Algebra (Master): An Introduction to Homological Algebra
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Contents

Invariants for topological spaces or other geometric objects are often expressed in terms of homological algebra. For instance, if A is an abelian group and X is a topological space, then the nth homology groups of X with coefficients in A can be calculated as

$$H_n(X; A) \cong H_n(X) \otimes A \oplus \mathsf{Tor}(H_{n-1}(X), A).$$

In topology you learn an elementary definition of the Tor-groups above that works in the context of abelian groups (also known as $\mathbb{Z}$-modules). As $\mathbb{Z}$ is a principal ideal domain, Tor-groups are particularly easy and the Tor above is actually a $\mathsf{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), A)$ and there are no higher Tor-groups.

We will see a more general version that works for modules over arbitrary rings. In these situations there can be Tor-groups $\mathsf{Tor}_*^R(M, N)$ for arbitrary $* \geqslant 0$.

In algebra, you have encountered groups, rings and fields. Algebraic objects also have invariants. For every group G one can consider the homology groups of G, $H_*(G)$, and for an algebra A over a commutative ring k we can determine its Hochschild homology groups, $\mathsf{HH}_*^k(A)$. These are invariants in the sense that

- if for two groups G_1 and G_2 we know that $H_*(G_1) \ncong H_*(G_2)$, then these two groups are not isomorphic, and
- if for two k-algebras A_1 and A_2 the groups $\mathsf{HH}_*^k(A_1)$ and $\mathsf{HH}_*^k(A_2)$ are not isomorphic, then the algebras are also not isomorphic.

These homology groups above have descriptions in terms of Tor-functors:

- For every group G
$$H_*(G) \cong \mathsf{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}),$$
- and if A is projective as a k-module we also have
$$\mathsf{HH}_*^k(A) \cong \mathsf{Tor}_*^{A \otimes_k A^{op}}(A, A).$$

So there are Tor-groups in these descriptions and other concepts that I'll explain later. This might look frightening, but in many cases these invariants are not that hard to calculate. In this lecture course we will develop the tools for defining such homology groups and Tor-functors. We will also study methods for calculating them and we will apply them in several examples. In particular, spectral sequences are important tools for calculations but they can also used for proofs. The plan is as follow:

- Chapter I: Basics
- Chapter II: Derived functors
- Chapter III: Homology of groups
- Chapter IV: Hochschild homology
- Chapter V: Spectral sequences

Some notation: We denote by $A \subset B$ the fact that A is a subset of B. That does not exclude $A = B$. The natural numbers are denoted by $\mathbb{N}$ and $\mathbb{N}_0$ are the natural numbers and zero.

CHAPTER I

Basics

I.1. Rings and modules

You all know what a ring is, but still:

Definition I.1.1.

(a) A set R with two maps

$$+\colon R \times R \to R \text{ and}$$
$$\cdot\colon R \times R \to R$$

is a *ring*, if

(1) $(R, +)$ is an abelian group,

(2) $\cdot$ is associative, *i.e.*, for all $x, y, z \in R$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

(3) For all $x, y, z \in R$:

$$x \cdot (y + z) = x \cdot y + x \cdot z,$$
$$(x + y) \cdot z = x \cdot z + y \cdot z.$$

(4) There is a unit element $1 = 1_R \in R$, such that

$$x \cdot 1 = x = 1 \cdot x$$

for all $x \in R$.

(b) A ring is *commutative*, if for all $x, y \in R$

$$x \cdot y = y \cdot x.$$

As usual, we will often write xy for $x \cdot y$.

Definition I.1.2.

- A map $f\colon R \to R'$ between two rings R and R' is a *ring homomorphism*, if for all $x, y \in R$
  (a) $f(x + y) = f(x) + f(y)$,
  (b) $f(xy) = f(x)f(y)$ and
  (c) $f(1_R) = 1_{R'}$.
  We denote the set of all ring homomorphisms from R to R' by $\mathsf{rings}(R, R')$.
- If R is a commutative ring and if $\eta\colon R \to A$ is a ring homomorphism, with $\eta(r)x = x\eta(r)$ for all $r \in R$ and $x \in A$, then we call A an *R-algebra*.

Remark I.1.3.

(a) There is the *zero ring* $R = \{0\}$. It is exceptional with the property that $0 = 1$. There is a unique ring homomorphism from every ring R' to 0.

(b) Let R be any ring. There is a unique ring homomorphism $\chi\colon \mathbb{Z} \to R$ from the ring of integers to R, determined by $\chi(1) = 1_R$.

Examples I.1.4.

