## Exercises in Algebra (master): Homological Algebra

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Exercise sheet no 5

for the exercise class on the 12th of May 2021

1 (Suspensions)

Let  $p \in \mathbb{Z}$  be arbitrary and let  $(C_*, d_C)$  be a chain complex. Define  $\Sigma^p C_*$  as

$$(\Sigma^p C)_n = C_{n-p}$$
 and  $d_{\Sigma^p C}(c) = (-1)^p d_C(c)$ .

Calculate  $H_n \Sigma^p C_*$  in terms of the homology groups of  $C_*$ .

## 2 (Mapping cones)

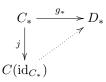
Let  $f_*: (C_*, d_*) \to (C'_*, d'_*)$  be a chain map. We define the mapping cone of f as  $C(f)_n = C_{n-1} \oplus C'_n$  and we let  $\tilde{d}: C(f)_n \to C(f)_{n-1}$  be the map that sends (c, c') to  $(-d_{n-1}(c), d'_n(c') - f_{n-1}(c))$ .

- (1) Check that this actually defines a chain complex.
- (2) Prove that there is a short exact sequence of chain complexes

$$0 \longrightarrow C'_* \xrightarrow{j} C(f)_* \xrightarrow{\varrho} \Sigma C_* \longrightarrow 0.$$

Identify the connecting homomorphism in this case.

(3) Show that a chain map  $g_*: C_* \to D_*$  is chain homotopic to the zero map if and only if  $g_*$  extends over the mapping cone of the identity map of  $C_*$ :



**3** (Simple but important chain complexes)

- (1) Let A be any abelian group and let n be an integer. We consider the chain complex  $S^n(A)$  that has A in degree n and is trivial in all other degrees. What is  $H_m(S^n(A))$  for all m? What is the abelian group of chain maps from  $S^n(\mathbb{Z})$  to any chain complex  $C_*$ ?
- (2) Let A and n be as above and let  $D^n(A)$  be the chain complex with  $D^n(A)_n = A = D^n(A)_{n-1}$  and  $D^n(A)_k = 0$  for all  $k \notin \{n, n-1\}$ . As the differential we take the identity map of A between chain degrees n and n-1 and take the zero map everywhere else. What is  $H_m(D^n(A))$  for all m? What is the abelian group of chain maps from  $D^n(\mathbb{Z})$  to any chain complex  $C_*$ ?
- (3) Let  $f \in Ab(A, B)$ . Consider f as a chain complex by placing A in degree one, B in degree zero and using f as the only non-trivial boundary operator. What are the homology groups of this chain complex?