ON THE HOMOLOGY AND HOMOTOPY OF COMMUTATIVE SHUFFLE ALGEBRAS

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Abstract. For commutative algebras there are three important homology theories, Harrison homology, André-Quillen homology and Gamma-homology. In general these differ, unless one works with respect to a ground field of characteristic zero. We show that the analogues of these homology theories agree in the category of pointed commutative monoids in symmetric sequences and that Hochschild homology always possesses a Hodge decomposition in this setting. In addition we prove that the category of pointed differential graded commutative monoids in symmetric sequences has a model structure and that it is Quillen equivalent to the model category of pointed simplicial commutative monoids in symmetric sequences.

1. Introduction

Symmetric sequences are used in many areas of mathematics: Joyal [J81] baptized them *species*, they are important in the theory of operads and one symmetric monoidal model of the stable homotopy category is given by symmetric spectra [HSS00] and these are spectra that are built out of symmetric sequences in spaces (simplicial sets).

Let k be an arbitrary commutative ring with unit. The aim of this paper is to understand homology theories for commutative monoids in symmetric sequences and to investigate the corresponding homotopy theory of simplicial and differential graded objects. A crucial ingredient for our investigation is a result by Stover. He proved [Sto93, 9.10] that the norm map $N = \sum_{\sigma \in \Sigma_n} \sigma$ induces an isomorphism between coinvariants and invariants of tensor powers

$$N\colon (V^{\odot n})_{\Sigma_n} \to (V^{\odot n})^{\Sigma_n} \qquad (1)$$

for all $n \geqslant 1$ and all reduced symmetric sequences V, thus the zeroth Tate cohomology group of Σ_n with coefficients in $V^{\otimes n}$ vanishes. Therefore, quite often arguments work in this context that otherwise only hold in characteristic zero. A second way to interpret Stover's result is that the difference between commutative monoids and commutative monoids with divided power structures disappears. In fact we show that there is a natural way to associate an ordinary graded divided power algebra to every commutative monoid in symmetric sequences (see Theorem 2.3).

For commutative algebras A over a commutative ring k there are several important homology theories: André-Quillen homology, AQ_*, is defined in terms of derived functors of indecomposables in the simplicial sense, Harrison homology, Harr_*, takes the indecomposables of the Hochschild complex and Gamma-homology, $H\Gamma_*$, views the commutative algebra as an E_∞-algebra and takes its homology in this setting. An identification [PR00] says that Gamma-homology can also we viewed as the stabilization of a commutative algebra: the category of simplicial commutative augmented algebras is enriched in the category of pointed simplicial sets, so $A \otimes \mathbb{S}^n$ makes sense for a simplicial model $\mathbb{S}^n$ of the n-sphere. Taking $\mathbb{S}^n = (\mathbb{S}^1)^{\wedge n}$ gives rise to natural stabilization maps $\pi_k(A \otimes \mathbb{S}^n) \to \pi_{k+1}(A \otimes \mathbb{S}^{n+1})$ and Gamma-homology can be identified with the stable homotopy groups of $A \otimes \mathbb{S}^\bullet$ [PR00, Theorem 1].

In general all three homology theories differ drastically: if k is $\mathbb{F}_2$ for instance, then André-Quillen homology of the polynomial ring $\mathbb{F}_2[x]$ vanishes above degree zero whereas Gamma-homology of $\mathbb{F}_2[x]$ with coefficients in $\mathbb{F}_2$ is isomorphic to $(H\mathbb{F}_2)_* H\mathbb{Z}$ [RiRo04, 3.2]. In contrast

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Harrison homology of $\mathbb{F}_2$ viewed as an $\mathbb{F}_2$-algebra is not trivial in positive degrees whereas André-Quillen homology and Gamma-homology are zero (see for instance [RoWh02, Example 6.7]). One reason for these differences is that the operad of commutative algebras is in general not Σ-free unless we work in characteristic zero: the operad Com over k is $\mathsf{Com}(n) = k$ for all n and this is k-projective but not projective as a module over the group algebra of the symmetric group Σ_n, $k[\Sigma_n]$. Usually one replaces the operad Com by an E_∞-operad to make things homotopy invariant. For instance Mike Mandell showed [M03, 1.8, 1.3] that the normalization functor induces an isomorphism between André-Quillen homology for simplicial E_∞-algebras and André-Quillen homology for differential graded E_∞-algebras and a Quillen equivalence between the corresponding model categories. We take the point of view that we want to keep the operad Com but work in an underlying symmetric monoidal category with enough Σ-freeness to make up for the algebraic defects of the operad Com.

We show that Harrison homology, André-Quillen homology and Gamma homology coincide if we consider reduced commutative monoids in symmetric sequences. In other contexts such algebras are called commutative shuffle algebras [Ron11]; shuffle algebras play an important rôle in the theory of combinatorial Hopf algebras. If A is such a monoid and if A is levelwise projective as a k-module we have

$$AQ_*(A;k) \cong \Sigma^{-1}\mathsf{Harr}_*(A) \cong H\Gamma_*(A)$$

(see Theorems 3.3 and 3.4). We also show that the Hodge decomposition for Hochschild homology is valid in our context in arbitrary characteristic (Theorem 3.6).

We establish a model category structure for pointed commutative monoids in symmetric sequences of chain complexes with fibrations and weak equivalences induced by the underlying category (see Corollary 5.8). This should be compared to commutative differential graded algebras, where such a model structure does not exist unless one works over the rationals.

We extend the classical Dold-Kan correspondence to a Quillen equivalence between the model category structures of pointed simplicial commutative shuffle algebras and pointed differential graded commutative shuffle algebras (Theorem 6.5). Here we call a monoid A pointed if its zeroth level consists precisely of the unit of the underlying category. This generalizes Quillen's result [Qu69, Remark on p. 223] in the characteristic zero setting.

Brooke Shipley showed [S07] that there is a Quillen equivalence between the model categories of Hk-algebra spectra and differential graded k-algebras for any commutative ring k. We use her chain of functors to show in Proposition 7.1 that commutative Hk-algebra spectra give rise to natural examples of differential graded and simplicial commutative shuffle algebras. In fact, we conjecture that there is a Quillen equivalence between commutative Hk-algebra spectra and commutative monoids in spectra of simplicial k-modules and commutative monoids in spectra of differential graded k-modules. We plan to address this question in future work.

Acknowledgement: I thank Teimuraz Pirashvili for several helpful discussions. The idea, that all common homology theories of commutative algebras should agree in the setting of symmetric sequences is due to him. John Rognes asked whether commutative shuffle algebras are divided power algebras in a suitable sense and thanks to this question I thought about the structures that lead to Theorem 2.3.

2. Commutative shuffle algebras

We will fix an arbitrary commutative ground ring k. If S is a set, then kS denotes the free k-module generated by S. We denote by Σ_n the symmetric group on n letters and by $k\Sigma_n$ its group algebra.

For general facts about symmetric sequences I recommend [AM10, Sto93]. For any category $\mathcal{C}$ we denote by $\mathcal{C}\Sigma$ the category of symmetric sequences in $\mathcal{C}$. Its objects are sequences $X = (X(\ell))_{\ell\in\mathbb{N}_0}$ of objects $X(\ell)$ of $\mathcal{C}$ such that every $X(\ell)$ carries a left Σ_ℓ-action with Σ_ℓ denoting the symmetric group on ℓ letters. We call $X(\ell)$ the ℓth level of X. A morphism $f\colon X \to Y$ between two objects X, Y of $\mathcal{C}\Sigma$ is a sequence of morphisms $f(\ell)\colon X(\ell) \to Y(\ell)$ in $\mathcal{C}$ such that every $f(\ell)$ is Σ_ℓ-equivariant.

If $(\mathcal{C}, \boxtimes, 1)$ is symmetric monoidal and possesses sums that distribute over the monoidal product, *e.g.*, if $\mathcal{C}$ is closed, and if $\mathcal{C}$ has coequalizers, then $\mathcal{C}\Sigma$ inherits a symmetric monoidal product from $\mathcal{C}$.

In the following the categories of k-modules, simplicial modules and chain complexes play an important rôle. If we denote the category of k-modules by mod, then the usual tensor product of k-modules, $\otimes$, gives rise to a symmetric monoidal category $(\mathsf{mod}\Sigma, \odot, \mathbb{1})$ such that for two $M, N \in \mathsf{mod}\Sigma$ we get in level ℓ

$$(M \odot N)(\ell) = \bigoplus_{p+q=\ell} k\Sigma_\ell \otimes_{k\Sigma_p \otimes k\Sigma_q} M(p) \otimes N(q).$$

Here the (p, q)-summand of the right-hand side denotes the coequalizer with respect to the $\Sigma_p \times \Sigma_q$-action on $M(p) \otimes N(q)$ and on Σ_ℓ by viewing $\Sigma_p \times \Sigma_q$ as a subgroup of $\Sigma_\ell = \Sigma_{p+q}$. This structure is usually called the tensor product of symmetric sequence of modules. We use the notation $\odot$ in an attempt to minimize confusion.

The unit for this symmetric monoidal structure is the symmetric sequence $\mathbb{1}$ with

$$\mathbb{1}(\ell) = 0 \text{ for } \ell \neq 0 \text{ and } \mathbb{1}(0) = k.$$

For the symmetry we denote by $\chi(q, p) \in \Sigma_{q+p}$ the permutation with

$$\chi(q,p)(i) = \begin{cases} i + p, & \text{for } 1 \leqslant i \leqslant q, \\ i - q, & \text{for } q < i \leqslant q + p. \end{cases}$$

The symmetry for $\odot$, $tw\colon (M \odot N)(p+q) \to (N \odot M)(q+p)$, is then defined by sending a class $[\sigma \otimes x \otimes y]$ with $\sigma \in \Sigma_{p+q}$, $x \in M(p)$ and $y \in N(q)$ to

$$tw[\sigma \otimes x \otimes y] = [(\sigma \circ \chi(q,p)) \otimes y \otimes x].$$

For graded modules the situation is similar but we introduce the sign $(-1)^{|x||y|}$ in the definition of tw in the (differential) graded setting.