- Let R be a commutative ring. Important R-algebras are the *polynomial ring*, $R[x]$, over R and the *ring of formal power series*, $R[[x]]$ over R. Elements of the latter are power series
$$\sum_{i\geqslant 0} a_i x^i$$
with $a_i \in R$ and the word 'formal' indicates that we don't consider convergence issues. You add polynomials and power series by adding the coefficients for every x^i. In both cases the multiplication is given by
$$\left(\sum_{i\geqslant 0} a_i x^i\right) \cdot \left(\sum_{j\geqslant 0} b_j x^j\right) = \sum_{n\geqslant 0} \left(\sum_{i+j=n} a_i b_j\right) x^n.$$
- Let $(M, \cdot)$ be a monoid and let R be a commutative ring. The *monoid algebra*, $R[M]$ is the R-algebra whose elements are of the form $\sum_{m\in M} a_m m$, with $a_m \in R$ and $m \in M$, where only finitely many $a_m \in R$ are non-trivial. We define
$$\sum_{m\in M} a_m m + \sum_{m\in M} b_m m := \sum_{m\in M} (a_m + b_m)m$$
and
$$(\sum_{m\in M} a_m m) \cdot (\sum_{m\in M} b_m m) = \sum_{m\in M} c_m m,$$
with $c_\ell = \sum_{mn=\ell} a_m b_n$.
Later, we will mostly study *group algebra*, *i.e.*, R-algebras $R[G]$, where G is a group. These algebras are crucial for defining and studying group homology.
- What is $R[(\mathbb{N}_0, +)]$? There is an isomorphism $R[\mathbb{N}_0] \to R[x]$ that sends i to x^i.
- The group algebra $R[\mathbb{Z}]$ is isomorphic to the ring of *Laurent polynomials*, $R[x^{\pm 1}]$. You'll work out the details in an exercise.
- If we take $R = \mathbb{R}$ and $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ the quaternion group, then $\mathbb{R}[Q_8]$ is *not* isomorphic to the skew field of the quaternions, $\mathbb{H}$.
- Let C_n be a cyclic group of order n with generator t. Then in $R[C_n]$ we have
$$(1 - t)(1 + t + \ldots + t^{n-1}) = 0,$$
and hence $R[C_n]$ has zero divisors.
- For a group G and a commutative ring R the group algebra $R[G]$ is commutative if and only if G is abelian. If G is the trivial group $\{e\}$, then $R[\{e\}] \cong R$.
- An open conjecture by Kaplansky is, that for a torsion-free group G and a field K the group algebra $K[G]$ has no non-trivial zero divisors.

Definition I.1.5. Let R be a ring. The *opposite ring of* R, R^{op}, has the same underlying abelian group as R, so $(R, +) = (R^{op}, +)$, but its multiplication is reversed: for $r_1, r_2 \in R$ we define
$$r_1 \cdot^{op} r_2 := r_2 r_1,$$
where the latter denotes the multiplication of r_2 and r_1 in R.

Please check, that this is a ring. If R is commutative, then $R^{op} \cong R$.

Modules are generalizations of vector spaces and of abelian groups. We allow arbitrary rings as scalars.

Definition I.1.6. Let R be a ring.

(a) A *left R-module* (or R-module for short) is an abelian group $(M, +)$ together with a biadditive map $\mu\colon R \times M \to M$ such that
  - $\mu(1_R, m) = m$ for all $m \in M$.
  - $\mu(r_1, \mu(r_2, m)) = \mu(r_1 r_2, m)$ for all $r_1, r_2 \in R$ and $m \in M$.

  We abbreviate $\mu(r, m)$ to rm.

(b) For two R-modules M and N we set
$$\mathsf{Hom}_R(M, N) := \{f\colon M \to N, f\ R\text{-linear}\}$$
to be the abelian group of R-module maps.

Examples I.1.7.
- For every ring R the trivial abelian group 0 is an R-module.
- Every K-vector space is a K-module.
- A $\mathbb{Z}$-module is nothing but an abelian group.
- If R is any ring, then it is an R-module, and R^n is an R-module for all $n \in \mathbb{N}$.
- Every left-ideal $I \subset R$ of any ring is an R-module.

Remark I.1.8. One can define *right R-modules* by requiring that they are left R^{op}-modules. Explicitly that means that

$$\mu(r_1, \mu(r_2, m)) = \mu(r_1 \cdot^{op} r_2, m) = \mu(r_2 r_1, m).$$

Writing mr for $\mu(r, m)$ in this case gives the defining property

$$(mr_2)r_1 = m(r_2 r_1).$$

Examples I.1.9.

(a) Let R be an arbitrary ring and consider the set of $n \times n$-matrices over R, $M_n(R)$. Then $M_n(R)$ is a ring with the usual addition of matrices and with matrix multiplication. The map

$$M_n(R) \times R^n \to R^n, \quad (A, v) \mapsto A \cdot v$$

turns the R-module R^n into an $M_n(R)$-module. We can also define a right $M_n(R)$-module structure on R^n by using

$$R^n \times M_n(R) \to R^n, \quad (v, A) \mapsto v^t \cdot A,$$

so we multiply the row vector v^t by A.

(b) A $\mathbb{Z}[G]$-module is an abelian group M together with an additive group action.

Lecture 2

I.2. Constructions for rings and modules

You already know direct sums and products from other contexts.

Definition I.2.1. Let R be a ring and let $(M_i)_{i \in I}$ be a family of R-modules.

(a) The *product* $\prod_{i \in I} M_i$ is the R-module whose underlying set is

$$\prod_{i \in I} M_i = \{(m_i)_{i \in I}, m_i \in M_i\}.$$

The addition and R-multiplication of the R-module structure is defined componentwise.

(b) The *direct sum* $\bigoplus_{i \in I} M_i$ is the subset of $\prod_{i \in I} M_i$ of all $(m_i)_{i \in I}$ such that $m_i = 0$ for almost all $i \in I$.

There are R-module homomorphisms

$$i_j \colon M_j \to \bigoplus_{i \in I} M_i \subset \prod_{i \in I} M_i$$

for all $j \in I$ and

$$\pi_j \colon \prod_{i \in I} M_i \to M_j.$$

Often, we will denote elements in $\bigoplus_{i \in I} M_i$ by $\sum_{i \in I} m_i$. This makes sense, because only finitely many m_i are non-trivial.

