LINEAR OPERADS

Operads occur when you want to study objects with certain algebraic properties, for instance, associative monoids. You can study them in the category of vector spaces or chain complexes or topological space or simplicial sets or... So the object might look rather different but the type of algebraic structure that you want to study is the same.

Today, we will only consider linear operads and that means that our underlying symmetric monoidal category is the one of k-vector spaces for a field k. Most things also work if k is just a commutative ring.

Later, we want to study Koszul duality and this duality for instance relates Lie algebras and commutative algebras whereas the operad that encodes an associative monoid structure turns out to be self-dual.

The term *operad* was coined by Peter May in the setting of topological spaces. He used them to study iterated loop spaces.

1. Symmetric sequences and Schur functors

Definition 1.1.

(1) A symmetric sequence over k is a sequence

$$M := (M(0), M(1), M(2), \dots)$$

where every $M(n) \in \mathsf{Vect}$ is a right $k[\Sigma_n]$-module.

(2) The corresponding category Vect-Σ has as morphisms $f\colon M \to M'$ sequences $f = (f_0, f_1, f_2, \dots)$ where every $f_n\colon M(n) \to M'(n)$ is $k[\Sigma_n]$-linear.

(3) M is called reduced if $M(0) = 0$.

Remark 1.2. We can view M as a functor $\Sigma^{op} \to \mathsf{Vect}$, where Σ is the category whose objects are $\mathbb{N}_0$ and whose morphisms are

$$\Sigma(n, m) = \begin{cases} \varnothing, & n \neq m, \\ \Sigma_n, & n = m. \end{cases}$$

Note that every $M(n)$ is a Σ_n-representation, so the theory of representations of the symmetric groups plays an important role in this topic. In characteristic zero this is well-understood but in finite characteristic it is hard and classifying such representations is an open question.

Definition 1.3. If M and M' are two symmetric sequences, then we can form

(1) their direct sum via $(M \oplus M')(n) := M(n) \oplus M'(n)$,

(2) their Hadamard tensor product is

$$M \otimes_H M'(n) := M(n) \otimes M'(n),$$

(3) their tensor product as

$$(M \odot M')(n) := \bigoplus_{p+q=n} M(p) \otimes M'(q) \otimes_{k[\Sigma_p \times \Sigma_q]} k[\Sigma_n]$$

(4) and their composite is

$$(M \circ M')(n) := \bigoplus_{\ell \geqslant 0} M(\ell) \otimes_{k[\Sigma_\ell]} (M')^{\odot \ell}(n).$$

Remark 1.4. The tensor product of symmetric sequences is actually the Day convolution product in the functor category $\mathsf{Fun}(\Sigma^{op}, \mathsf{Vect})$. Its unit is the symmetric sequence $(k, 0, 0, \ldots)$.

One can be very explicit: for $p + q = n$ the subgroup $\Sigma_p \times \Sigma_q < \Sigma_n$ has the (p, q)-shuffle permutations as coset representatives for $\Sigma_p \times \Sigma_q \backslash \Sigma_n$.

The composite of two symmetric sequences is a bit involved, but can be made explicit:

$$\begin{aligned}(M \circ M')(n) &= \bigoplus_{\ell \geqslant 0} M(\ell) \otimes_{k[\Sigma_\ell]} (M')^{\odot \ell}(n) \\ &= \bigoplus_{\ell \geqslant 0} M(\ell) \otimes_{k[\Sigma_\ell]} \left(\bigoplus_{p_1 + \ldots + p_\ell = n} M'(p_1) \otimes \ldots \otimes M'(p_\ell) \otimes_{k[\Sigma_{p_1} \times \ldots \times \Sigma_{p_\ell}]} k[\Sigma_n] \right).\end{aligned}$$

Definition 1.5. If M is a symmetric sequence, then its Schur functor is

$$\widetilde{M} \colon \mathsf{Vect} \to \mathsf{Vect}, \quad V \mapsto \widetilde{M}(V) = \bigoplus M(n) \otimes_{k[\Sigma_n]} V^{\otimes n}.$$

Here, $\otimes = \otimes_k$ and Σ_n acts on the left on $V^{\otimes n}$ by

$$\sigma.(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}.$$

With respect to the tensor product and the composite of symmetric sequences one can show the following compatibility results for Schur functors:

- $\widetilde{M \odot M'} \cong \widetilde{M} \otimes \widetilde{M'}$, and
- $\widetilde{M \circ M'} \cong \widetilde{M} \circ \widetilde{M'}$.