These symmetric monoidal structures on mod and $\mathsf{mod}\Sigma$ transfer to symmetric monoidal structures on the category dgmod of non-negatively graded chain complexes and on dg$\mathsf{mod}\Sigma$, so that for $X_*, Y_* \in$ dg$\mathsf{mod}\Sigma$ (with $*$ denoting the chain degree) we get in chain degree n

$$(X_* \odot Y_*)(\ell)_n = \bigoplus_{p+q=\ell} \bigoplus_{r+s=n} k\Sigma_\ell \otimes_{k\Sigma_p \otimes k\Sigma_q} X_r(p) \otimes Y_s(q).$$

Thus

$$(X_* \odot Y_*)(\ell) = \bigoplus_{p+q=\ell} k\Sigma_\ell \otimes_{k\Sigma_p \otimes k\Sigma_q} X_*(p) \otimes Y_*(q)$$

if we follow the usual grading convention for tensor products of chain complexes.

Similarly, for the category $s\mathsf{mod}$ of simplicial modules and the corresponding category of symmetric sequences therein, $s\mathsf{mod}\Sigma$, we get for $A_\bullet, B_\bullet \in s\mathsf{mod}\Sigma$

$$(A_\bullet \hat{\odot} B_\bullet)(\ell) = \bigoplus_{p+q=\ell} k\Sigma_\ell \otimes_{k\Sigma_p \otimes k\Sigma_q} A_\bullet(p) \hat{\otimes} B_\bullet(q)$$

where $\hat{\otimes}$ denotes the symmetric monoidal product for simplicial modules, *i.e.*, in simplicial degree n this yields

$$(A_\bullet(p) \hat{\otimes} B_\bullet(q))_n = A_n(p) \otimes B_n(q)$$

with a diagonal action of face and degeneracy operators. As we will use these categories frequently in the rest of the paper we simplify notation by abbreviating dg$\mathsf{mod}\Sigma$ to $\mathsf{dg}\boldsymbol{\Sigma}$ and $s\mathsf{mod}\Sigma$ to $\mathsf{s}\boldsymbol{\Sigma}$. We call an object M *reduced* if $M(0) = 0$.

We will make frequent use of the following two constructions. The free associative monoid generated by M is

$$T(M) = \bigoplus_{i \geqslant 0} M^{\odot i}$$

and the free commutative monoid generated by M is

$$C(M) = \bigoplus_{i \geqslant 0} M^{\odot i}/\Sigma_i.$$

Sometimes, we need the reduced version of $C(M)$, so let $\bar{C}(M)$ be the free commutative non-unital monoid in symmetric sequences generated by M. Then $\bar{C}(M) = \bigoplus_{i \geqslant 1} M^{\odot i}/\Sigma_i$. Unravelling the definitions shows that these objects deserve their names.

Note that elements in $C(M)$ behave like polynomials in every level, but globally they can differ. Take for instance as M the symmetric sequence that is concentrated in level 1 and is equal to k there. Then we can define an element d in $C(M)$ by $d(\ell) = [\mathrm{id}_\ell \otimes 1^{\otimes \ell}] \in M^{\odot \ell}/\Sigma_\ell(\ell)$. So d is non-trivial in every level and if we would like to assign a degree to this 'polynomial' we could do that levelwise with the degree of $d(\ell)$ being ℓ, but this assignment does not give rise to any reasonable notion of global degree.

A monoid in A in $\mathsf{mod}\Sigma$ has a multiplication map $\mu\colon A \odot A \to A$. For a fixed level ℓ the map $\mu(\ell)$ is a Σ_ℓ-equivariant map

$$\mu(\ell)\colon \bigoplus k\Sigma_\ell \otimes_{k\Sigma_p \otimes k\Sigma_q} A(p) \otimes A(q) \to A(\ell).$$

As the coset $\Sigma_\ell/\Sigma_p \times \Sigma_q$ has the set of (p,q)-shuffles, $\mathsf{Sh(p,q)}$, as a set of representatives, one can also define a monoid in symmetric sequences of modules by declaring that there is a map $\mu(\sigma)\colon A(p) \otimes A(q) \to A(p+q)$ for every (p,q)-shuffle σ and that these maps satisfy certain coherence conditions as spelled out in [Ron11, Definition 2.1].

Definition 2.1.
- A monoid in $\mathsf{mod}\Sigma$ is called a *shuffle algebra*.
- A commutative monoid in $\mathsf{mod}\Sigma$ is a *commutative shuffle algebra*.

Before we move on to the differential graded and simplicial context we give three examples of commutative shuffle algebras in the category of (graded) modules.

- Let V be a k-module. Then we can define the symmetric sequence generated by V as
$$\mathrm{Sym}(V)(\ell) = V^{\otimes \ell}$$
where Σ_ℓ acts by permuting the tensor coordinates. Then $\mathrm{Sym}(V)$ is a commutative shuffle algebra despite the fact that its underlying graded object is the tensor algebra generated by V, *i.e.*, the free associative algebra generated by V.
- If A_* is a graded commutative k-algebra, then Stover defines a symmetric sequence $A_*^\pm$ [Sto93, p. 323] with
$$A_*^\pm(\ell) = A_\ell$$
with the Σ_ℓ-action given by the sign-action:
$$\sigma.a := \mathrm{sgn}(\sigma)a.$$
With this convention, $A_*^\pm$ is actually a commutative shuffle algebra. Note that just placing A_ℓ in level ℓ does *not* define a commutative monoid.
- If $\varepsilon\colon A \to k$ is an augmented commutative unital k-algebra, then we can define the symmetric sequence $gr^\Sigma(A)$ in k-modules by $gr^\Sigma(A)(\ell) = I^\ell/I^{\ell+1}$, where I denotes the augmentation ideal of A and where $gr^\Sigma(A)(\ell)$ carries the trivial Σ_ℓ-action. Then gr^Σ actually defines a functor from the category of augmented commutative unital k-algebras to commutative shuffle algebras.

We also get interesting functors from commutative shuffle algebras to ordinary (graded) commutative algebras. The following notion is a variant of Stover's definition [Sto93, 14.4].

Definition 2.2. Let A be an augmented commutative unital shuffle algebra. Then we define $\Psi(A)$ to be the graded module with

$$\Psi(A)_n = \begin{cases} 0, & n \text{ odd}, \\ A(m), & n = 2m. \end{cases}$$

We let $\Psi(A)$ carry a symmetrized multiplication by setting

$$a \cdot b := \sum_{\sigma \in \mathsf{Sh(p,q)}} \sigma\mu(\mathrm{id} \otimes a \otimes b) = \sum_{\sigma \in \mathsf{Sh(p,q)}} \mu(\sigma \otimes a \otimes b) \tag{2}$$

for homogeneous $a \in \Psi(A)_{2p} = A(p)$ and $b \in \Psi(A)_{2q}$.

In the following we use the abbreviation $\mathsf{Sh(p;n)}$ for the set of shuffles $\mathsf{Sh}(\underbrace{p,\ldots,p}_{\mathsf{n}})$. This set carries an action of the symmetric group on n letters where an element $\sigma \in \Sigma_n$ acts via the precomposition with the corresponding block permutation $\sigma^b \in \Sigma_{pn}$ which permutes the blocks $\{1,\ldots,p\}$, $\{p+1,\ldots,2p\}$,. . .,$\{p(n-1)+1,\ldots,pn\}$ as σ permutes the numbers $\{1,\ldots,n\}$. We denote by $S(p;n)$ a set of representatives for the quotient of $\mathsf{Sh(p;n)}$ by Σ_n. We refer to [E95, A2.4] for basics on graded divided power algebras.

Theorem 2.3. *For every augmented commutative shuffle algebra with $A(0) = k$, $\Psi(A)$ is a graded divided power algebra.*

Proof. For any element $x \in \Psi(A)$ of positive degree $2p$ we define

$$\gamma_n(x) := \sum_{\sigma \in S(p;n)} \mu(\sigma \otimes x^{\otimes n}).$$

As $x^{\otimes n}$ is invariant under the Σ_n-action, this is well-defined and $n!\gamma_n(x) = x^n$. We have to show that this definition gives a divided power structure on $\Psi(A)$.

- The product $\gamma_n(x) \cdot \gamma_m(x)$ is equal to

$$\sum_{\sigma \in \mathsf{Sh(np,mp)}} \sum_{\tau_1 \in S(p;n)} \sum_{\tau_2 \in S(p;m)} \mu^{\sigma}((\sigma \circ (\tau_1 \oplus \tau_2)) \otimes x^{\otimes(n+m)}).$$

  On the other hand $\gamma_{n+m}(x)$ is

$$\sum_{\xi \in S(p;n+m)} \mu^{\Sigma}(\xi \otimes x^{\otimes(n+m)}).$$

  Composing elements in $\mathsf{Sh(np,mp)}$ with the sum of elements in $\mathsf{Sh(p;n)}$ and $\mathsf{Sh(p;m)}$ gives precisely the set $\mathsf{Sh(p;n+m)}$. In $\gamma_n(x) \cdot \gamma_m(x)$ we divide out by the action of Σ_n and Σ_m whereas for $\gamma_{n+m}(x)$ we take the quotient with respect to Σ_{n+m} and therefore we obtain

$$\gamma_n(x) \cdot \gamma_m(x) = \binom{n+m}{n} \gamma_{n+m}(x).$$

- The nth divided power of a sum of two homogeneous elements of positive degree $2p$ is

$$\begin{aligned}\gamma_n(x+y) &= \sum_{\sigma \in S(p;n)} \mu^{\Sigma}(\sigma \otimes (x+y)^{\otimes n}) \\ &= \sum_{\sigma \in S(p;n)} \sum_{i=0}^{n} \sum_{\tau \in \mathsf{Sh(i,n-i)}} \mu^{\Sigma}(\sigma \otimes \tau(x^{\otimes i} \otimes y^{\otimes(n-i)})).\end{aligned}$$

  For $\sum_{i=0}^{n} \gamma_i(x) \cdot \gamma_{n-i}(y)$ we obtain

$$\sum_{i=0}^{n} \sum_{\xi \in \mathsf{Sh(ip,(n-i)p)}} \sum_{\tau_1 \in S(p;i)} \sum_{\tau_2 \in S(p;n-i)} \mu^{\Sigma}((\xi \circ (\tau_1 \oplus \tau_2)) \otimes x^{\otimes i} \otimes y^{\otimes(n-i)}).$$

  As

$$\mu^{\Sigma}(\sigma \otimes \tau(x^{\otimes i} \otimes y^{\otimes(n-i)})) = \mu^{\Sigma}((\sigma \circ \tau^b) \otimes x^{\otimes i} \otimes y^{\otimes(n-i)})$$

  we can again finish the comparison by a bijection of the indexing sets in the two summations.

