Mackey and Tambara functors

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  we have $M^*(\delta) \circ M_*(\gamma) = M_*(\beta) \circ M^*(\alpha)$,
- for every pair of finite G-sets X and Y, applying M_* to $X \to X \sqcup Y \leftarrow Y$ gives the component maps of an isomorphism $\underline{M}(X) \oplus \underline{M}(Y) \cong \underline{M}(X \sqcup Y)$.

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The transfer tr_K^H for $p\colon G/K \to G/H$ sends an $f \in G\text{-maps}(G/K, B)$ to $\mathrm{tr}_K^H(f)(gH) = \sum_{x \in p^{-1}(gH)} f(x)$.

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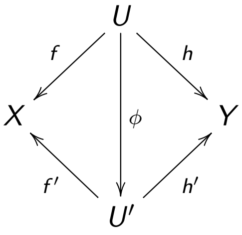
Here the above span is equivalent to $X \xleftarrow{f'} U' \xrightarrow{h'} Y$, if there is a bijection of finite G-sets $\phi\colon U \to U'$ such that



commutes.

Composition in B_G^+ is defined via pullbacks:

$$[\,Y \xleftarrow{g} V \xrightarrow{k} Z\,] \circ [\,X \xleftarrow{f} U \xrightarrow{h} Y\,] := [\,X \xleftarrow{\xi} W \xrightarrow{\zeta} Z\,]$$

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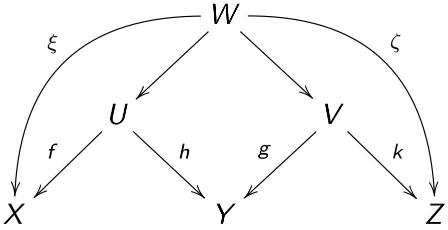

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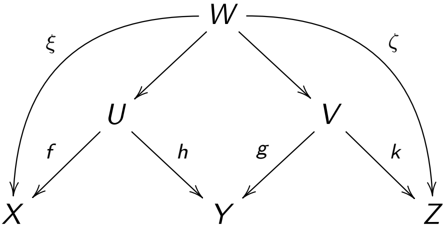

The set $B_G^+(X, Y)$ carries an abelian monoid structure defined via

$$[X \xleftarrow{f_1} U_1 \xrightarrow{h_1} Y] + [X \xleftarrow{f_2} U_2 \xrightarrow{h_2} Y] := [\,X \xleftarrow{(f_1, f_2)} U_1 \sqcup U_2 \xrightarrow{(h_1, h_2)} Y\,]$$

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The equivalence of the two definitions can be seen as follows: For a map $f\colon X \to Y$ of finite G-sets we consider the span

$X = X \xrightarrow{f} Y$ which gives $M_*(f)$, and the span

$Y \xleftarrow{f} X = X$ which yields $M^*(f)$.

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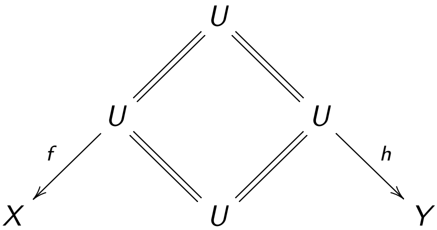

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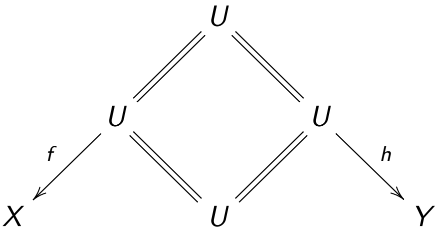

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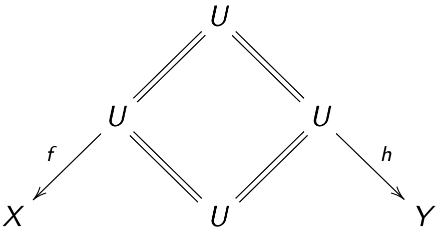

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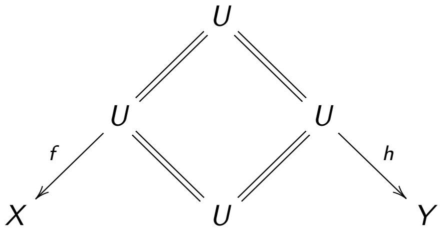

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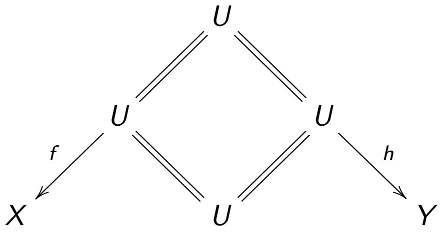

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  Beware that this is only constant on objects! For instance, the transfer map tr_K^H for $K < H < G$ multiplies with the index of K in H.
- The *Burnside Mackey functor*, $\underline{A}$, for G sends a finite G-set X to the Grothendieck group of the abelian monoid of isomorphism classes of finite G-sets over X.

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We can also induce an H-Mackey functor $\underline{N}$ up to a G-Mackey functor by sending a G-set S to $\mathrm{ind}_H^G \underline{N}(S) := \underline{N}(S)$ where we view S just as an H-set.