The first isomorphism of functors is rather straightforward, but the last one is more painful. Evaluated on a vector space V they say

- $\widetilde{M \odot M'}(V) \cong \widetilde{M}(V) \otimes \widetilde{M'}(V)$, and
- $\widetilde{M \circ M'}(V) \cong \widetilde{M}(\widetilde{M'}(V))$.

2. Linear operads and their algebras

Lemma 2.1. *The category* $\mathsf{Vect}\text{-}\Sigma$ *with the composition* $\circ$ *is a monoidal category (not symmetric!). The unit of this structure is the symmetric sequence* $e = (0, k, 0, \ldots)$.

We can use this lemma in order to define operads and algebras over them in the most elegant and concise (but maybe obscure?) manner:

Definition 2.2. We consider the monoidal category $(\mathsf{Vect}\text{-}\Sigma, \circ, e)$:

(1) A (linear) operad O is a monoid in this category.
(2) Let A be a vector space. We consider the symmetric sequence $\mathbb{A} = (A, 0, 0, \ldots)$. We say that A is an algebra over the operad O, if $\mathbb{A}$ is a left O-module in

Vect-Σ, *i.e.*, if there is a map $\theta_A \colon O \circ \mathbb{A} \to \mathbb{A}$ that is associative and the map $\eta \colon e \to O$ sits in a commuting diagram:

$$\begin{array}{ccc} e \circ \mathbb{A} & \xrightarrow{\eta \circ \mathrm{id}} & O \circ \mathbb{A} \\ & \searrow^{\cong} & \downarrow{\scriptstyle \theta_A} \\ & & \mathbb{A} \end{array}$$

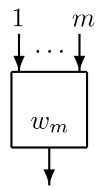
So what does that say?

First, let's unravel the definition of an operad: We have a collection of vector spaces $O(n)$ with a right Σ_n-action for $n \in \mathbb{N}_0$, together with a unit morphism $\eta \colon k \to O(1)$, and composition morphisms

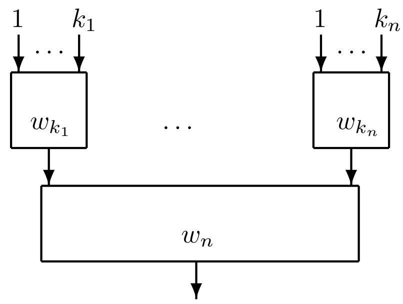
$$\gamma \colon O(n) \otimes O(k_1) \otimes \ldots \otimes O(k_n) \to O\left(\sum_{i=1}^{n} k_i\right)$$

for $n \geqslant 1$ and $k_i \geqslant 0$.

You might want to think about an $w_m \in O(m)$ as a device that can digest m inputs and gives back one output:



We can stack n such devices with k_i inputs and one output on top of a device with n inputs, in order to create something with $k_1 + \ldots + k_n$ inputs and one output:



The composition maps are associative, unital, and equivariant in the following sense:

- Let k be $\sum_{i=1}^{n} k_i$ and let m_i be the sum $k_1 + \ldots + k_i$. Starting from
$$O(n) \otimes \left(\bigotimes_{i=1}^{n} O(k_i)\right) \otimes \left(\bigotimes_{j=1}^{k} O(\ell_j)\right),$$
one can first use γ on $O(n) \otimes (\bigotimes_{i=1}^{n} O(k_i))$ to get to $O(k) \otimes \left(\bigotimes_{j=1}^{k} O(\ell_j)\right)$ and then use another instance of γ to map to $O(\sum_{j=1}^{k} \ell_j)$. We require that

this morphism is equal to the one, where we first just permute the tensor factors

$$\left(\bigotimes_{i=1}^{n} O(k_i)\right) \otimes \left(\bigotimes_{j=1}^{k} O(\ell_j)\right),$$

such that every $O(k_i)$ comes next to $O(\ell_{m_{i-1}+1}) \otimes \ldots \otimes O(\ell_{m_i})$. We can evaluate γ on these terms to get to $O(n) \otimes (\bigotimes_{i=1}^{n} O(\ell_{m_{i-1}+1} + \ldots + \ell_{m_i}))$ and then apply γ again to end up in $O(\sum_{j=1}^{k} \ell_j)$:

$$\begin{array}{ccc}
O(n)\otimes(\bigotimes_{i=1}^{n} O(k_i))\otimes(\bigotimes_{j=1}^{\sum_{i=1}^{n} k_i} O(\ell_j)) & \xrightarrow{\gamma\otimes 1} & O(k)\otimes(\bigotimes_{j=1}^{\sum_{i=1}^{n} k_i} O(\ell_j)) \\
\downarrow{\scriptstyle \text{shuffle}} & & \downarrow{\scriptstyle \gamma} \\
O(n)\otimes(\bigotimes_{i=1}^{n}(O(k_i)\otimes O(\ell_{m_{i-1}+1})\otimes\ldots\otimes O(\ell_{m_i}))) & & O(\sum_{j=1}^{k} \ell_j). \\
\downarrow{\scriptstyle 1\otimes\gamma^{\otimes n}} & & \uparrow{\scriptstyle \gamma} \\
O(n)\otimes(\bigotimes_{i=1}^{n} O(\ell_{m_{i-1}+1}+\ldots+\ell_{m_i})) & = & O(n)\otimes(\bigotimes_{i=1}^{n} O(\ell_{m_{i-1}+1}+\ldots+\ell_{m_i}))
\end{array}$$

- The unit map $\eta\colon k \to O(1)$ fits into the following commutative diagrams:

$$\begin{array}{ccc}
O(n) \otimes k^{\otimes n} & \xrightarrow{\cong} & O(n) \\
\downarrow{\scriptstyle \mathrm{id}\otimes\eta^{\otimes n}} & \nearrow{\scriptstyle \gamma} & \\
O(n) \otimes O(1)^{\otimes n} & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
k \otimes O(k) & \xrightarrow{\cong} & O(k). \\
\downarrow{\scriptstyle \eta\otimes\mathrm{id}} & \nearrow{\scriptstyle \gamma} & \\
O(1) \otimes O(k) & &
\end{array}$$

- We require the following two equivariance conditions:

  (1) If $\sigma \in \Sigma_n$, then we denote by $\sigma(k_1, \ldots, k_n)$ the permutation in Σ_k that permutes the blocks $k_{i-1}+1, \ldots, k_i$ for $1 \leqslant i \leqslant n$ as σ permutes the numbers $1, \ldots, n$. Then, the following diagram must commute:

$$\begin{array}{ccc}
O(n) \otimes O(k_1) \otimes \ldots \otimes O(k_n) & \xrightarrow{\sigma\otimes\sigma^{-1}} & O(n) \otimes O(k_{\sigma(1)}) \otimes \ldots \otimes O(k_{\sigma(n)}) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\
O(k) & \xrightarrow{\sigma(k_{\sigma(1)},\ldots,k_{\sigma(n)})} & O(k).
\end{array}$$

  (2) We also need the permutation $\tau_1 \oplus \ldots \oplus \tau_n \in \Sigma_{k_1+\ldots+k_n}$ for $\tau_i \in \Sigma_{k_i}$ for $1 \leqslant i \leqslant n$, which is the concatenation of the τ_is. Then the diagram

$$\begin{array}{ccc}
O(n) \otimes O(k_1) \otimes \ldots \otimes O(k_n) & \xrightarrow{1_{O(n)}\otimes\tau_1\otimes\ldots\otimes\tau_n} & O(n) \otimes O(k_1) \otimes \ldots \otimes O(k_n) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\
O(k) & \xrightarrow{\tau_1\oplus\ldots\oplus\tau_n} & O(k)
\end{array}$$

is commutative.

The vector space $O(n)$ is often called the *n-ary part of the operad.*

There is a particularly sleak description of operads due to Martin Markl in terms of $\circ_i$ products: one requires maps

$$\circ_i \colon O(m) \otimes O(n) \to O(m+n-1)$$

that have to satisfy several coherence conditions. In terms of the classical definition, you can define the $\circ_i$ map as

$$w \circ_i \nu := \gamma(w \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes \nu \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id})$$

where you insert the operation ν into the i-th spot of w.