- For the iteration of divided powers we express $\gamma_n(\gamma_m(x))$ as

$$\sum_{\sigma\in S(pm;n)} \sum_{\rho_1\in S(p;m)} \dots \sum_{\rho_n\in S(p;m)} \mu^\Sigma((\sigma\circ(\rho_1\oplus\dots\oplus\rho_n))\otimes(x^{\otimes m})^{\otimes n}).$$

  The indexing set is in bijection with the quotient of $\mathsf{Sh(p;mn)}$ by Σ_n acting by block permutations composed with $\Sigma_m\times\dots\times\Sigma_m$ whereas for $\gamma_{nm}(x)$ we get the quotient of the same set of shuffles by the group Σ_{mn} and therefore

$$\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x).$$

- If we apply γ_n to a product of of two homogeneous elements $x\in\Psi_{2p}(A)$ and $y\in\Psi_{2q}(A)$, then we obtain

$$\sum_{\sigma in S(p+q;n)} \sum_{\tau_1\in\mathsf{Sh(p,q)}} \sum_{\tau_2\in\mathsf{Sh(p,q)}} \mu^\Sigma((\sigma\circ(\tau_1\oplus\tau_2))\otimes(x\otimes y)^{\otimes n}).$$

  In contrast to this we get

$$x^n\cdot\gamma_n(y) = \sum_{\xi\in\mathsf{Sh(pn,qn)}} \sum_{\rho_1\in\mathsf{Sh(p;n)}} \sum_{\rho_2\in S(q;n)} \mu^\Sigma(\xi\circ(\rho_1\oplus\rho_2)\otimes x^{\otimes n}\otimes y^{\otimes n})$$

  so the x- and y-terms appear in a different order. Using the relation

$$\mu^\Sigma(\xi\circ(\rho_1\oplus\rho_2)\otimes x^{\otimes n}\otimes y^{\otimes n}) = \mu^\Sigma(\xi\circ(\rho_1\oplus\rho_2)\circ\chi\otimes(x\otimes y)^{\otimes n})$$

  for the permutation χ that shuffles the x-terms past the y-terms we can again compare the two terms by a bijection of the indexing sets and hence we obtain

$$\gamma_n(x\cdot y) = x^n\cdot\gamma_n(y)$$

  for elements in positive degrees. The multilinearity of the tensor product also ensures that the above equation holds for x in $\Psi(A)_0 = A(0) = k$.

□

Note that $\Psi(\mathrm{Sym}(V))$ is the tensor algebra $T(V[2])$ on the k-module $V[2]$ (V concentrated in degree 2) with the shuffle multiplication. Its divided power structure is made explicit in [Rob68, §5]. For instance, one obtains for $v,w\in V$ that $\gamma_2(v\otimes w) = v\otimes w\otimes v\otimes w + 2v\otimes v\otimes w\otimes w$.

3. Homology theories

3.1. **Harrison homology.** Benoit Fresse introduced a version of Harrison homology for reduced commutative monoids in the category of symmetric sequences of k-modules by mimicking the classical definition, thus it is defined as the homology of the indecomposables $\mathrm{Indec}B(A)$ of the bar complex [F11, 6.8]: if A is a reduced commutative monoid in symmetric sequences in mod, then

$$\mathsf{Harr}_*(A) := H_*(\mathrm{Indec}B(A)).$$

Here, working with reduced monoids corresponds to the setting where one considers augmented monoids and takes coefficients in the ground ring. In this sense Harrison homology as above corresponds to Harrison homology of $\mathbb{1}\oplus A$ with coefficients in $\mathbb{1}$. In the following we assume that M is a reduced symmetric sequence in mod such that every $M(\ell)$ is k-projective. Fresse proves [F11, Proposition 6.9] that Harrison homology, Harr_*, of $\bar{C}(M)$ yields the generators, if M is levelwise k-projective, *i.e.*,

$$\mathsf{Harr}_*(\bar{C}(M)) \cong \Sigma M.$$

3.2. **Gamma homology.** Gamma homology [RoWh02] can be identified with E_∞-homology [F11, Theorem 9.5], $H_*^{E_\infty}$, which in turn can be defined as the stabilization of E_n-homology. If A is a non-unital commutative algebra (in k-modules or reduced symmetric sequences of k-modules), then Fresse shows [F11]

$$H_*^{E_\infty}(A) = \operatorname{colim} H_*^{E_n}(A) \cong \operatorname{colim} H_*\Sigma^{-n}B^n(A) =: \Sigma^{-\infty}B^\infty(A).$$

We assume again that M is a reduced symmetric sequence which is degreewise k-projective. Fresse calculates the homology of the bar construction applied to $\bar{C}(M)$ [F11, Proposition 6.2] and by iterating this result we obtain:

$$H_*B^n\bar{C}(M) \cong \bar{C}(\Sigma^n M) \text{ for all } n \geqslant 1.$$

The isomorphism is induced by a map ∇_n. For $\nabla_1 \colon \bar{C}\Sigma M \to B\bar{C}M$ one considers the inclusion $M \to \bar{C}M$. This map combined with the natural map from the suspension to the bar construction yields $\Sigma M \to B\bar{C}M$, and as $B\bar{C}M$ is commutative, we can extend this map to $C\Sigma M$ [F11, 6.2]. Note that the case $n = 1$ is the Hochschild-Kostant-Rosenberg Theorem in disguise.

As E_n-homology of $\bar{C}(M)$ is isomorphic to $H_*(\Sigma^{-n}B^n\bar{C}(M))$, an immediate consequence of [F11, 6.3] is:

Proposition 3.1. *For every M as above*

$$H_*^{E_n}(\bar{C}(M)) \cong \Sigma^{-n}\bar{C}(\Sigma^n M).$$

With the help of this identification, we can show that Gamma-homology of $\bar{C}(M)$ behaves like Harrison homology, up to a suspension.

Theorem 3.2. *For every reduced symmetric module M which is levelwise k-projective we have*

$$H\Gamma_*(\bar{C}(M)) = H_*^{E_\infty}(\bar{C}(M)) \cong M.$$

This isomorphism is natural in M.

Proof. We have to understand the maps in the direct system for $H_*^{E_\infty}(\bar{C}(M))$, *i.e.*, we have to understand the squares of the form

$$\begin{array}{ccc} \Sigma^{-n}\bar{C}(\Sigma^n M) & \xrightarrow{\psi_n} & \Sigma^{-(n+1)}\bar{C}(\Sigma^{(n+1)}M) \\ \Big\downarrow{\scriptstyle \Sigma^{-n}\nabla_n} & & \Big\downarrow{\scriptstyle \Sigma^{-(n+1)}\nabla_{n+1}} \\ H_*\Sigma^{-n}B^n(\bar{C}(M)) & \xrightarrow{\sigma_n} & H_*\Sigma^{-(n+1)}B^{n+1}(\bar{C}(M)), \end{array}$$

and in particular, we have to determine what the effect of the maps ψ_n is. The map σ_n is the stabilization map from E_n-homology to E_{n+1}-homology. It is induced by the canonical map

$$\Sigma B^n\bar{C}(M) \to B^{n+1}\bar{C}(M).$$

Let $s_{-n}(s_n m_1 \cdot \ldots \cdot s_n m_\ell)$ denote an element in $\Sigma^{-n}\bar{C}(\Sigma^n M)$. Under $\Sigma^{-n}\nabla_n$ it is sent to $s_{-n}[m_1]_n \cdot \ldots \cdot [m_\ell]_n$, where $\cdot$ denotes the product in B^n and $[-]_n$ denotes an n-fold iterated bracket in the bar construction. Therefore

$$\sigma_n(\Sigma^{-n}\nabla_n(s_{-n}[m_1]_n \cdot \ldots \cdot [m_\ell]_n)) = s_{-(n+1)}[[m_1]_n \cdot \ldots \cdot [m_\ell]_n].$$

For a product of monomial length one we obtain

$$\sigma_n(\Sigma^{-n}\nabla_n(s_{-n}[m_1]_n) = s_{-(n+1)}[m_1]_{n+1}$$

and this is in the image of $\Sigma^{-(n+1)\nabla_{n+1}}$ with

$$\psi_n(s_{-n}s_n m_1) = \psi_n(m_1) = s_{-(n+1)}s_{n+1}m_1 \in \Sigma^{-(n+1)}\bar{C}(\Sigma^{(n+1)}M).$$

Elements of higher monomial length cannot be in the image of $\Sigma^{-(n+1)\nabla_{n+1}} \circ \psi_n$ for degree reasons. Therefore the maps in the stabilization sequence

$$\psi_n \colon \Sigma^{-n}\bar{C}(\Sigma^n M) \to \Sigma^{-(n+1)}\bar{C}(\Sigma^{(n+1)}M)$$

are given by identifying the summand $\Sigma^{-n}\Sigma^n M = M \subset \Sigma^{-n}\bar{C}(\Sigma^n M)$ with the summand $\Sigma^{-(n+1)}\Sigma^{(n+1)}M = M \subset \Sigma^{-(n+1)}\bar{C}(\Sigma^{(n+1)}M)$ and by projecting all other summands to zero. Thus M is the direct limit of the stabilization process. □

3.3. **André-Quillen homology.** We can define André-Quillen homology for reduced commutative monoids in symmetric sequences as usual: for every such A there is a standard free simplicial resolution

$$\dots \overset{\longrightarrow}{\underset{\longrightarrow}{\Longleftarrow\!\!\Longrightarrow}} \bar{C}^3(A) \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow\!\!\longleftarrow}} \bar{C}^2(A) \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} \bar{C}(A)$$

and we define André-Quillen homology of A (with trivial coefficients) to be

$$\mathsf{AQ}_*(A) = H_*(Q_a(\bar{C}^{\bullet+1}(A))$$

where $Q_a(-)$ denotes the module of indecomposables and we take the homology of the corresponding chain complex. Note that the canonical inclusion $A \to \bar{C}(A)$ is a section to the augmentation $\bar{C}(A) \to A$ that codifies the commutative monoid structure on A.

