

Exercises in Algebra (master): Homological Algebra

Prof. Dr. Birgit Richter

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Exercise sheet no 4

for the exercise class on the 5th of May 2021

1 (Quasi-isomorphisms)

- (1) Is “being quasi-isomorphic” an equivalence relation on chain complexes?
- (2) Let k be a field and consider chain complexes in k -vector spaces, $\text{Ch}(k)$. For a $C_* \in \text{Ch}(k)$ we consider $H_*(C_*)$ as a chain complex with zero differential. Prove that every $C_* \in \text{Ch}(k)$ is quasi-isomorphic to $H_*(C_*)$.

2 (Projective and injective modules over group algebras)

Let G be a group and consider \mathbb{Z} as a trivial $\mathbb{Z}[G]$ -module, *i.e.*, every $g \in G$ acts as the identity on \mathbb{Z} .

- (1) For which G is \mathbb{Z} a projective $\mathbb{Z}[G]$ -module?
- (2) Let P be a projective $\mathbb{Z}[G]$ -module. Show that P is also projective as an $\mathbb{Z}[H]$ -module for all subgroups $H < G$.
- (3) Assume now that G is finite. Prove that every module over $\mathbb{Q}[G]$ is projective. Does that also hold for injectivity? Do you really need \mathbb{Q} or what is a condition on k such that the same thing works for $k[G]$?

3 (Cyclic groups)

- (1) Let C_n be the cyclic group with n elements and let t be its multiplicative generators, *i.e.*, $C_n = \langle t | t^n = 1 \rangle$. Set $N := \sum_{i=0}^{n-1} t^i$. This is often called the *norm element*. Prove that in $\mathbb{Z}[C_n]$

$$N(1 - t) = (1 - t)N = 0.$$

- (2) Construct a projective resolution of \mathbb{Z} as a $\mathbb{Z}[C_n]$ -module. Here, the C_n -action on \mathbb{Z} is trivial.