The R-modules $\bigoplus_{i \in I} M_i$ and $\prod_{i \in I} M_i$ have the following universal properties.

Proposition I.2.2. *For all R-modules $(M_i)_{i \in I}$ and N there are isomorphisms*

(a)

$$\mathsf{Hom}_R(\bigoplus_{i \in I} M_i, N) \cong \prod_{i \in I} \mathsf{Hom}_R(M_i, N).$$

(b)
$$\mathsf{Hom}_R(N, \prod_{i\in I} M_i) \cong \prod_{i\in I} \mathsf{Hom}_R(N, M_i).$$

Proof. In (a) the isomorphism is given by sending an $f \in \mathsf{Hom}_R(\bigoplus_{i\in I} M_i, N)$ to the family $(f \circ i_j)_{j\in I}$ and its inverse is
$$(g_i)_{i\in I} \mapsto \sum_{i\in I} g_i \circ \pi_i.$$

In (b) we send an $f \in \mathsf{Hom}_R(N, \prod_{i\in I} M_i)$ to the family $(\pi_i \circ f)_{i\in I}$ and the inverse maps a family $(g_i)_{i\in I}$ to the map that sends an $n \in N$ to $(g_i(n))_{i\in I}$. □

You've also seen subobjects in several contexts:

Definition I.2.3. Let M be an R-module and let $N \subset M$ be a subset. Then N is a *R-submodule of M*, if $(N, +)$ is a subgroup of the abelian group $(M, +)$ and if rn is an element in N for all $n \in N$ and $r \in R$.

Remark I.2.4. In this case N is itself an R-module and the inclusion map $N \hookrightarrow M$ is an R-module map.

Proposition I.2.5. *Let N be an R-submodule of M, then the factor group $M/N := (M, +)/(N, +)$ is an R-module by defining*
$$r(m + N) := rm + N.$$

Proof. The R-module structure is well-defined. If $m + N = m' + N$, then this is equivalent to $m - m'$ being an element of N. Therefore $r(m - m') \in N$ and therefore $rm + N = rm' + N$. □

We call M/N the *quotient of M by N*.
As usual, there is a canonical projection $\pi\colon M \to M/N$ with $\pi(m) = m + N$.
You've also seen the analogue of the following result in other contexts.

Proposition I.2.6. *Let $f\colon M_1 \to M_2$ be R-linear and let $N \subset M_1$ be an R-submodule. If $N \subset \ker(f)$, then there is a unique R-linear map $\bar{f}\colon M_1/N \to M_2$ with $\bar{f} \circ \pi = f$*

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "\pi"'] & M_2 \\ M_1/N \arrow[ur, dotted, "\bar{f}"'] & \end{tikzcd}$$

Proof. As usual we have no choice but to define
$$\bar{f}(m_1 + N) = \bar{f}(\pi(m_1)) = f(m_1).$$
□

We can deduce the usual isomorphism from this.

Corollary I.2.7. *If $f\colon M_1 \to M_2$ is an R-linear map, then the induced map*
$$\bar{f}\colon M_1/\ker(f) \to \mathsf{im}(f)$$
is an isomorphism.

Definition I.2.8. Let N be an R-module and let M be a right R-module. The *tensor product $M \otimes_R N$* is the quotient of the free abelian group F generated by the elements elements (m, n) for $m \in M$ and $n \in N$ modulo the subgroup U generated by

- $((m_1 + m_2), n) - (m_1, n) - (m_2, n)$ for $m_1, m_2 \in M$ and $n \in N$.
- $(m, (n_1 + n_2)) - (m, n_1) - (m, n_2)$ for $m \in M$ and $n_1, n_2 \in N$.
- $((mr), n) - (m, (rn))$ for $r \in R$, $m \in M$ and $n \in N$.

We denote $(m, n) + U$ by $m \otimes n$.

Remark I.2.9. So you build the free abelian group

$$\bigoplus_{m \in M, n \in N} \mathbb{Z}\{(m,n)\}$$

and then you form the factor group with respect to the subgroup generated by the elements as in Definition I.2.8. You force the elements $m \otimes n$ to be additive in both components and to satisfy the relation $mr \otimes n = m \otimes rn$. Note that the relations imply, for instance, that $0 \otimes n = 0 = m \otimes 0$ because

$$0 \otimes n = (m - m) \otimes n = m \otimes n - m \otimes n = 0.$$

We collect some properties of tensor products:

Proposition I.2.10. *Let M, M_i be right R-modules and N, N_i be left R-modules.*

(a) $0 \otimes_R N \cong 0 \cong M \otimes_R 0$.
(b) *The ring R is a unit up to isomorphism:*

$$R \otimes_R N \cong N \text{ and } M \otimes_R R \cong M.$$

(c) *We have the following distributivity laws:*

$$\left(\bigoplus_{i \in I} M_i\right) \otimes_R N \cong \bigoplus_{i \in I} (M_i \otimes_R N),$$

$$M \otimes \left(\bigoplus_{i \in I} N_i\right) \cong \bigoplus_{i \in I} (M \otimes_R N_i).$$

(d) *Tensor products are associative up to isomorphism, so if M is a right R_1-module, P is a left R_2-module and N is a left R_1-module and simultaneously a right R_2-module such that $(r_1 n) r_2 = r_1 (n r_2)$ for all $r_1 \in R_1$, $n \in N$ and $r_2 \in R_2$, then*

$$(M \otimes_{R_1} N) \otimes_{R_2} P \cong M \otimes_{R_1} (N \otimes_{R_2} P).$$