Definition 6.1 will describe a model category structure on commutative monoids in symmetric sequences of simplicial modules, and one can prove then that any free simplicial resolution, $P_\bullet$, gives rise to the same homology groups, *i.e.*, $\mathsf{AQ}_*(A) = H_*(Q_a(P_\bullet))$.

3.4. **Comparison.** We obtain the following comparison result.

Theorem 3.3. *If A is a reduced commutative shuffle algebra such that every $A(\ell)$ is projective as a k-module, then*

$$\mathsf{AQ}_*(A) \cong \Sigma^{-1}\mathsf{Harr}_*(A).$$

Proof. Let $P_\bullet$ be the free simplicial resolution in the category of reduced commutative monoids in symmetric sequences with $P_t = \bar{C}^{t+1}(A)$. We consider the bicomplex with $\mathrm{Indec}(B(P_t))_s$ in bidegree (s,t). Taking homology in s-direction and using Fresse's result gives

$$H_s(\mathrm{Indec}(B(P_t))_* = \mathsf{Harr}_s(P_t) = \begin{cases} 0, & s > 0, \\ \Sigma Q_a(P_t), & s = 0, \end{cases}$$

and therefore

$$H_t H_s(\mathrm{Indec}(B(P_t))_* \cong H_t \Sigma Q_a(P_\bullet) \cong \mathsf{AQ}_{t-1}(A).$$

On the other hand, the section $s\colon A \to P_0$ ensures that $H_t\mathrm{Indec}(B(P_\bullet))$ reduces to $\mathrm{Indec}(B(A))$ in degree $t = 0$. Thus the total complex of our bicomplex calculates the Harrison homology groups of A and hence we obtain

$$\mathsf{AQ}_r(A) \cong \mathsf{Harr}_{r+1}(A) \text{ for all } r \geqslant 0.$$

□

In a similar manner we can show that André-Quillen homology is isomorphic to Gamma-homology:

Theorem 3.4. *Let A be a reduced commutative shuffle algebra such that every $A(\ell)$ is projective as a k-module, then*

$$\mathsf{AQ}_*(A) \cong H_*^{E_\infty}(A).$$

Proof. In this case we consider the bicomplex which is $C_p^{E_\infty}(\bar{C}^{q+1}(A))$ in bidegree (p,q). Its total complex computes E_∞-homology of A, but taking homology in p-direction first yields

$$H_p^{E_\infty}(\bar{C}^{q+1}(A)) \cong \bar{C}^q(A) \cong Q_a(\bar{C}^{q+1}(A))$$

and thus if we then take homology in q-direction we obtain André-Quillen homology of A. □

Therefore, all three homology theories coincide for reduced commutative monoids in symmetric sequences:

$$\Sigma^{-1}\mathsf{Harr}_*(A) \cong \mathsf{AQ}_*(A) \cong H_*^{E_\infty}(A).$$

3.5. **Hodge decomposition.** Let k be any field and assume that A is a non-unital reduced commutative shuffle algebra. For any symmetric sequence M we denote by $\bar{C}_q M$ the qth homogeneous part of $\bar{C}M$, *i.e.*, $\bar{C}_q M = M^{\odot q}/\Sigma_q$.

Definition 3.5. For $q > 0$ the *pth q-André-Quillen homology group of A* is defined as

$$\mathsf{AQ}_p^{(q)}(A) := H_p \Sigma^{-1} \bar{C}_q \Sigma Q_a(\bar{C}^{\bullet+1}(A)).$$

This definition should be compared to the definition of q-André-Quillen homology of an augmented commutative algebra with coefficients in the ground ring in the usual setting as the homology of $\Lambda^q Q_a(C^{\bullet+1}(A))$ ([L98, §3.5],[Qu68, §6]). Note that $\mathsf{AQ}_p^{(1)}(A) \cong \mathsf{AQ}_p(A)$.

Theorem 3.6. *For every reduced commutative shuffle k-algebra A we have*

$$H_n^{E_1}(A) \cong \bigoplus_{p+q=n} \mathsf{AQ}_p^{(q)}(A).$$

Proof. We consider the bicomplex in symmetric sequences given by $\Sigma^{-1} B_p(\bar{C}^{q+1}A)$ in bidegree (p, q). Taking vertical homology first yields $\Sigma^{-1} B_p(A)$ and taking the pth horizontal homology afterwards gives $H_p^{E_1}(A)$. Thus the spectral sequence converges to the E_1-homology groups of A.

If we take horizontal homology first we obtain

$$H_q^{E_1}(\bar{C}^{p+1}A) = H_q(\Sigma^{-1} B\bar{C}(\bar{C}^p(A))) \cong (\Sigma^{-1}\bar{C}(\Sigma\bar{C}^p A))_q \cong \Sigma^{-1}\bar{C}_q \Sigma Q_a(\bar{C}^{p+1}A).$$

The vertical homology of this yields $\mathsf{AQ}_p^{(q)}(A)$. Therefore we obtain a spectral sequence

$$E^2_{p,q} = \mathsf{AQ}_p^{(q)}(A) \Rightarrow H_{p+q}^{E_1}(A).$$

We have to show that there are no higher differentials in this spectral sequence. To this end we consider the map of bicomplexes induced by ∇:

$$\Sigma^{-1}\bar{C}_p \Sigma Q_a(\bar{C}^{q+1}A) \cong \Sigma^{-1}\bar{C}_p\Sigma\bar{C}^q A \to \Sigma^{-1} B_p \bar{C}^{q+1} A.$$

The only non-trivial differentials on the left hand side are the vertical ones. The horizontal homology groups

$$\Sigma^{-1}\bar{C}\Sigma\bar{C}^q A \cong H_*(\Sigma^{-1} B\bar{C}^{q+1}A)$$

are isomorphic and we also get an isomorphism on the associated E^2-terms. Thus we know that the homology groups of the associated total complexes agree and hence the E^2-term is already isomorphic to the E^∞-term. As we are working over a field, there are no extension issues. □

4. Symmetric sequences in the dg and simplicial context

From now on let k denote an arbitrary commutative unital ring. For a chain complex X_* and an integer $r \geqslant 0$, we denote by $G_r(X_*)$ the symmetric sequence

$$G_r(X_*)(n) = \begin{cases} 0, & n \neq r, \\ k[\Sigma_r] \otimes X_*, & n = r. \end{cases}$$

As usual we denote by $\mathbb{D}^n$ the n-disc complex, *i.e.*, the chain complex which has k in degrees n and $n-1$ and whose only non-trivial differential is the identity map. Similarly, let $\mathbb{S}^n$ denote the n-sphere complex which has k as the only non-trivial entry in chain degree n.

A standard result [Hi03, Theorem 11.6.1, 11.5] turns the category of symmetric sequences of non-negatively graded chain complexes, into a cofibrantly generated model category:

Proposition 4.1. *The category* $\mathsf{dg}\Sigma$*is a cofibrantly generated model category. Its generating cofibrations are given by*

$$\mathcal{I}_\Sigma = \{G_r(\mathbb{S}^{n-1}) \to G_r(\mathbb{D}^n), n \geqslant 0, r \geqslant 0\}$$

and its generating acyclic cofibrations are

$$\mathcal{J}_\Sigma = \{G_r(0) = 0 \to G_r(\mathbb{D}^n), n \geqslant 1, r \geqslant 0\}.$$

A map $f\colon X_* \to Y_*$ in $\mathsf{dg\Sigma}$ is a weak equivalence (fibration) if the maps $f(n)\colon X_*(n) \to Y_*(n)$ are weak equivalences (fibrations) in the model category structure of non-negatively graded chain complexes for all $n \geqslant 0$.

Similarly, as the model category of simplicial k-modules is cofibrantly generated with generating cofibrations and generating acyclic cofibrations induced from the ones in the model structure of simplicial sets, we can transfer the model structure to symmetric sequences of simplicial k-modules and obtain the following structure.

Proposition 4.2. *There is a cofibrantly generated model category structure on the category of symmetric sequences in simplicial k-modules, $s\Sigma$, such that a map $f\colon A_\bullet \to B_\bullet$ is a weak equivalence (fibration) if and only if $f(n)\colon A_\bullet(n) \to B_\bullet(n)$ is a weak equivalence (fibration) for all $n \geqslant 0$.*

The Dold-Kan correspondence establishes an equivalence of categories between non-negatively graded differential graded objects in an abelian category $\mathcal{A}$ and simplicial objects in $\mathcal{A}$. The category $\mathsf{s\Sigma}$ is the category of simplicial objects in $\mathsf{mod\Sigma}$ and $\mathsf{dg\Sigma}$ is the category of non-negatively graded chain complexes in $\mathsf{mod\Sigma}$. Since $\mathsf{mod\Sigma}$ is an abelian category, we can apply the Dold-Kan theorem:

Proposition 4.3. *The normalization functor induces an equivalence of categories*

$$N\colon \mathsf{s\Sigma} \to \mathsf{dg\Sigma}.$$

If we denote by Γ the inverse of N, then the pair (N, Γ) gives rise to a Quillen equivalence between the model categories $\mathsf{s\Sigma}$ and $\mathsf{dg\Sigma}$. As the abelian structure in $\mathsf{mod\Sigma}$ is formed levelwise, the normalization of any $A_\bullet$ in $\mathsf{s\Sigma}$ is given by

$$(N(A_\bullet))(n) = N(A_\bullet(n)).$$

Consequently Γ is also formed levelwise and therefore if we start with a chain complex X_* then

$$\Gamma(G_r X_*) = G_r \Gamma(X_*), \tag{3}$$

where G_r is the functor that assigns to a simplicial k-module $A_\bullet$ the symmetric sequence in simplicial modules with

$$(G_r A_\bullet)(n) = \begin{cases} 0, & n \neq r, \\ k[\Sigma_r] \otimes A_\bullet, & n = r. \end{cases}$$

Note that we can express $k[\Sigma_r] \otimes A_\bullet$ also as

$$\underline{k[\Sigma_r]} \hat{\otimes} A_\bullet$$

where $\underline{k[\Sigma_r]}$ denotes the constant simplicial object with value $k[\Sigma_r]$ in every simplicial degree and $\hat{\otimes}$ is the tensor product of simplicial modules.

