Exercises in Algebra (master): Homological Algebra

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Exercise sheet no 12 for the exercise class on the 30th of June 2021

1 (Very sparse spectral sequences)

Let (E^r, d^r) be a first quadrant spectral sequence with $E_{p,q}^r = 0$ if $p \neq 0, n$ for some fixed $n > 0$. What can you say about possible differentials? What about the E^∞-term?

Dually, assume that you have a first quadrant spectral sequence with $E_{p,q}^r = 0$ unless $q = 0, n$. What about non-trivial differentials? What about the E^∞-term?

(Such spectral sequences arise for the homology of fiber bundles whose base or fiber is a sphere.)

2 (Comparison theorem)

Assume that $f \colon (E_1^r, d^r) \to (E_2^r, d^r)$ is a morphism of spectral sequences and assume that $f^i \colon E_1^i \to E_2^i$ is an isomorphism for some $i \geqslant 2$. Prove that then all f^r's are isomorphisms for all $r \geqslant i$.

Prove that in the above situation $f^\infty \colon E_1^\infty \to E_2^\infty$ is an isomorphism if both spectral sequences are bounded.

3 (Tor is balanced)

We proved that one can calculate Tor by deriving either variable and we did that in a rather ad-hoc manner. Revisit the proof and redo it using the spectral sequences associated to a bicomplex.