Proof. Properties (a) and (b) follow directly from the definition of the tensor product. For the first claim in (c) we define a module map

$$\psi \colon \bigoplus_{i \in I} (M_i \otimes_R N) \to \left(\bigoplus_{i \in I} M_i\right) \otimes_R N.$$

By the universal property of the direct sum, it suffices to define

$$\psi_j := \psi \circ i_j \colon M_j \otimes_R N \to \left(\bigoplus_{i \in I} M_i\right) \otimes_R N.$$

On a generator $m_j \otimes n$ we set

$$\psi_j(m_j \otimes n) := i_j(m_j) \otimes n.$$

An inverse to ψ is then given by $\phi \colon \left(\bigoplus_{i \in I} M_i\right) \otimes_R N \to \bigoplus_{i \in I} (M_i \otimes_R N)$. Note that we can write elements in the source as linear combinations of elements of the form

$$\left(\sum_{j \in I} i_j(m_j)\right) \otimes n.$$

As this sum is finite, we can rewrite it as

$$\sum_{j \in I} i_j(m_j) \otimes n$$

and we define ϕ by

$$\phi(i_j(m_j) \otimes n) := i_j(m_j \otimes n).$$

The proof of the second claim in (c) is similar.

For (d) we have to specify an R_1-module structure on $N \otimes_{R_2} P$ and a right R_2-module structure on $M \otimes_{R_1} N$ and we define these as

$$r_1(n \otimes p) := (r_1 n) \otimes p, \quad (m \otimes n) r_2 := m \otimes (n r_2)$$

for $r_1 \in R_1$, $r_2 \in R_2$, $m \in M$, $n \in N$ and $p \in P$. This is well-defined because of the compatibility condition that we required in (d). On generators $(m \otimes n) \otimes p$ the isomorphism

$$\alpha \colon (M \otimes_{R_1} N) \otimes_{R_2} P \to M \otimes_{R_1} (N \otimes_{R_2} P)$$

is then given by $\alpha((m \otimes n) \otimes p) = m \otimes (n \otimes p)$. □

Lecture 3

Examples I.2.11.

- For every finite n we have $R^n = \prod_{i=1}^n R = \bigoplus_{i=1}^n R$ and therefore

$$R^n \otimes_R R^m = \left(\bigoplus_{i=1}^n R\right) \otimes_R \left(\bigoplus_{i=1}^m R\right) \cong \bigoplus_{i=1}^n \bigoplus_{j=1}^m R \otimes_R R \cong R^{nm}.$$

- Let $n, m \in \mathbb{N}$, then

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/gcd(n,m)\mathbb{Z}.$$

  This can be seen as follows: Let $d = gcd(n,m)$ and define

$$f \colon \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$$

  as $f(\bar{i} \otimes \bar{j}) := \overline{ij}$. Then f is well-defined and an surjective. Let x be a general element of $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$, so

$$x = \overline{a_1} \otimes \overline{b_1} + \ldots + \overline{a_n} \otimes \overline{b_n}$$

  for some natural n and $a_i, b_i \in \mathbb{Z}$. As the tensor product is bilinear, we can rewrite x as

$$x = (a_1 b_1 + \ldots + a_n b_n)\bar{1} \otimes \bar{1}$$

  and therefore $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ is a cyclic group generated by $\bar{1} \otimes \bar{1}$. By definition we know that $f(a(\bar{1} \otimes \bar{1})) = \bar{a}$ is trivial if and only if d divides a. By the lemma of Bézout we find $y, z \in \mathbb{Z}$ with $d = yn + zm$ and therefore $a = \alpha n + \beta m$ for $\alpha, \beta \in \mathbb{Z}$. Hence

$$a(\bar{1} \otimes \bar{1}) = \alpha \bar{n} \otimes \bar{1} + \beta \bar{1} \otimes \bar{m} = 0.$$

  Therefore f is an isomorphism.
- For every natural number n, the tensor product $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial. A generator of $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ is of the form $\bar{i} \otimes \frac{a}{b}$ with $i, a, b \in \mathbb{Z}$ and $b \neq 0$. But then

$$\bar{i} \otimes \frac{a}{b} = \bar{i} \otimes \frac{an}{bn} = n\bar{i} \otimes \frac{a}{bn} = \overline{ni} \otimes \frac{a}{bn} = 0.$$

  Therefore $\prod_{n \in \mathbb{N}} (\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}) \cong 0$.

  In contrast, $\left(\prod_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q})$ is *not* trivial. The element $x := (\bar{1})_{n \in \mathbb{N}}$ is not a torsion element and $x \otimes 1 \neq 0 \in \left(\prod_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q})$. In particular, tensor products don't distribute over products in general.

Proposition I.2.12 (Universal property of tensor products)**.** *Let R be a ring, M a right R-module and N a left R-module. Let P be an abelian group. We denote by $Bil_R(M \times N, P)$ the abelian group of all biadditive maps $f \colon M \times N \to R$ with the property that $f(mr, n) = f(m, rn)$ for $r \in R$, $m \in M$ and $n \in N$. Then*

$$Bil_R(M \times N, P) \cong \mathsf{Hom}_{\mathbb{Z}}(M \otimes_R N, P).$$

Proof. For a biadditive $f \colon M \times N \to P$ with $f(mr, n) = f(m, rn)$ for $r \in R$, $m \in M$ and $n \in N$ we define

$$\phi(f) \colon M \otimes_R N \to P, \quad m \otimes n \mapsto f(m, n)$$

and for $g \in \mathsf{Hom}_{\mathbb{Z}}(M \otimes_R N, P)$ we set

$$\psi(g)(m, n) := g(m \otimes n).$$

Then ϕ and ψ are inverse to each other and are group homomorphisms. □

Remark I.2.13.