5. A model structure on reduced commutative monoids in symmetric sequences

In characteristic zero there is a nice model structure on the category of differential graded commutative algebras where the fibrations and weak equivalences are determined by the forgetful functor to (non-negatively graded) chain complexes (compare [BG76]). In positive characteristic, or for a commutative ground ring k such a model structure does not exist in general. Stanley constructed a model structure with different features in [Sta∞].

One crucial problem is that the free commutative algebra generated an acyclic complex doesn't have to be acyclic. For instance for the n-disc complex, $\mathbb{D}^n$, for even n the free commutative algebra on $\mathbb{D}^n$ is $k[x_n] \otimes \Lambda(x_{n-1})$. The differential is determined by $\partial(x_n) = x_{n-1}$ and the derivation property. If we consider $\partial(x_n^2)$ then this gives $2x_{n-1}x_n$, and if 2 is not invertible in k, then this phenomenon causes non-trivial homology. We will see that such problems do not arise for symmetric sequences in chain complexes.

Definition 5.1.

(1) We denote the category of unital commutative monoids in symmetric sequences of non-negatively graded chain complexes by $C_{\mathsf{dg\Sigma}}$ and the category of unital commutative monoids in symmetric sequences of simplicial modules by $C_{\mathsf{s\Sigma}}$. We call objects in $C_{\mathsf{s\Sigma}}$ *simplicial commutative shuffle algebras* and objects in $C_{\mathsf{dg\Sigma}}$ *differential graded commutative shuffle algebras*
(2) We call an object $A \in C_{\mathsf{s\Sigma}}$ (or $C_{\mathsf{dg\Sigma}}$) *pointed* if $A(0) = \underline{k}$ ($A(0) = \mathbb{S}^0$). The full subcategory of pointed objects is denoted by $C^+_{\mathsf{s\Sigma}}$ (or $C^+_{\mathsf{dg\Sigma}}$).
(3) The category of *reduced commutative monoids in symmetric sequences* in $\mathsf{dg\Sigma}$ or $\mathsf{s\Sigma}$ consists of commutative non-unital monoids A with $A(0) = 0$. We denote the corresponding categories by $C^-_{\mathsf{dg\Sigma}}$ and $C^-_{\mathsf{s\Sigma}}$.

Before we discuss a suitable model category structure we give some examples of such algebras.

- Let C_* be a non-negatively graded chain complex of k-modules. The symmetric sequence
$$\mathrm{Sym}(C_*)(\ell) = C_*^{\otimes \ell}$$
is a pointed differential graded commutative shuffle algebra and we can reduce it by setting its zero level to 0.
- Let $\varepsilon\colon A_* \to k$ be an augmented differential graded commutative unital k-algebra with augmentation ideal I_*. We define a symmetric sequence $gr^{\Sigma}(A_*)$ as
$$gr^{\Sigma}(A_*)(\ell) = I_*^{\ell}/I_*^{\ell+1},$$
but now we let Σ_ℓ act on $gr^{\Sigma}(A_*)(\ell)$ by the sign-action. Then $gr^{\Sigma}(A_*)$ is a differential graded commutative shuffle algebra and we can define a reduced version by setting its zero level to 0. In fact gr^{Σ} defines a functor from the category of augmented differential graded commutative unital k-algebras to differential graded commutative shuffle algebras.

Given an object $A_* \in C^+_{\mathsf{dg\Sigma}}$, we can define a variant of the functor Ψ from Definition 2.2 by setting $\Psi(A_*)_r := \bigoplus_{2p+m=r} A_m(p)$. Then the differential on A_* yields a differential on $\Psi(A_*)$ and this satisfies the derivation property with respect to the symmetrized multiplication on $\Psi(A_*)$ as in (2).

Lemma 5.2. *With the above convention, $\Psi(A_*)$ is a differential graded divided power algebra.*

Proof. We have to show that for any $a \in A(p)_m$ we get
$$d\gamma_n(a) = \gamma_{n-1}(a) \cdot da.$$
The left hand side is equal to
$$\sum_{i=1}^{n} \sum_{\xi \in S(p;n)} (-1)^{m(i-1)} \mu^{\Sigma}(\xi \otimes a^{\otimes(i-1)} \otimes da \otimes a^{\otimes(n-i)})$$
whereas the right hand side gives
$$\sum_{\tau \in \mathsf{Sh}((\mathsf{n}-1)\mathsf{p},\mathsf{p})} \sum_{\sigma \in S(p;n-1)} \mu^{\Sigma}(\tau \circ (\sigma \oplus \mathrm{id}_p) \otimes a^{\otimes(n-1)} \otimes da).$$
Both indexing sets are in bijection with the set $\mathsf{Sh}(\mathsf{p};\mathsf{n})$ divided by the Σ_{n-1}-action on the first $n-1$ blocks of p numbers and as we have that
$$\mu^{\Sigma}(\tau \circ (\sigma \oplus \mathrm{id}_p) \otimes a^{\otimes(n-1)} \otimes da) = (-1)^{m(i-1)} \mu^{\Sigma}(\tau \circ (\sigma \oplus \mathrm{id}_p) \circ \chi_i \otimes a^{\otimes(i-1)} \otimes da \otimes a^{\otimes(n-i)})$$
for a suitable block permutation χ_i, we can split the right hand side into n parts where the ith part corresponds to the summand on the left hand side where da is in spot number i. $\square$

We consider the free commutative monoid functor from $\mathsf{dg\Sigma}$ to the category of commutative monoids in $\mathsf{dg\Sigma}$, $C_{\mathsf{dg\Sigma}}$,
$$C\colon \mathsf{dg\Sigma} \to C_{\mathsf{dg\Sigma}}, \qquad V_* \mapsto C(V_*) = \bigoplus_{\ell \geqslant 0} V_*^{\odot \ell}/\Sigma_\ell.$$

There is a canonical map from the unit $\mathbb{1}$ to $C(V_*)$ for any V_*, given by the inclusion into the summand for $\ell = 0$. Note that for any k-module V we can identify $\mathrm{Sym}(V)$ with $C(G_1 V)$.

A crucial auxiliary result is the following.

Lemma 5.3. *Let X_* be $\bigodot_{i\in S}\bigodot_{j\in T} C(G_{r_i}\mathbb{D}^{n_j})$ where S and T are some arbitrary indexing sets and $S \ni r_i \neq 0 \neq n_j \in T$ for all i, j. Then the canonical map*

$$\mathbb{1} \to H_*(X_*)$$

is an isomorphism of symmetric sequences in graded k-modules.

Remark 5.4. Note that the case $r = 0$ is excluded: if Y_* is a chain complex, then $G_0 Y_*$ is a symmetric sequence concentrated in level zero, and $C(G_0 Y_*)$ is also concentrated in level zero, where it is the free graded commutative algebra generated by Y_*, so in that case we cannot expect the result to hold.

Proof. By definition $G_r\mathbb{D}^n$ is concentrated in level r with value $k\Sigma_r \otimes \mathbb{D}^n$, and this implies

$$(G_r\mathbb{D}^n)^{\odot a}(m) \cong \begin{cases} k[\Sigma_{ar}] \otimes (\mathbb{D}^n)^{\otimes a}, & m = ar, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $C(G_r\mathbb{D}^n)$ is only non-trivial in levels of the form $\ell = ar$, and

$$C(G_r(\mathbb{D}^n))(ar) \cong (k[\Sigma_{ar}] \otimes (\mathbb{D}^n)^{\otimes a})/\Sigma_a.$$

However, as chain complexes

$$(k[\Sigma_{ar}] \otimes (\mathbb{D}^n)^{\otimes a})/\Sigma_a \cong \bigoplus_{\Sigma_{ar}/\Sigma_a} (\mathbb{D}^n)^{\otimes a},$$

with Σ_{ar}/Σ_a denoting the coset of the subgroup Σ_a of Σ_{ar}, where Σ_a permutes the r-blocks of numbers of cardinality a in $\{1, \ldots, ar\}$. Therefore the homology of $C(G_r\mathbb{D}^n)$ is trivial for all levels $\ell > 0$, and the only contribution to homology arises from $\mathbb{1} \subset C(G_r(X_*))$ in level zero.

Note that

$$\bigodot_{i\in S}\bigodot_{j\in T} C(G_{r_i}\mathbb{D}^{n_j}) \cong C(\bigoplus_{i\in S}\bigoplus_{j\in T} G_{r_i}\mathbb{D}^{n_j}),$$

because $\odot$ is the categorical sum in $C_{\mathsf{dg\Sigma}}$ and $C(-)$ is left adjoint to the forgetful functor from commutative monoids to $\mathsf{mod}\Sigma$. An induction shows that finite $\odot$-products of factors of the form $C(G_r\mathbb{D}^n)$ have homology isomorphic to $\mathbb{1}$ and as $\bigodot_{i\in S}\bigodot_{j\in T} C(G_{r_i}\mathbb{D}^{n_j})$ is a colimit over finite $\odot$-products we get the claim. □

Recall that $C^+_{\mathsf{dg\Sigma}}$ denotes the full subcategory of $C_{\mathsf{dg\Sigma}}$ consisting of commutative monoids A in $C_{\mathsf{dg\Sigma}}$ with $A(0) = \mathbb{S}^0$. Note that the objects $C(G_r(X_*))$ are in $C^+_{\mathsf{dg\Sigma}}$ for all $r > 0$. We denote by $\mathsf{dg\Sigma}_+$ the full subcategory of $\mathsf{dg\Sigma}$ consisting of reduced objects. Let $C^-_{\mathsf{dg\Sigma}}$ be the category of commutative non-unital monoids B in $(\mathsf{dg\Sigma}, \odot)$ with $B(0) = 0$. We obtain $C^+_{\mathsf{dg\Sigma}}$ from $C^-_{\mathsf{dg\Sigma}}$ by adding the unit $\mathbb{1}$. In fact there is an equivalence of categories between $C^+_{\mathsf{dg\Sigma}}$ and $C^-_{\mathsf{dg\Sigma}}$

$$C^-_{\mathsf{dg\Sigma}} \underset{(\bar{\ })}{\overset{(\ )_+}{\rightleftarrows}} C^+_{\mathsf{dg\Sigma}}$$

where for any $A \in C^+_{\mathsf{dg\Sigma}}$ the non-unital algebra $\bar{A}$ consists of the augmentation ideal of A and where $B_+ = \mathbb{1} \oplus B$.

