Exercises in Algebra (master): Homological Algebra

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Exercise sheet no 8 for the exercise class on the 2nd of June 2021

1 (Free groups)
Let F_n be again a free group on n generators for $n \geqslant 1$ and let $\mathbb{Z}$ be the trivial F_n-module. Calculate $H^k(F_n;\mathbb{Z})$ and $H_k(F_n;\mathbb{Z})$ for all $k \geqslant 0$.

2 (Induced maps)

(1) Let G be a group and let $\tilde{g} \in G$ be a fixed element. Consider the group homomorphism $c\colon G \to G$, $g \mapsto \tilde{g}g\tilde{g}^{-1}$. For a G-module M we consider the map $f\colon M \to M$, $f(m) = \tilde{g}m$. Show that (c,f) induces a self-map on $H_*(G;M)$ and prove that this map is trivial.

(2) Let $G_1 = \mathbb{Z}$ and $G_2 = \mathbb{Z}/n$ for some $n \geqslant 2$ and consider $M = \mathbb{Z}$ as the trivial G_1- and G_2-module. We denote by $\pi\colon \mathbb{Z} \to \mathbb{Z}/n$ the canonical projection. What is

$$H_1(\pi;\mathbb{Z})\colon H_1(G_1;\mathbb{Z}) \to H_1(G_2;\mathbb{Z})?$$

3 (Extensions)

(1) How many extensions are there of $G = \mathbb{Z}/2$ by $M = \mathbb{Z}/3$? What does that say about $H^2(\mathbb{Z}/2;\mathbb{Z}/3)$?

(2) Let k be a field and let $n \geqslant 2$ be an integer. We consider the *projective linear group* $PGL_n(k)$ which is defined by the short exact sequence

$$1 \to k^\times \hookrightarrow GL_n(k) \twoheadrightarrow PGL_n(k) \to 1.$$

Here, the units of k, $k^\times$, are embedded into $GL_n(k)$ as the diagonal copy of $k^\times$ in $GL_n(k)$. A *projective representation of* G is a group homomorphism $\varrho\colon G \to PGL_n(k)$. Consider the diagram

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle \varrho} \\ GL_n(k) & \twoheadrightarrow & PGL_n(k). \end{array}$$

Prove that its pullback

$$\begin{array}{ccc} X & \overset{\pi}{\dashrightarrow} & G \\ {\scriptstyle r}\,\vdots & & \downarrow{\scriptstyle \varrho} \\ GL_n(k) & \twoheadrightarrow & PGL_n(k) \end{array}$$

fits in a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & k^\times & \xrightarrow{i} & X & \xrightarrow{\pi} & G & \to & 1. \\ & & \| & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \varrho} & & \\ 1 & \to & k^\times & \to & GL_n(k) & \to & PGL_n(k) & \to & 1. \end{array}$$

whose rows are extensions. In particular, ϱ gives rise to a representation $r\colon X \to GL_n(k)$. (So if $H^2(G;k^\times) = 0$, then the upper extension splits and we get a representation of G, so we can lift a projective representation to an 'honest' representation of G.)