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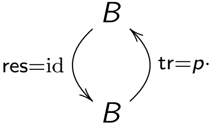

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$$\begin{array}{c} B \\ \text{res}=\mathrm{id} \downarrow \;\; \uparrow \text{tr}=p\cdot \\ B \end{array}$$

For the C_2-fixed point Mackey functor of $\mathbb{Z}[i]$ the Lewis diagram is

$$\begin{array}{c} \mathbb{Z} = \mathbb{Z}[i]^{C_2} \\ \text{res}=\text{inclusion} \downarrow \;\; \uparrow \text{tr}(1)=2, \text{tr}(i)=i-i=0 \\ \mathbb{Z}[i] \end{array}$$

$W_{C_2}(e) = C_2$, $1 \mapsto 1$, $i \mapsto -i$.

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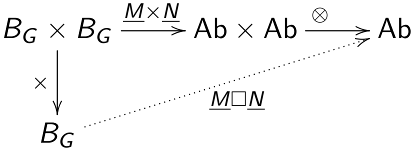

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This is also called a *Day convolution product* after Day, who defined this is his thesis in 1970:

For abelian groups we have the symmetric monoidal structure induced by the tensor product. For G-Mackey functors there is a *box-product*. Formally, it can be defined as a left Kan extension: Assume that $\underline{M}$ and $\underline{N}$ are in G-Mack, then $\underline{M} \Box \underline{N}$ is defined via the following diagram:

$$\begin{array}{ccccc} B_G \times B_G & \xrightarrow{\underline{M} \times \underline{N}} & \mathrm{Ab} \times \mathrm{Ab} & \xrightarrow{\otimes} & \mathrm{Ab} \\ \times \big\downarrow & & & \nearrow_{\underline{M} \Box \underline{N}} & \\ B_G & & & & \end{array}$$

Explicitly,

$$\underline{M} \Box \underline{N}(X) = \mathrm{colim}_{Y \times Z \to X} \underline{M}(Y) \otimes \underline{N}(Z).$$

This is also called a *Day convolution product* after Day, who defined this is his thesis in 1970: You merge the symmetric monoidal structures $\times$ for finite G-sets and $\otimes$ for abelian groups to get $\Box$ for the functor category.

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Commutative monoids in $(G\text{-Mack}, \square)$ are commutative Green functors. They *cannot* be tensored with finite G-sets in a functorial manner. In order to have for instance maps $G/K \otimes \underline{R} \to G/H \otimes \underline{R}$ for $K < H < G$ we need more structure.

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A diagram isomorphic to

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xleftarrow{\varepsilon} & X \times_Y \prod_f A \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle \varrho} \\ Y & & \xleftarrow{\qquad q \qquad} & & \prod_f A \end{array}$$

is called an *exponential diagram*. (Here, $\varepsilon(x, (y, s)) = s(x)$, $\varrho(x, (y, s)) = (y, s)$.)

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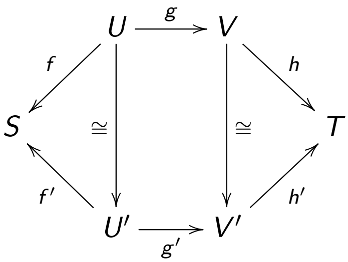

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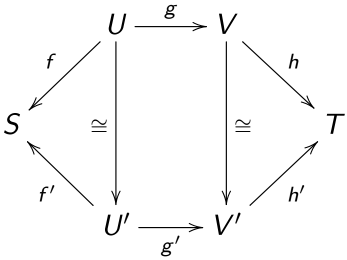


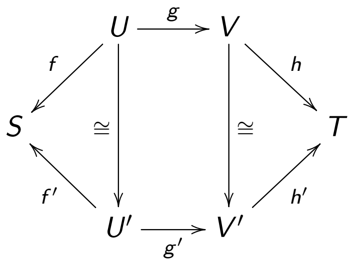
Composition of morphisms is a bit involved.

For the definition of G-Tambara functors we need the following category:

Definition The category $\mathcal{P}^G$ has as objects finite G-sets and a morphism from S to T is an isomorphism class of diagrams

$$S \xleftarrow{f} U \xrightarrow{g} V \xrightarrow{h} T$$

where two diagrams are isomorphic if there is a commutative diagram



Composition of morphisms is a bit involved. We define restriction, norm and transfer associated to a map:

Let $f: S \to T$ be a map of finite G-sets.

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Proposition

- R is a contravariant functor from finite G-sets to $\mathcal{P}^G$ and N, T are covariant ones.
- For a pullback diagram in finite G sets

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xrightarrow{f} & T \end{array}$$

$$R_g \circ N_f = N_{f'} \circ R_{g'} \text{ and } R_g \circ T_f = T_{f'} \circ R_{g'}$$

- For every exponential diagram

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xleftarrow{\varepsilon} & X \times_Y \prod_f A \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle \varrho} \\ Y & & \xleftarrow{\quad q \quad} & & \prod_f A \end{array}$$

$$N_f \circ T_p = T_q \circ N_\varrho \circ R_\varepsilon.$$

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Theorem [Kristen Mazur 2013, Rolf Hoyer 2014] There is a functor

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2. There is a natural isomorphism $X\otimes(Y\otimes\underline{R}) \cong (X\times Y)\otimes\underline{R}$.
3. On the category with objects finite sets with trivial G-action and morphisms consisting only of isomorphisms, the functor restricts to exponentiation $X\otimes\underline{R} = \Box_{x\in X}\underline{R}$.

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The pair (N_H^G, i_H^*) is an adjoint functor pair.