(a) We can define an R-module structure on $\mathsf{Hom}_{\mathbb{Z}}(M, P)$ as $(r.f)(m) := f(mr)$ and an R^{op}-module structure on $\mathsf{Hom}_{\mathbb{Z}}(N, P)$ as $(g.r)(n) := g(rn)$. Then Proposition I.2.12 implies

$$\mathsf{Hom}_{\mathbb{Z}}(M \otimes_R N, P) \cong \mathsf{Hom}_{R^{op}}(M, \mathsf{Hom}_{\mathbb{Z}}(N, P)) \cong \mathsf{Hom}_R(N, \mathsf{Hom}_{\mathbb{Z}}(M, P)).$$

(b) If R is a commutative ring, then $M \otimes_R N$ carries the structure of an R-module: We can define

$$r(m \otimes n) := mr \otimes n = m \otimes rn.$$

This is well-defined because

$$(r_1 r_2)(m \otimes n) = m(r_1 r_2) \otimes n = m(r_2 r_1) \otimes n = (mr_2)r_1 \otimes n = r_1(r_2(m \otimes n)).$$

Thus, we won't distinguish between left and right R-module structures in this case.

For any two R-modules M and N the abelian group $\mathsf{Hom}_R(M, N)$ is then an R-module via $(rf)(m) := f(rm)$ and $M \otimes_R N$ can then also be defined with the same additive relations and with $rm \otimes n = m \otimes rn$ replacing $mr \otimes n = m \otimes rn$.

In addition to the properties from Proposition I.2.10 we obtain that the tensor product of two modules is commutative in this case:

$$M \otimes_R N \cong N \otimes_R M.$$

(c) So for a commutative ring R and R-modules M, N, P we obtain the string of isomorphisms of R-modules

$$\mathsf{Hom}_R(M \otimes_R N, P) \cong \mathsf{Hom}_R(M, \mathsf{Hom}_R(N, P)) \cong \mathsf{Hom}_R(N, \mathsf{Hom}_R(M, P)).$$

Definition I.2.14. If R_1 and R_2 are rings, then $R_1 \otimes_{\mathbb{Z}} R_2$ is a ring with multiplication

$$(r_1 \otimes r_2)(r_1' \otimes r_2') := r_1 r_1' \otimes r_2 r_2'.$$

For a family of rings $(R_i)_{i \in I}$ the product $\prod_{i \in I}$ is a ring with componentwise addition and multiplication.

Note that for an infinite indexing set I the direct sum $\bigoplus_{i \in I} R_i$ is not a ring in general because it doesn't have a unit element unless only finitely many R_is are not the zero ring.

Remark I.2.15. You can check that ring homomorphisms from a ring T to $\prod_{i \in I} R_i$ are the same as $\prod_{i \in I} \mathsf{rings}(T, R_i)$, using the universal property of the product.

If R_1 and R_2 are commutative, then $R_1 \otimes_{\mathbb{Z}} R_2$ is their *coproduct* in the sense that for any other commutative ring T and every pair of ring homomorphisms $f \colon R_1 \to T$ and $g \colon R_2 \to T$ there is a unique ring homomorphism $\xi \colon R_1 \otimes_{\mathbb{Z}} R_2 \to T$ such that the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\chi_{R_2}"] \arrow[d, "\chi_{R_1}"'] & R_2 \arrow[d, "i_2"] \arrow[ddr, bend left, "g"] & \\
R_1 \arrow[r, "i_1"] \arrow[drr, bend right, "f"'] & R_1 \otimes_{\mathbb{Z}} R_2 \arrow[dr, dotted, "\xi"] & \\
& & T
\end{tikzcd}$$

commutes. Here $i_1(r_1) = r_1 \otimes 1_{R_2}$ and $i_2(r_2) = 1_{R_1} \otimes r_2$. You have no choice but to define $\xi(r_1 \otimes r_2)$ as $f(r_1)g(r_2)$.

I.3. Properties of modules

The following definitions are crucial for homological algebra.

Definition I.3.1. Let R be a ring. A sequence

$$\ldots \xrightarrow{f_{i+2}} M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \xrightarrow{f_{i-1}} \ldots$$

of R-modules M_i and $f_i \in \mathsf{Hom}_R(M_i, M_{i-1})$ for $i \in \mathbb{Z}$ is

- a *chain complex*, if $f_i \circ f_{i+1} = 0$ for all i and

- *exact*, if $\mathsf{im}(f_{i+1}) = \ker(f_i)$ for all i.

Remark I.3.2. So exact sequences are in particular chain complexes. Often, we will consider sequences that are not necessarily indexed over the intergers. A sequence $M_n \xrightarrow{f_n} \ldots \xrightarrow{f_1} M_0$ is exact, if $\mathsf{im}(f_{i+1}) = \ker(f_i)$ for all i.

You know short exact sequences of groups, so the following should be familiar.

Definition I.3.3. A *short exact sequence of R-modules* is an exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

of R-modules and R-linear maps.

To spell out what that means: The map i is a monomorphism, p is an epimorphism and $\mathsf{im}(i) = \ker(p)$.

