Exercises in Algebra (master): Homological Algebra

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Summer term 2021

Exercise sheet no 5 for the exercise class on the 12th of May 2021

1 (Suspensions)
Let $p \in \mathbb{Z}$ be arbitrary and let (C_*, d_C) be a chain complex. Define $\Sigma^p C_*$ as

$$(\Sigma^p C)_n = C_{n-p} \text{ and } d_{\Sigma^p C}(c) = (-1)^p d_C(c).$$

Calculate $H_n \Sigma^p C_*$ in terms of the homology groups of C_*.

2 (Mapping cones)

Let $f_*\colon (C_*, d_*) \to (C'_*, d'_*)$ be a chain map. We define the *mapping cone of* f as $C(f)_n = C_{n-1} \oplus C'_n$ and we let $\tilde{d}\colon C(f)_n \to C(f)_{n-1}$ be the map that sends (c, c') to $(-d_{n-1}(c), d'_n(c') - f_{n-1}(c))$.

(1) Check that this actually defines a chain complex.

(2) Prove that there is a short exact sequence of chain complexes

$$0 \longrightarrow C'_* \xrightarrow{j} C(f)_* \xrightarrow{\varrho} \Sigma C_* \longrightarrow 0.$$

Identify the connecting homomorphism in this case.

(3) Show that a chain map $g_*\colon C_* \to D_*$ is chain homotopic to the zero map if and only if g_* extends over the mapping cone of the identity map of C_*:

$$\begin{tikzcd} C_* \arrow[r, "g_*"] \arrow[d, "j"'] & D_* \\ C(\mathrm{id}_{C_*}) \arrow[ur, dotted] & \end{tikzcd}$$

3 (Simple but important chain complexes)

(1) Let A be any abelian group and let n be an integer. We consider the chain complex $S^n(A)$ that has A in degree n and is trivial in all other degrees. What is $H_m(S^n(A))$ for all m? What is the abelian group of chain maps from $S^n(\mathbb{Z})$ to any chain complex C_*?

(2) Let A and n be as above and let $D^n(A)$ be the chain complex with $D^n(A)_n = A = D^n(A)_{n-1}$ and $D^n(A)_k = 0$ for all $k \notin \{n, n-1\}$. As the differential we take the identity map of A between chain degrees n and $n-1$ and take the zero map everywhere else. What is $H_m(D^n(A))$ for all m? What is the abelian group of chain maps from $D^n(\mathbb{Z})$ to any chain complex C_*?

(3) Let $f \in \mathsf{Ab}(A, B)$. Consider f as a chain complex by placing A in degree one, B in degree zero and using f as the only non-trivial boundary operator. What are the homology groups of this chain complex?