Exercises in Algebra (master): Homological Algebra

Prof. Dr. Birgit Richter

Summer term 2021

Exercise sheet no 9 for the exercise class on the 9th of June 2021

1 (Adjunctions) Let $\mathcal{C}\underset{R}{\overset{L}{\rightleftarrows}}\mathcal{D}$ be a pair of functors.

(1) Prove that L and R are adjoint to each other if and only if there are natural transformations $\varepsilon\colon LR \Rightarrow \mathrm{Id}_{\mathcal{D}}$ and $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$ such that the composites

$$L(C)\xrightarrow{L(\eta_C)}LRL(C)\xrightarrow{\varepsilon_{LC}}L(C) \text{ and } R(D)\xrightarrow{\eta_{R(D)}}RLR(D)\xrightarrow{R(\varepsilon_D)}R(D)$$

are the identity for all objects C of $\mathcal{C}$ and all D of $\mathcal{D}$.

(2) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Assume that an additive functor $R\colon \mathcal{A} \to \mathcal{B}$ is right adjoint to an exact functor L. Show that for any injective object I of $\mathcal{A}$ the object $R(I)$ is injective in $\mathcal{B}$. Dually, if an additive functor $L\colon \mathcal{B} \to \mathcal{A}$ is left adjoint to an exact functor R and if P is a projective object of $\mathcal{B}$, show that $L(P)$ is projective in $\mathcal{A}$.

(3) Let Top be the category of topological spaces and continuous maps. Show that the forgetful functor from Top to the category of sets, Sets, has both a left and a right adjoint.

2 (Hilbert's Theorem 90 for cyclic Galois extensions) Let $K \subset L$ be a Galois extension with $\langle t, t^n = 1\rangle = C_n = \mathrm{Gal}(L/K)$. In the context of Galois extensions the *trace of an* $x \in L$ is the element $\mathrm{tr}(x) = x + tx + \ldots + t^{n-1}x$. Deduce that the inclusion $i\colon K \to L$ and the trace fit into an exact sequence

$$0\longrightarrow K\xrightarrow{i}L\xrightarrow{t-1}L\xrightarrow{\mathrm{tr}}K\longrightarrow 0.$$

3 (Shapiro and transfer)

(1) Let G be a finite group with $|G| = n$. Show that for any G-module M multiplication by n annihilates $H^k(G;M)$ and $H_k(G;M)$ for all $k \geqslant 1$.

(2) We know by Shapiro's Lemma that $H_*(C_3;\mathbb{Z}) \cong H_*(\Sigma_3;\mathrm{Ind}_{C_3}^{\Sigma_3}\mathbb{Z})$. Show that $\mathrm{Ind}_{C_3}^{\Sigma_3}\mathbb{Z}$ is isomorphic to a free abelian group on two generators and identify the corresponding Σ_3-module structure on $\mathbb{Z} \oplus \mathbb{Z}$.