Examples I.3.4.

- For all $n \in \mathbb{N}$ the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is a short exact sequence.

- Let p be a prime. The sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

is a short exact sequence. Does that also work if we allow arbitrary natural numbers instead of a prime p?

Proposition I.3.5. *Let*

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

be a short exact sequence of R-modules and R-linear maps. The following are equivalent:

(a) *The map i has a retraction,* i.e., *there is an $r \in \mathsf{Hom}_R(M, M')$ with $r \circ i = \mathrm{id}_{M'}$.*
(b) *The map p has a section,* i.e., *there is an $s \in \mathsf{Hom}_R(M'', M)$ with $p \circ s = \mathrm{id}_{M''}$.*
(c) *There is an isomorphism $\phi\colon M \to M' \oplus M''$ such that*

$$\begin{tikzcd}
 & & M \arrow[dd, "\phi"] \arrow[dr, "p"] & & \\
0 \arrow[r] & M' \arrow[ur, "i"] \arrow[dr, "i_{M'}"'] & & M'' \arrow[r] & 0 \\
 & & M' \oplus M'' \arrow[ur, "p_{M''}"'] & &
\end{tikzcd}$$

commutes. Here, $i_{M'}$ is the inclusion into the first summand and $p_{M''}(m', m'') = m''$ for $m' \in M'$ and $m'' \in M''$.

Definition I.3.6. In the situation of Proposition I.3.5 the sequence is called a short *split-exact* sequence.

Proof. If we assume (c), then we get retractions and sections as in (a) and (b) by setting $r := \tilde{r} \circ \phi$ and $s := \phi^{-1} \circ \tilde{s}$, where $\tilde{r}\colon M' \oplus M'' \to M'$ is the projection map $\tilde{r}(m', m'') = m'$ and $\tilde{s}\colon M'' \to M' \oplus M''$ is the inclusion map $\tilde{s}(m'') = (0, m'')$.

Let's assume that (a) holds. Note that we can write every $m \in M$ as

$$m = m - ir(m) + ir(m).$$

Here, $ir(m)$ is in the image of i and $m - ir(m)$ is in the kernel of r because

$$r(m - ir(m)) = r(m) - rir(m) = r(m) - r(m) = 0.$$

We claim that $\mathsf{im}(i) \cap \ker(r) = \{0\}$. Assume that $m \in \mathsf{im}(i) \cap \ker(r)$, so there is an m' with $i(m') = m$ and $r(m) = 0$. But then we get

$$0 = r(m) = ri(m') = m'$$

so that $m' = 0$. But then $m = i(0) = 0$.

Therefore $M \cong \mathsf{im}(i) \oplus \ker(r)$ via the map $m \mapsto (ir(m), m - ir(m))$.

We know that $\mathsf{im}(i) \cong M'$ because i is a monomorphism. We claim that the kernel of r is isomorphic to M''. As p is an epimorphism we find for every $m'' \in M''$ an $m \in M$ with $m'' = p(m)$. Again, we write $m = m - ir(m) + ir(m)$ so

$$m'' = p(m) = p(m - ir(m)) + pir(m).$$

But $p \circ i = 0$, so we obtain that the restriction of p to the kernel of r is surjective.

Assume that $p(a) = 0$. Then by the exactness of the sequence we get that a is in the image of i, but the intersection $\mathsf{im}(i) \cap \ker(r)$ is trivial, so $p|_{\ker(r)}$ is also injective and hence an isomorphism.

The proof that (b) implies (c) is similar. □

Lecture 4

Definition I.3.7. An R-module M is *free*, if there is a set I such that M is isomorphic to $\bigoplus_{i \in I} R$. In this case $|I|$ is called the rank of M.

Remark I.3.8. Beware that the rank might not be a well-defined number if R is not commutative! You will see an example as an exercise. A ring R has *invariant basis number, IBN*, if for all positive integers m and n, $R^m \cong R^n$ implies that $n = m$.

Proposition I.3.9. *Let P be an R-module. The following are equivalent:*

(a) *For all diagrams*

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ N \arrow[r, "\pi"] & Q \arrow[r] & 0 \end{tikzcd}$$

with exact row there is an R-linear $\xi\colon P \to N$ with $\pi \circ \xi = f$:

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dotted, "\xi"'] & \\ N \arrow[r, "\pi"] & Q \arrow[r] & 0 \end{tikzcd}$$

(b) *There is an R-module $\tilde{P}$ such that $P \oplus \tilde{P}$ is a free R-module.*

(c) *Every short exact sequence*

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{\pi} P \longrightarrow 0$$

splits.

(d) *For every short exact sequence*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

the induced sequence

$$0 \longrightarrow \mathsf{Hom}_R(P, M') \xrightarrow{\mathsf{Hom}_R(P,f)} \mathsf{Hom}_R(P, M) \xrightarrow{\mathsf{Hom}_R(P,g)} \mathsf{Hom}_R(P, M'') \longrightarrow 0$$

is short exact.

Definition I.3.10. An R-module P satisfying the requirements of Proposition I.3.9 is called *projective*.

Proof. Let us assume that (a) holds and consider the diagram

$$\begin{tikzcd} & & & P \arrow[d, equal] & \\ 0 \arrow[r] & M' \arrow[r, "i"] & M \arrow[r, "\pi"] & P \arrow[r] & 0. \end{tikzcd}$$

By assumption we find an R-linear $\xi\colon P \to M$ with $\pi \circ \xi = \mathrm{id}_P$ and this gives the desired splitting as in (c).

