phi(X)$ equiv of FMP
by Prop STS $\Theta := u_X(E) : X(E) \longrightarrow \Psi\Phi(X)(E)$ equiv of spectra
$= e(\Phi(X))$

homotopy colim ie fat realization

Fact: $\exists X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathrm{Moduli}^{\Gamma}/_X)$ st. - $\|X_\bullet\| \simeq X \in \mathrm{Moduli}^{\Gamma}$
- each X_n is prorepresentable

note since e, Φ pres sifted colim
$e(\Phi(\|X_\bullet\|)) \simeq \|e(\Phi(X_\bullet))\|$

filtered colim of representables in $\mathrm{Fun}(\Gamma, \mathrm{Ani})$

$\Longrightarrow$ have $\Theta_\bullet : X_\bullet(E) \longrightarrow e(\Phi(X_\bullet))$ with $\|\Theta_\bullet\| = \Theta$
$\Longrightarrow$ STS each $\Theta_n : X_n(E) \longrightarrow e(\Phi(X_n))$ is equiv or equivalently
each $u_{X_n} : X_n \longrightarrow \Psi\Phi(X_n)$ is equiv

$\Xi(\Phi\mathrm{Spec}A, K) = \mathrm{Mod}^{\Gamma}(\mathrm{Spec}A, \Psi K)$
$= \Psi(K)(A)$
$= \Xi(DA, K)$ $\forall K$

Since Ψ, Φ pres filtered colim
STS $u_{\mathrm{Spec}A} : \mathrm{Spec}A \longrightarrow \Psi\Phi\mathrm{Spec}A = \Psi(DA)$ equiv $\forall A \in \Gamma$

RTS $\forall B \in \Gamma^{sm}$ $\mathrm{Spec}(A)(B) = \mathrm{Map}_{\Gamma}(A,B) \underset{\eta_*}{\xrightarrow{\sim}} \mathrm{Map}_{\Xi}(DB, DA) \simeq \mathrm{Map}_{\Gamma}(A, D'DB)$

$\eta : \mathrm{id}_{\Gamma^{sm}} \overset{\sim}{\Rightarrow} D'DB$

□