2.1. **Examples of operads.**

- Let V be a vector space. The endomorphism operad on V, $\mathit{End}(V)$, has as the n-ary part the vector space

$$\mathit{End}(V)(n) = \mathrm{Hom}_k(V^{\otimes n}, V).$$

  The operad structure is just given by insertion of homomorphisms:

$$\gamma(f \otimes h_1 \otimes \ldots \otimes h_n) = f \circ (h_1 \otimes \ldots \otimes h_n), \quad f \in \mathit{End}(V)(n), \quad h_i \in \mathit{End}(V)(k_i).$$

- The operad Com has $\mathrm{Com}(n) = k$ for all n with trivial Σ_n-action. The composition uses iterations of the canonical identification $k \otimes_k k \cong k$.
- The operad As has the k vector space with basis Σ_n as n-ary part: $\mathrm{As}(n) = k\{\Sigma_n\}$. The composition is dictated by the equivariance condition in any operad: Explicitly, $(\sigma, \tau_1, \ldots, \tau_n) \in \Sigma_n \times \Sigma_{k_1} \times \ldots \times \Sigma_{k_n}$ is sent to

$$\gamma(\sigma, \tau_1, \ldots, \tau_n) = (\tau_{\sigma^{-1}(1)} \oplus \ldots \oplus \tau_{\sigma^{-1}(n)}) \circ \sigma(k_1, \ldots, k_n).$$

- The operad Lie has as $\mathrm{Lie}(n)$ the k-sub vector space of the free Lie algebra on n generators $\mathrm{Lie}(x_1, \ldots, x_n)$, generated by Lie-words in which every generator occurs exactly once. For instance for $n = 4$ you have

$$[[[x_3, x_2], x_1], x_4] + [[x_4, x_3], [x_1, x_2]] \in \mathrm{Lie}(4).$$

  Beware of the antisymmetry condition if the characteristic is two! In finite characteristic you might want to model restricted Lie algebras. These need operadic algebras with divided powers.

  Free Lie algebras have many different bases, called Hall bases. Reutenauer's book gives a comprehensive account. Some of these are more amenable to generalizations to non-linear situations than others.

We unravel what an algebra A over an operad O is: A vector space A is an O-algebra if there are linear maps $\theta_n \colon O(n) \otimes A^{\otimes n} \to A$ for all n that are associative, unital, and equivariant in the following sense:

(1) The action maps are associative. For all $k = \sum_{i=1}^{n} k_i$, the diagram

$$\begin{array}{ccc}
O(n) \otimes O(k_1) \otimes \ldots \otimes O(k_n) \otimes A^{\otimes k} & \xrightarrow{\gamma \otimes 1} & O(k) \otimes A^{\otimes k} \\
\downarrow{\scriptstyle \text{shuffle}} & & \\
O(n) \otimes O(k_1) \otimes A^{\otimes k_1} \otimes \ldots \otimes O(k_n) \otimes A^{\otimes k_n} & & \downarrow{\scriptstyle \theta_k} \\
\downarrow{\scriptstyle 1 \otimes \theta_{k_1} \otimes \ldots \otimes \theta_{k_n}} & & \\
O(n) \otimes A^{\otimes n} & \xrightarrow{\theta_n} & A
\end{array}$$

commutes.

(2) The action is unital:

$$\begin{array}{ccc}
k \otimes A & \xrightarrow{\cong} & A. \\
\downarrow{\scriptstyle \eta \otimes 1} & \nearrow{\scriptstyle \theta_1} & \\
O(1) \otimes A & &
\end{array}$$

(3) The symmetric group action on the operad and on n-fold tensor powers of A is compatible for all n:

(1)
$$\begin{array}{ccc}
O(n) \otimes A^{\otimes n} & \xrightarrow{\sigma \otimes \sigma^{-1}} & O(n) \otimes A^{\otimes n} \\
{\scriptstyle \theta_n} \searrow & & \swarrow {\scriptstyle \theta_n} \\
& A &
\end{array}$$

commutes for all $\sigma \in \Sigma_n$.