Assume that (c) holds. Note that for every R-module M there is an R-linear surjective map from a free module to M, for instance

$$\varrho\colon \bigoplus_{m \in M} R \to M$$

where ϱ sends the 1_R in component m to $m \in M$. In particular for P there is a free R-module F with

$$F \xrightarrow{\varrho} P \longrightarrow 0$$

and

$$0 \longrightarrow \ker(\varrho) \xrightarrow{i} F \xrightarrow{\varrho} P \longrightarrow 0$$

is a short exact sequence. As we assume (c), we know that this sequence splits, so there is an $s\colon P \to F$ with $\varrho \circ s = \mathrm{id}_P$ and with Proposition I.3.5 we obtain $F \cong \ker(\varrho) \oplus P$ and this shows (b).

We will show that (b) implies (d). First assume that P is free, so $P \cong \bigoplus_{i \in I} R$. Then

$$\mathsf{Hom}_R(P, M) \cong \mathsf{Hom}_R(\bigoplus_{i \in I} R, M) \cong \prod_{i \in I} \mathsf{Hom}_R(R, M) \cong \prod_{i \in I} M.$$

For R-modules, an arbitrary product of short exact sequence is short exact [**12**, Appendix A.5]. Therefore, if

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is a short exact sequence, the induced sequence is isomorphic to

$$0 \longrightarrow \prod_{i \in I} M' \xrightarrow{\prod_{i \in I} f} \prod_{i \in I} M \xrightarrow{\prod_{i \in I} g} \prod_{i \in I} M'' \longrightarrow 0$$

and is again short exact.

If we now assume that (b) holds, then we can choose an R-module $\tilde{P}$ with $P \oplus \tilde{P} \cong \bigoplus_{i \in I} R$. From the argument above we know that the induced sequence

$$0 \longrightarrow \mathsf{Hom}_R(P \oplus \tilde{P}, M') \xrightarrow{\mathsf{Hom}_R(P \oplus \tilde{P}, f)} \mathsf{Hom}_R(P \oplus \tilde{P}, M) \xrightarrow{\mathsf{Hom}_R(P \oplus \tilde{P}, g)} \mathsf{Hom}_R(P \oplus \tilde{P}, M'') \longrightarrow 0$$

is exact. But this sequence again splits as a product of sequences because $\mathsf{Hom}_R(P \oplus \tilde{P}, M) \cong \mathsf{Hom}_R(P, M) \times \mathsf{Hom}_R(\tilde{P}, M)$, so the above sequence is isomorphic to the sequence
(I.3.1)

$$0 \longrightarrow \mathsf{Hom}_R(P, M') \times \mathsf{Hom}_R(\tilde{P}, M') \xrightarrow{\mathsf{Hom}_R(P, f) \times \mathsf{Hom}_R(\tilde{P}, f)} \mathsf{Hom}_R(P, M) \times \mathsf{Hom}_R(\tilde{P}, M)$$

$$\xrightarrow{\mathsf{Hom}_R(P, g) \times \mathsf{Hom}_R(\tilde{P}, g)} \mathsf{Hom}_R(P, M'') \times \mathsf{Hom}_R(\tilde{P}, M'') \longrightarrow 0$$

If one of the sequences

$$0 \longrightarrow \mathsf{Hom}_R(P, M') \xrightarrow{\mathsf{Hom}_R(P, f)} \mathsf{Hom}_R(P, M) \xrightarrow{\mathsf{Hom}_R(P, g)} \mathsf{Hom}_R(P, M'') \longrightarrow 0$$

or

$$0 \longrightarrow \mathsf{Hom}_R(\tilde{P}, M') \xrightarrow{\mathsf{Hom}_R(\tilde{P}, f)} \mathsf{Hom}_R(\tilde{P}, M) \xrightarrow{\mathsf{Hom}_R(\tilde{P}, g)} \mathsf{Hom}_R(\tilde{P}, M'') \longrightarrow 0$$

weren't exact, the sequence (I.3.1) would not be exact.

For (d) implies (a), consider the diagram

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ N \arrow[r, "\pi"] & Q \arrow[r] & 0 \end{tikzcd}$$

which results in the short exact sequence

$$0 \longrightarrow \ker(\pi) \longrightarrow N \xrightarrow{\pi} Q \longrightarrow 0.$$

If we assume that (d) holds, then the induced sequence

$$0 \longrightarrow \mathsf{Hom}_R(P, \ker(\pi)) \xrightarrow{\mathsf{Hom}_R(P,i)} \mathsf{Hom}_R(P, N) \xrightarrow{\mathsf{Hom}_R(P,\pi)} \mathsf{Hom}_R(P, Q) \longrightarrow 0$$

is a short exact sequence, so in particular, $\mathsf{Hom}_R(P, \pi)$ is surjective. Thus there is a $\xi \in \mathsf{Hom}_R(P, N)$ with $\pi \circ \xi = f$ which proves (a). $\square$

Examples I.3.11.

- Every free R-module is projective; just set $\tilde{P} = 0$.
- If the ring R splits as $R = R_1 \times R_2$, then the R-modules $R_1 \times \{0\}$ and $\{0\} \times R_2$ are projective, but they are not free. A $\tilde{P}$ for $R_1 \times \{0\}$ is for instance $\{0\} \times R_2$.
- There are several highly non-trivial examples coming from number theory. Consider the ring $\mathbb{Z}[\sqrt{-5}]$ and consider the ideal I generated by 2 and $1 + \sqrt{-5}$. We claim that I is not free as an R-module but it is projective.