Categorical sums are straightforward. In the category $C^+_{\mathsf{dg\Sigma}}$ the sum of two objects is given by their $\odot$-product. For reduced commutative monoids we obtain an induced structure.

Lemma 5.5. *Let B_1, B_2 be two objects in $C^-_{\mathsf{dg\Sigma}}$. Then their categorical sum is given by the symmetric sequence $B_1 \diamond B_2$ with*

$$(B_1 \diamond B_2)(\ell) = B_1(\ell) \oplus B_2(\ell) \oplus \bigoplus_{p+q=\ell, p,q\geqslant 1} k[\Sigma_\ell] \otimes_{k[\Sigma_p\times\Sigma_q]} B_1(p) \otimes B_2(q).$$

Proof. Note that $B_1 \diamond B_2$ is isomorphic to the augmentation ideal of $((B_1)_+) \odot ((B_2)_+)$. As the augmentation ideal functor is part of an equivalence of categories, it preserves sums. This proves the universal property and also determines the multiplication on $B_1 \diamond B_2$ as that inherited from $((B_1)_+) \odot ((B_2)_+)$. □

The reduced sequence 0 which consists of the zero module in every level is a unit for $\diamond$ (compare [AM10, p. 267]).

Remark 5.6. Note that the proof of Lemma 5.3 also implies that $\Diamond_{i \in S} \Diamond_{j \in T} C(G_{r_i} \mathbb{D}^{n_j})$ has trivial homology.

Theorem 5.7. *The category $C^-_{\mathsf{dg\Sigma}}$ has a Quillen model category structure such that a morphism is a fibration (weak equivalence) if its underlying map in $\mathsf{dg\Sigma}$ is a fibration (weak equivalence).*

Proof. Let $\bar{C}(X_*)$ be the reduced free commutative monoid generated by $X_* \in \mathsf{dg\Sigma}_+$,

$$\bar{C}(X_*) = \bigoplus_{\ell > 0} (X_*)^{\odot \ell} / \Sigma_\ell.$$

We consider the following two sets.

$$\mathcal{I}_- := \{\bar{C}(G_r(\mathbb{S}^{n-1})) \to \bar{C}(G_r(\mathbb{D}^n)); r \geqslant 1, n \geqslant 0\},$$
$$\mathcal{J}_- := \{0 = \bar{C}(G_r(0)) \to \bar{C}(G_r(\mathbb{D}^n)); r, n \geqslant 1\}.$$

We show that $C^-_{\mathsf{dg\Sigma}}$ is a cofibrantly generated model category with generating cofibrations $\mathcal{I}_-$ and generating acyclic cofibrations $\mathcal{J}_-$. The weak equivalences are the maps inducing quasi-isomorphisms in each level. We use Hovey's criterion [Ho99, 2.1.19].

The domains of our generators are small, and the weak equivalences satisfy the 2-out-of-3 property and closure under retracts. We have to understand maps with the right lifting property (RLP) with respect to $\mathcal{I}_-$ and $\mathcal{J}_-$, $\mathcal{I}_-$-inj and $\mathcal{J}_-$-inj.

A diagram of the form

$$\begin{array}{ccc} \bar{C}(G_r(\mathbb{S}^{n-1})) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \bar{C}(G_r(\mathbb{D}^n)) & \longrightarrow & Y \end{array}$$

is adjoint to the diagram

$$\begin{array}{ccc} \mathbb{S}^{n-1} & \longrightarrow & UX(r) \\ \downarrow & & \downarrow Uf(r) \\ \mathbb{D}^n & \longrightarrow & UY(r) \end{array}$$

in the category of chain complexes. Here UX denotes the underlying object in $\mathsf{dg\Sigma}$ of X. Thus the RLP is equivalent to $Uf(r)$ being an acyclic fibration in the category of chain complexes for all $r \geqslant 1$.

Analogously we get that the RLP with respect to $\mathcal{J}_-$ is equivalent to $Uf(r)$ being a fibration of chain complexes for all $r \geqslant 1$. Therefore we obtain that $\mathcal{I}_-$-inj equals the intersection of $\mathcal{J}_-$-inj with the class of weak equivalences.

It remains to show that $\mathcal{J}_-$-cells are weak equivalences and $\mathcal{I}_-$-cofibrations. Remark 5.6 ensures that each building block of a $\mathcal{J}_-$-cell object is acyclic. By induction we see that the nth stage in the building process of a cell object is acyclic and so are sequential colimits of those, thus we get the acyclicity. By definition $\mathcal{I}_-$-cofibrations are the maps with the left lifting property (LLP) with respect to the maps that have the RLP with respect to $\mathcal{I}_-$. Hence we are looking for maps with the LLP with respect to maps f with $Uf(r)$ being an acyclic fibration in chain complexes for all $r \geqslant 1$. The maps in $\mathcal{J}_-$ have this LLP and this property is preserved by pushouts and (transfinite) composition. □

Corollary 5.8. *The category $C^+_{\mathsf{dg\Sigma}}$ possesses a model category structure.*

Proof. A morphism f in $C^+_{\mathsf{dg\Sigma}}$ is a weak equivalence, fibration or cofibration if and only if $\mathbb{1} \oplus f$ is one. □

Note that the model structure on $C^+_{\mathsf{dg\Sigma}}$ is also cofibrantly generated with

$$\mathcal{I}_+ := \{C(G_r(\mathbb{S}^{n-1})) \to C(G_r(\mathbb{D}^n)); r \geqslant 1, n \geqslant 0\}$$

as generating cofibrations and

$$\mathcal{J}_+ := \{\mathbb{1} = C(G_r(0)) \to C(G_r(\mathbb{D}^n)); r, n \geqslant 1\}$$

as generating acyclic cofibrations.

6. A Dold-Kan correspondence for commutative shuffle algebras

It is a folklore result that the model categories of reduced simplicial commutative algebras over $\mathbb{Q}$, $s\mathcal{C}\mathrm{om}_{\mathbb{Q}}$, and of reduced differential graded commutative algebras over $\mathbb{Q}$, $d\mathcal{C}\mathrm{om}_{\mathbb{Q}}$, are Quillen equivalent. A proof is given in [Qu69, p.223]. We adapt Quillen's argument to the setting of pointed commutative shuffle algebras and show that the implementation of the symmetric groups into the monoidal structure allows us to drop the characteristic zero assumption.

We discussed the model category structure on $C^+_{\mathsf{dg\Sigma}}$ earlier. On $C^+_{\mathsf{s\Sigma}}$ we take the Quillen model structure for simplicial objects in a nice category [Qu67, II §4]. As usual, free maps are crucial:

Definition 6.1.

- A morphism $f\colon A_\bullet \to B_\bullet$ in $C^+_{\mathsf{s\Sigma}}$ is *free* if for each $q \geqslant 0$ there is a subsymmetric sequence in sets Z_q of B_q such that $B_q = A_q \odot C(kZ_q)$ and $f_q\colon A_q \to B_q$ is the inclusion of A_q into this sum. In addition, the Z_q are closed under degeneracies in $B_\bullet$, *i.e.*, $s_i(Z_q) \subset Z_{q+1}$ for all degeneracies s_i of $B_\bullet$ and for all q.
- An object $A_\bullet$ in $C^+_{\mathsf{s\Sigma}}$ is *free* if the map from the initial object $\underline{k}$ to $A_\bullet$ is free, *i.e.*, if $A_q \cong C(kZ_q)$ for all q with $Z_q \subset A_q$ as above.

Let W denote the forgetful functor $W\colon C^+_{\mathsf{s\Sigma}} \to \mathsf{s\Sigma}$.

Definition 6.2. (compare [Qu68, 2.9]) A morphism $f\colon A_\bullet \to B_\bullet$ in $C^+_{\mathsf{s\Sigma}}$ is

- a weak equivalence if $W(f)$ is a weak equivalence in $\mathsf{s\Sigma}$.
- a cofibration if f is a retract of a free map, and
- a fibration if it has the right lifting property with respect to acyclic cofibrations.

With these definitions, $C^+_{\mathsf{s\Sigma}}$ is a model category.

The normalization functor $N\colon \mathsf{s\Sigma} \to \mathsf{dg\Sigma}$ passes to a functor $N\colon C^+_{\mathsf{s\Sigma}} \to C^+_{\mathsf{dg\Sigma}}$:

Lemma 6.3. *The functor $N\colon \mathsf{s\Sigma} \to \mathsf{dg\Sigma}$ is lax symmetric monoidal.*

Proof. Let $A_\bullet, B_\bullet$ be any two objects in $\mathsf{s\Sigma}$. We have to show that the diagram

$$\begin{array}{ccc} N(A_\bullet) \odot N(B_\bullet) & \xrightarrow{s_{A,B}} & N(A_\bullet \hat{\odot} B_\bullet) \\ \downarrow{\scriptstyle tw} & & \downarrow{\scriptstyle N(tw)} \\ N(B_\bullet) \odot N(A_\bullet) & \xrightarrow{s_{B,A}} & N(B_\bullet \hat{\odot} A_\bullet) \end{array}$$

commutes for a suitable binatural map s. Here, tw denotes the corresponding symmetry isomorphism. For a fixed level ℓ we define s as the composite

$$\begin{array}{c}\bigoplus_{p+q=\ell} k\Sigma_\ell \otimes_{k\Sigma_p \otimes k\Sigma_q} \otimes N(A_\bullet(p)) \otimes N(B_\bullet(q)) \\ \Big\downarrow \mathrm{id}\otimes sh_{A_\bullet(p),B_\bullet(q)} \\ \bigoplus_{p+q=\ell} k\Sigma_\ell \otimes_{k\Sigma_p \otimes k\Sigma_q} \otimes N(A_\bullet(p) \hat{\otimes} B_\bullet(q)) \\ \Big\downarrow \cong \\ N(\bigoplus_{p+q=\ell} \underline{k\Sigma_\ell} \hat{\otimes}_{\underline{k\Sigma_p} \hat{\otimes} \underline{k\Sigma_q}} A_\bullet(p) \hat{\otimes} B_\bullet(q)).\end{array}$$