There are many equivalent ways of saying what an algebra over an operad is. I'll mention one more:

A vector space A is an algebra over an operad O, if there is an action map $\theta_A \colon \widetilde{O}(A) \to A$ from the Schur functor for O and A to A such that the diagrams

$$\begin{array}{ccccc}
\widetilde{O \circ O}(A) & \xrightarrow{\cong} & \widetilde{O}(\widetilde{O}(A)) & \xrightarrow{\widetilde{O}(\theta_A)} & \widetilde{O}(A) \\
\downarrow{\scriptstyle \widetilde{\gamma}(A)} & & & & \downarrow{\scriptstyle \theta_A} \\
\widetilde{O}(A) & & \xrightarrow{\theta_A} & & A
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\widetilde{e}(A) & \xrightarrow{i_1} & \widetilde{O}(A) \\
& \searrow\!\!\!= & \downarrow{\scriptstyle \theta_A} \\
& & A
\end{array}$$

commute.

2.2. **Examples of algebras over operads.**

- Let V be a vector space. Then V is an algebra over its endomorphism operad, $End(V)$. The action map

$$\theta_V \colon End(V)(n) \otimes V^{\otimes n} = \mathrm{Hom}_k(V^{\otimes n}, V) \otimes V^{\otimes n} \to V$$

just evaluates a linear map $f \colon V^{\otimes n} \to V$ on $V^{\otimes n}$.

  In fact, this gives an alternative way of defining an algebra over an operad: A vector space A is an O-algebra if and only if there is a morphism of operads $\alpha \colon O \to End(A)$.

- An algebra over the operad Com is a commutative algebra A: As $\mathrm{Com}(n) = k$ for all n, we just have one operation (up to scalar multiple) $\mu_n \in \mathrm{Com}(n)$ and for every permutation $\sigma \in \Sigma_n$ we have $\mu_n \circ \sigma = \mu_n$ because the Σ_n-action was trivial. If we abbreviate $\theta_n(\mu_n \otimes a_1 \otimes \ldots \otimes a_n)$ by $\mu_n(a_1 \otimes \ldots \otimes a_n) = a_1 \cdot \ldots \cdot a_n$, then the equivariance condition on θ says that
$$a_1 \cdot \ldots \cdot a_n = a_{\sigma^{-1}(1)} \cdot \ldots \cdot a_{\sigma^{-1}(n)} \text{ for all } \sigma \in \Sigma_n$$
and hence the multiplication in A is commutative and associative.
- An algebra A over the operad As is an associative algebra: The multiplication in A corresponds to
$$a \cdot b = \theta_2(\mathrm{id}_2 \otimes a \otimes b) \text{ for } a, b \in A$$
but we also have
$$b \cdot a = \theta_2((1,2) \otimes a \otimes b).$$
For larger n
$$\theta_n(\sigma \otimes a_1 \otimes \ldots \otimes a_n) = a_{\sigma^{-1}(1)} \cdot \ldots \cdot a_{\sigma^{-1}(n)}$$
so the multiplication is associative but not necessarily commutative.
- An algebra $\mathfrak{g}$ over the operad Lie is a Lie-algebra with
$$[x, y] := \theta_2([x_1, x_2] \otimes x \otimes y) \text{ for } x, y \in \mathfrak{g}.$$
Anti-symmetry and the Jacobi relation hold because they hold in the free Lie-algebra.

Remark 2.3. A morphism of operads $\beta\colon O \to P$ induces a functor from the category of P-algebras to the category of O-algebras: if A is a P-algebra, then we can precompose its structure map $P \to End(A)$ with β to obtain $O \to End(A)$.

Every commutative and associative algebra is an associative algebra. This can be encoded by the map of operads
$$\mathrm{As} \to \mathrm{Com}, \text{ with } \mathrm{As}(n) = k[\Sigma_n] \to k = \mathrm{Com}(n), \sigma \mapsto 1 \text{ for all } \sigma \in \Sigma_n.$$
Every associative algebra A is a Lie algebra with the commutator bracket
$$[a, b] := ab - ba.$$
The Schur functor of an operad plays another important role:

Proposition 2.4. *Let O be a linear operad. The functor $\widetilde{O}\colon \mathsf{Vect} \to O$-algs, $V \mapsto \tilde{O}(V)$, is left adjoint to the forgetful functor $U\colon O$-algs $\to \mathsf{Vect}$.*

So $\tilde{O}(V) = \bigoplus_{n \geqslant 0} O(n) \otimes_{k[\Sigma_n]} V^{\otimes n}$ is the free O-algebra generated by V.