Here, $sh_{A_\bullet(p),B_\bullet(q)}$ denotes the ordinary shuffle transformation of the simplicial modules $A_\bullet(p)$ and $B_\bullet(q)$. For a fixed pair (p,q) with $p+q=\ell$, a homogeneous element $[\sigma \otimes x \otimes y]$ with $\sigma \in \Sigma_\ell$, $x \in N(A_\bullet(p))$ and $y \in N(B_\bullet(q))$ is sent via s to $[\sigma \otimes sh(x \otimes y)]$. If we twist first and then apply s we get $[\sigma \circ \chi(p,q) \otimes sh(y \otimes x)]$. As the shuffle transformation is lax symmetric monoidal, this is the image of $[\sigma \otimes sh(x \otimes y)]$ under $N(tw)$. □

Lemma 6.4. *The functor* $N\colon C^+_{\mathsf{s\Sigma}} \to C^+_{\mathsf{dg\Sigma}}$ *possesses a left adjoint*

$$L_N\colon C^+_{\mathsf{dg\Sigma}} \to C^+_{\mathsf{s\Sigma}}.$$

Proof. The construction is standard: if X_* is a reduced object in $\mathsf{dg\Sigma}$, we define $L_N(C(X_*))$ as $C(\Gamma(X_*))$. Every object $A_* \in C^+_{\mathsf{dg\Sigma}}$ can be written as a coequalizer

$$C(\overline{C(\bar{A}_*)}) \rightrightarrows C(\bar{A}_*) \longrightarrow A_*.$$

As a left adjoint, L_N has to respect colimits and hence we define $L_N(A_*)$ as the coequalizer of

$$C(\Gamma(\overline{C(\bar{A}_*)})) = L_N C(\overline{C(\bar{A}_*)}) \rightrightarrows L_N C(\bar{A}_*) = C(\Gamma(\bar{A}_*)).$$

□

Transferring Quillen's sketch of proof [Qu69, p. 223] to the setting of pointed commutative shuffle algebras yields the following result.

Theorem 6.5. *The pair* (N, L_N) *induces a Quillen equivalence between the model categories* $C^+_{\mathsf{dg\Sigma}}$ *and* $C^+_{\mathsf{s\Sigma}}$ *for every commutative ground ring* k.

Before we prove the theorem, we state a few lemmata. First, we need to understand the associated graded of a free commutative monoid.

Lemma 6.6. *Let* S_* *be a symmetric sequence in graded sets with* $S_*(0) = \varnothing$. *Then the associated graded of* $C(kS_*)$ *with respect to the filtration coming from powers of the augmenation ideal* $m \subset C(kS_*)$ *is isomorphic to* $C(kS_*)$*:*

$$grC(kS_*) = C(kS_*)/m \oplus m/m^2 \oplus m^2/m^3 \oplus \ldots \cong C(kS_*).$$

This result is the analog of the well-known fact that the associated graded of a free commutative algebra is isomorphic to the very same free commutative algebra.

As $S_*(0) = \varnothing$, we find that $grC(kS_*)(\ell) = \bigoplus_{i=0}^{\ell} m^i/m^{i+1}$ and that $(kS_*^{\odot i}/\Sigma_i)(\ell) = 0$ for $i > \ell$.

Proof. The indecomposables of $C(kS_*)$ are $m/m^2 \cong kS_*$. The inclusion map $m/m^2 \to grC(kS_*)$ extends to a morphism of commutative monoids

$$\xi\colon C(kS_*) \to grC(kS_*).$$

In every level ℓ elements $p(\ell)$ in

$$C(kS_*)(\ell) = I(\ell) \oplus kS_*(\ell) \oplus (kS_*^{\odot 2}/\Sigma_2)(\ell) \oplus \ldots$$

have only finitely many non-trivial summands, and we denote $p(\ell)$ by $(p_0(\ell), \dots, p_\ell(\ell), 0, \dots)$ with $p_i(\ell) \in (kS_*^{\odot i}/\Sigma_i)(\ell)$. The map ξ is then given by

$$\xi(\ell)(p_0(\ell), \dots, p_\ell(\ell), 0, \dots) = ([p_0(\ell)], [p_1(\ell)], \dots, [p_\ell(\ell)], 0, \dots),$$

where $[p_i(\ell)]$ denotes the equivalence class of $p_i(\ell)$ with respect to m^{i+1}.

As the intersection $(kS_*^{\odot i}/\Sigma_i) \cap m^{i+1}$ is zero, the map ξ is injective.

Let z be an arbitrary element of $grC(kS_*)$. Then $z(\ell)$ has finitely many non-trivial summands and we write

$$z(\ell) = ([z_0(\ell)], [z_1(\ell)], \dots, [z_\ell(\ell)], 0, \dots).$$

Here $z_i(\ell) \in m^i(\ell)$ and thus we know that $z_i(\ell) \in \bigoplus_{r \geqslant i}(kS_*^{\odot r}/\Sigma_r)(\ell)$, so we can express $z_i(\ell)$ as $q_i^i(\ell) + \dots + q_\ell^i(\ell)$ with $q_j^i(\ell) \in (kS_*^{\odot j}/\Sigma_j)(\ell) \subset m^j$. Therefore,

$$\xi(q_0^0(\ell), \dots, q_\ell^\ell(\ell), 0, \dots) = ([q_0^0(\ell)], \dots, [q_\ell^\ell(\ell)], 0, \dots) = ([z_0(\ell)], \dots, [z_\ell(\ell)], 0, \dots).$$

□

Lemma 6.7.

(1) *Let C_* be a cell object in $C^+_{\mathsf{dg}\Sigma}$ and I its augmentation ideal. Then $grC_* \cong C(I/I^2)$ in $C^+_{\mathsf{dg}\Sigma}$.*

(2) *If $A_\bullet$ is a free object in $C^+_{\mathsf{s}\Sigma}$ and $\hat{I}$ is its augmentation ideal, then*

$$grA_\bullet \cong C(\hat{I}/\hat{I}^2) \in C^+_{\mathsf{s}\Sigma}.$$

Proof. The underlying commutative monoids in symmetric sequences of graded modules of C_* and $A_\bullet$ are of the form $C(X_*)$ and $C(Y_\bullet)$, respectively, where X_* and $Y_\bullet$ are trivial in level zero and are degreewise free k-modules. Thus, by Lemma 6.6 we know that the canonical maps

$$C(I/I^2) \to grC_*$$

and

$$C(\hat{I}/\hat{I}^2) \to grA_\bullet$$

are isomorphisms of underlying commutative monoids in symmetric sequences in graded k-modules. It remains to show that these isomorphism are compatible with the differential on C_* and the simplicial structure maps of $A_\bullet$.

Let $[\sigma \otimes x_1 \otimes \dots \otimes x_r]$ denote a generator in $(I/I^2)^{\odot r}/\Sigma_r)(\ell)$ (*i.e.*, $\sigma \in \Sigma_r$, $x_i \in I/I^2(p_i)$ for some suitable p_i) and let d be the differential in C_*. Then

$$d[\sigma \otimes x_1 \otimes \dots \otimes x_r] = \pm \sum_{j=1}^{r} [\sigma \otimes x_1 \otimes \dots \otimes dx_j \otimes \dots \otimes x_r]$$

and the canonical map ξ sends this element to $\pm \sum_{j=1}^r \mu(\sigma \otimes x_1 \otimes \dots \otimes dx_j \otimes \dots \otimes x_r)$, where μ denotes the multiplication in C_*. As d is a derivation, the latter is equal to $d(\mu(\sigma \otimes x_1 \otimes \dots \otimes x_r))$.

The argument for the simplicial structure maps is similar. We spell it out for the face maps. The ith face map, d_i, sends a generator $[\sigma \otimes x_1 \otimes \dots \otimes x_r]$ in $((\hat{I}/\hat{I}^2)^{\odot r}/\Sigma_r)(\ell)$ to $[\sigma \otimes d_i x_1 \otimes \dots \otimes d_i x_r]$, and applying ξ yields $\mu(\sigma \otimes d_i x_1 \otimes \dots \otimes d_i x_r)$. As $A_\bullet$ is a simplicial monoid, this is equal to $d_i(\mu(\sigma \otimes x_1 \otimes \dots \otimes x_r))$. □

Lemma 6.8. *If $C_* \in C^+_{\mathsf{dg}\Sigma}$ is a cell object, then $L_N C_*$ is free.*

Proof. Recall that a cell object C_* in $C^+_{\mathsf{dg}\Sigma}$ is a sequential limit of pushouts of the form

$$\begin{array}{ccc}
\bigodot_{r \in R} \bigodot_{d \in D} C(G_r \mathbb{S}^{d-1}) & \longrightarrow & C_*^n \\
\downarrow & & \downarrow \\
\bigodot_{r \in R} \bigodot_{d \in D} C(G_r \mathbb{D}^d) & \longrightarrow & C_*^{n+1},
\end{array}$$

where the left vertical map is induced by the inclusions of spheres into disks. It therefore suffices to show that one such map $L_N C(G_r \mathbb{S}^{d-1}) \to L_N C(G_r \mathbb{D}^d)$ is free. This works similar to

Quillen's argument [Qu69, Proof of 4.4]: $L_N C(G_r \mathbb{S}^{d-1})$ is $CG_r\Gamma(\mathbb{S}^{d-1})$ and this in turn can be identified with $CG_r(\bar{k}\Delta^{d-1}/\partial\Delta^{d-1})$. Similarly the simplicial model of the d-disc can be chosen as Δ^d/Λ^d_d where Λ^d_d is the d-horn of dimension d, *i.e.*, the simplicial set that is generated by all top faces of $\mathrm{id}_d \in \Delta^d$ but the last one. The inclusion of $\mathbb{S}^{d-1}$ into $\mathbb{D}^d$ can then be modelled by the map $d_d\colon \Delta^{d-1} \to \Delta^d$. We can then choose Z_q to be the symmetric sequence in sets that is concentrated in level r and is generated as a module by all simplices in Δ^d/Λ^d_d that are not in the image of $\Delta^{d-1}/\partial\Delta^{d-1}$ under d_d. □

For an inductive step we need the following auxiliary result about homology and free objects.

Lemma 6.9. *For every $X_* \in \mathsf{dg\Sigma}$ the canonical map $\beta\colon H_*(CX_*) \to H_*(NC(\Gamma(X_*)) = \pi_* C(\Gamma(X_*))$ is an isomorphism.*

Proof. Stover shows [Sto93, 9.10] that for any reduced symmetric sequence M in mod the free commutative algebra generated by M, $C(M)$, embeds into the free associative algebra $T(M) = \bigoplus_{n\geqslant 0} M^{\odot n}$ via a split inclusion $j\colon C(M) \to T(M)$. Thus we can transfer Quillen's retract argument [Qu69, Proof of 4.5] to our context and consider the commutative square

$$\begin{array}{ccc} H_*(CX_*) & \xrightarrow{\beta} & H_*NC(\Gamma(X_*)) \\ j_* \downarrow \uparrow \varrho & & j_* \downarrow \uparrow \varrho \\ H_*(TX_*) & \xrightarrow{\beta'} & H_*NT(\Gamma(X_*)). \end{array}$$

Here ϱ is induced by a splitting of j. In every level the tensor algebra TX_* consists of copies of tensor powers of X_*. The Eilenberg-Zilber equivalence turns β' into an isomorphism. As β is a retract of β', it is an isomorphism as well. □

Proof of Theorem 6.5. This proof is an adaptation of [Qu69, Proof of 4.6].

Let $C_* \in C^+_{\mathsf{dg\Sigma}}$ and denote by $\hat{I}$ the augmentation ideal of $L_N C_*$ and by I the augmentation ideal of C_*. The powers of $\hat{I}$ filter $\hat{I}$ and this filtration respects the multiplication: $\hat{I}^r \cdot \hat{I}^s \subset \hat{I}^{r+s}$. As N is lax monoidal, $N\hat{I}^r \cdot N\hat{I}^s \subset N\hat{I}^{r+s}$ and the unit of the adjunction $\eta\colon C_* \to NL_NC_*$ satisfies $\eta(I^r) \subset N\hat{I}^r$.

We consider the associated graded of C_*, $grC_* = \bigoplus_{r\geqslant 0} I^r/I^{r+1}$, and similarly $gr(L_NC_*) = \bigoplus_{r\geqslant 0} \hat{I}^r/\hat{I}^{r+1}$. As N is exact we obtain

$$Ngr(L_NC_*) = N(\bigoplus_{r\geqslant 0} \hat{I}^r/\hat{I}^{r+1}) \cong \bigoplus_{r\geqslant 0} N\hat{I}^r/N\hat{I}^{r+1}.$$

We denote by triv the full subcategory of $C^-_{\mathsf{dg\Sigma}}$ consisting of objects with trivial multiplication and by i the inclusion functor from triv to $C^-_{\mathsf{dg\Sigma}}$. Let A_* be an object of triv and and let $C_* \in C^+_{\mathsf{dg\Sigma}}$. Then the morphisms in $C^-_{\mathsf{dg\Sigma}}$ from $\bar{C}$ to $i(A_*)$ are precisely the maps in triv from $\bar{C}/\bar{C}^2$ to A_*.

Thus the morphisms in $C^-_{\mathsf{dg\Sigma}}$ from $N\hat{I}$ to $i(A_*)$ are in bijection with the morphisms in triv from $N\hat{I}/N\hat{I}^2$ to A_*, but as $N\hat{I}/N\hat{I}^2$ is isomorphic to $N(\hat{I}/\hat{I}^2)$ and as the category triv is equivalent to the category of reduced symmetric sequences of chain complexes, we can identify this set of morphisms with the morphisms in symmetric sequences of simplicial modules from $\hat{I}/\hat{I}^2$ to $\Gamma(A_*)$, and these in turn correspond to maps in $C^-_{\mathsf{s\Sigma}}$ from $\hat{I}$ to $i(\Gamma(A_*))$. As L_N is left adjoint to N we finally get a bijection with the morphisms from I to $N(i(\Gamma(A_*)))$ and the latter is isomorphic to $i(A_*)$. Therefore, I/I^2 and $\hat{I}/\hat{I}^2$ satisfy the same universal property concerning maps from I to $i(A_*)$ and hence they are isomorphic.

Note that an adjunction argument also shows that $\hat{I}/\hat{I}^2 \cong \Gamma(I/I^2)$. The induced map on the associated graded induced by the unit of the (L_N, N)-adjunction is therefore of the form

$$gr(\eta)\colon gr(C_*) \cong C(I/I^2) \to Ngr(L_NC_*) \cong N(C(\hat{I}/\hat{I}^2)) \cong NC\Gamma(I/I^2).$$

As it sends the generators I/I^2 to $\hat{I}/\hat{I}^2$ it is of the form as β in Lemma 6.9 and thus it is a weak equivalence. A levelwise 5-lemma argument and an induction then shows that $H_*(I/I^r)(\ell) \cong H_*(N(\hat{I}/\hat{I}^r))(\ell)$ for all $r \geqslant 2$. If we fix an ℓ, then – as I and $\hat{I}$ are reduced – for all $r \geqslant \ell+1$ we get $I^r(\ell) = \hat{I}^r(\ell) = 0$ and thus

$$H_*(I(\ell)) \cong H_*(I/I^r)(\ell) \cong H_*(N(\hat{I}/\hat{I}^r))(\ell) \cong H_*N(\hat{I})(\ell),$$

so η is a weak equivalence. □

Remark 6.10. Of course, it is natural to ask whether one can extend the result above and establish a Quillen equivalence between (reduced) E_∞-monoids and commutative monoids in symmetric sequences, or more generally, whether for (certain types of) operads P, homotopy P-algebras and P-algebras have equivalent homotopy categories. We plan to pursue this question in future work.

7. Commutative Hk-algebra spectra

Brooke Shipley proved in [S07] that there is a chain of Quillen equivalences between the model categories of Hk-algebra spectra and differential graded k-algebras for every commutative ring k. This chain is derived from a composite of functors from Hk-module spectra in symmetric spectra via the category of symmetric spectra in simplicial k-modules and symmetric spectra in non-negatively graded chain complexes:

$$Hk\text{-mod} \xrightarrow{Z} \mathsf{Sp}^\Sigma(s\mathsf{mod}) \xrightarrow{\phi^*N} \mathsf{Sp}^\Sigma(\mathrm{dg}\mathsf{mod}).$$

Here $\mathsf{Sp}^\Sigma(s\mathsf{mod})$ is the category of symmetric sequences in simplicial k-modules that are modules over the commutative monoid $\tilde{k}(\mathbb{S})$, with $\tilde{k}(\mathbb{S})(\ell) = \tilde{k}(\mathbb{S}^1)^{\hat{\otimes}\ell}$, where $\tilde{k}(\mathbb{S}^1)$ is the simplicial free k-module generated by the non-basepoint simplices of the 1-sphere. Similarly, $\mathsf{Sp}^\Sigma(\mathrm{dg}\mathsf{mod})$ is the category of symmetric sequences in non-negatively graded chain complexes of k-modules with a module structure over the commutative monoid $k[\bullet]$, with $k[\bullet](\ell) = k[\ell]$ being the chain complex with k concentrated in chain degree ℓ with trivial Σ_ℓ-action. These categories of symmetric spectra are symmetric monoidal with respect to the smash product which is nothing but the coequalizer of the tensor product of symmetric sequences where the action of the respective commutative monoid on the left and right factor is identified.

The functor Z is defined as

$$Z(M) = \tilde{k}(M) \wedge_{\tilde{k}(Hk)} Hk$$

and

$$\phi^*N(A_\bullet)(\ell) = N(A_\bullet(\ell))$$

with $k[\bullet]$-module structure given by the equivalences

$$\phi(\ell)\colon k[\ell] \cong k[1]^{\otimes \ell} \cong N(\tilde{k}(\mathbb{S}^1))^{\otimes \ell} \longrightarrow N(\tilde{k}(\mathbb{S}^1)^{\hat{\otimes}\ell}) = N(\tilde{k}(\mathbb{S})(\ell)).$$

Shipley shows [S07, p. 372] that Z is strong symmetric monoidal and that ϕ^*N is lax symmetric monoidal.

Proposition 7.1. *The forgetful functors $V_1\colon \mathsf{Sp}^\Sigma(s\mathsf{mod}) \to \mathsf{s}\Sigma$ and $V_2\colon \mathsf{Sp}^\Sigma(\mathsf{Ch}_{\geqslant 0}) \to \mathsf{dg}\Sigma$ are lax symmetric monoidal. Hence every commutative Hk-algebra spectrum A gives rise to a commutative simplicial shuffle algebra $V_1(ZA)$ and a commutative differential graded shuffle algebra $V_2(\phi^*N(ZA))$.*

Proof. For $A_\bullet, B_\bullet \in \mathsf{Sp}^\Sigma(s\mathsf{mod})$ there is a binatural projection map

$$\pi_1(A_\bullet, B_\bullet)\colon V_1(A_\bullet)\hat{\odot}V_1(B_\bullet) = A_\bullet \hat{\odot} B_\bullet \to A_\bullet \hat{\odot}_{\tilde{\mathbb{Z}}(\mathbb{S})} B_\bullet = A_\bullet \wedge B_\bullet.$$

As $\tilde{k}(\mathbb{S})$ is a commutative monoid and as $\hat{\odot}$ is a symmetric monoidal structure, π_1 turns V_1 into a lax symmetric monoidal functor. An analogous argument applies to V_2. □

Remark 7.2. If A is a commutative Hk-algebra spectrum, then $V_1(ZA)$ is pointed if $ZA(0) = \underline{k}$, for instance if $A = Hk \vee M$ is a square-zero extension, where M is an Hk-module concentrated in positive levels.

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