3. Cooperads and coalgebras over them

Beware, that there are different conventions about cooperads and their coalgebras. The one below follows Loday-Vallette.

There is a slightly different monoidal structure on Vect-Σ than $\circ$ and cooperads are comonoids in that structure.

Definition 3.1.

(1) For two symmetric sequences M, N we set

$$(M \tilde{\circ} N)(n) = \bigoplus_{\ell \geqslant 0} (M(\ell) \otimes N^{\odot \ell})^{\Sigma_\ell}(n).$$

The unit for this structure is still $e = (0, k, 0, \ldots)$.

(2) A (linear) cooperad P is a symmetric sequence in the category of k-vector spaces that is a comonoid with respect to $\tilde{\circ}$.

(3) If $\mathcal{C}$ is a cooperad and if C is a k-vector space, then we set

$$\hat{\mathcal{C}}(C) := \prod_{n \geqslant 0} (\mathcal{C}(n) \otimes C^{\otimes n})^{\Sigma_n}.$$

(4) A coalgebra C over a cooperad $\mathcal{C}$ is a vector space with a cooperation $\Delta_C \colon C \to \hat{\mathcal{C}}(C)$ such that this cooperation is coassociative and counital in the sense that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta_C} & \hat{\mathcal{C}}(C) \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_{\hat{\mathcal{C}}(C)}} \\ \hat{\mathcal{C}}(C) & \xrightarrow{\hat{\mathcal{C}}(\Delta_C)} & \hat{\mathcal{C}}(\hat{\mathcal{C}}(C)) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C & & \\ {\scriptstyle \Delta_C}\downarrow & \searrow\!\!= & \\ \hat{\mathcal{C}}(C) & \xrightarrow{\varrho_1} & C \end{array}$$

commute.

Explicitly, a cooperad has decompositions

$$\chi \colon \mathcal{C}(\sum_{i=1} k_i) \to \mathcal{C}(n) \otimes \mathcal{C}(k_1) \otimes \ldots \otimes \mathcal{C}(k_n)$$

and these are co-unital, coassociative and satisfy an equivariance condition, dual to the ones of an operad.

Similarly, for a coalgebra C over a cooperad $\mathcal{C}$ we have linear coaction maps

$$C \to (\mathcal{C}(n) \otimes C^{\otimes n})^{\Sigma_n}$$

satisfying the dual axioms to those of an algebra over an operad.

Note that the linear dual $O(n) := \mathrm{Hom}_k(\mathcal{C}(n), k)$ of a cooperad $\mathcal{C}$ is an operad. The dual $O(n)$ then naturally carries a left Σ_n-action but you can turn this into a right action using the usual trick: $w.\sigma := \sigma^{-1}.w$ for $w \in O(n)$, $\sigma \in \Sigma_n$.

The converse is tricky: Even if we consider an operad O such that every $O(n)$ is finite dimensional, the dual $\mathcal{C}$ will have some completed coproduct, landing in $\mathcal{C} \hat{\circ} \mathcal{C}$ with

$$(\mathcal{C} \hat{\circ} \mathcal{C})(n) = \prod_{\ell \geqslant 0} \left(\mathcal{C}(\ell) \otimes \left(\prod_{i_1 + \ldots + i_\ell = n} (\mathcal{C}(i_1) \otimes \ldots \otimes \mathcal{C}(i_\ell)) \right) \right)^{\Sigma_\ell}.$$

Spoiler: We will later see that for a so called quadratic operad there is a Koszul dual cooperad.

References:

- Jean-Louis Loday, La renaissance des opérades, Séminaire Bourbaki, Vol. 1994/95, Astérisque 237, 1996, Exp. No. 792, 3, 47–74.

- Jean-Louis Loday, Bruno Vallette, Algebraic operads. Grundlehren der mathematischen Wissenschaften 346. Springer, Heidelberg, 2012. xxiv+634 pp.
- Christophe Reutenauer, Free Lie algebras. London Mathematical Society Monographs. New Series, 7. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. xviii+269 pp.
- Birgit Richter, From Categories to Homotopy Theory, Cambridge Studies in Advanced Mathematics No 188, Cambridge University Press, Cambridge, 2020. x+390 pp.