  Assume that I were free. As I has two generators, we know that the rank is $\leqslant 2$. If the rank were 2, then 2 and $1 + \sqrt{-5}$ would be $\mathbb{Z}[\sqrt{-5}]$-linearly independent, but

  $$3 \cdot 2 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

  So the rank would have to be 1 and hence I would have to be a principal ideal. So assume that $I = (a)$ for some a. Then we know that a divides 2 and a divides $1 + \sqrt{-5}$. We consider the multiplicative norm map

  $$N \colon \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}, \quad x + y\sqrt{-5} \mapsto x^2 + 5y^2.$$

  Then $N(a)$ has to divide $N(2) = 4$ and $N(1 + \sqrt{-5}) = 6$, so $N(a) = 2$. But 2 is not of the form $x^2 + 5y^2$ with integral x, y. So I is not free.

  We define $\pi \colon (\mathbb{Z}[\sqrt{-5}])^2 \to I$ as $\pi(z, w) := 2z + (1 + \sqrt{-5})w$. This map π has a section $s \colon I \to (\mathbb{Z}[\sqrt{-5}])^2$,

  $$s(2\alpha + (1 + \sqrt{-5})\beta) = (-2\alpha - (1 + \sqrt{-5})\beta, (1 - \sqrt{-5})\alpha + 3\beta).$$

  Please check that s is R-linear. We calculate:

  $$\begin{aligned} \pi \circ s(2\alpha + (1 + \sqrt{-5})\beta) &= \pi(-2\alpha - (1 + \sqrt{-5})\beta, (1 - \sqrt{-5})\alpha + 3\beta) \\ &= 2(-2\alpha - (1 + \sqrt{-5})\beta) + (1 + \sqrt{-5})(1 - \sqrt{-5})\alpha + 3\beta) \\ &= -4\alpha - 2(1 + \sqrt{-5})\beta + 6\alpha + 3(1 + \sqrt{-5})\beta \\ &= 2\alpha + (1 + \sqrt{-5})\beta. \end{aligned}$$

  Therefore I is projective.

Note that tensoring with an R-module does not have to preserve exactness:

Example I.3.12. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

If we tensor every term in this sequence with $\mathbb{Z}/2\mathbb{Z}$, then up to isomorphism we obtain the sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

But $\cdot 2$ is the zero map, so this sequence is *not* exact any more.

Proposition I.3.13. *Let M be a right R-module and let*

$$0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0.$$

be a short exact sequence of left R-modules.

(a) *Then the sequence*

$$M \otimes_R N' \xrightarrow{M \otimes_R \alpha} M \otimes_R N \xrightarrow{M \otimes_R \beta} M \otimes_R N'' \longrightarrow 0.$$

is exact.

(b) *If M is projective as an R^{op}-module, then the sequence*

$$0 \longrightarrow M \otimes_R N' \xrightarrow{M \otimes_R \alpha} M \otimes_R N \xrightarrow{M \otimes_R \beta} M \otimes_R N'' \longrightarrow 0.$$

is a short exact sequence.

Lecture 5

Proof. For (a) we note that $M \otimes_R \beta$ is surjective: a generator $m \otimes n''$ can be written as $m \otimes \beta(n)$ for some $n \in N$ because β is surjective.

In order to show that $\mathsf{im}(M \otimes_R \alpha) = \ker(M \otimes_R \beta)$ we consider the cokernel

$$M \otimes_R N/\mathsf{im}(M \otimes_R \alpha) = \mathrm{coker}(M \otimes_R \alpha).$$

Define $f\colon M \otimes_R N'' \to M \otimes_R N/\mathsf{im}(M \otimes_R \alpha)$ as

$$f(m \otimes n'') := m \otimes n + \mathsf{im}(M \otimes_R \alpha)$$

for a choice of n with $\beta(n) = n''$. This map is well-defined: Assume that $\tilde{n}$ also satisfies $\beta(\tilde{n}) = n''$, then $n - \tilde{n} \in \ker(\beta) = \mathsf{im}(\alpha)$ and hence

$$m \otimes n + \mathsf{im}(M \otimes_R \alpha) = m \otimes \tilde{n} + \mathsf{im}(M \otimes_R \alpha).$$

Define $g\colon \mathrm{coker}(M \otimes_R \alpha) \to M \otimes N''$ as

$$g(m \otimes n + \mathsf{im}(M \otimes_R \alpha)) := m \otimes \beta(n).$$

Then g is an inverse for f:

$$(f \circ g)(m \otimes n + \mathsf{im}(M \otimes_R \alpha)) = f(m \otimes \beta(n)) = m \otimes n + \mathsf{im}(M \otimes_R \alpha)$$

and

$$(g \circ f)(m \otimes n'') = g(m \otimes n + \mathsf{im}(M \otimes_R \alpha)) = m \otimes \beta(n).$$

But as n was chosen with $\beta(n) = n''$, this proves the claim.

For (b) we now assume that P is projective as an R^{op}-module. We have to show that $P \otimes_R \alpha$ is still injective. Let $\tilde{P}$ be an R^{op}-module such that $P \oplus \tilde{P} = \bigoplus_{i \in I} R^{op} =: F$ is free. The diagram

$$F \otimes_R N' \xrightarrow{F \otimes_R \alpha} F \otimes_R N$$

$$\bigoplus_{i \in I} N' \xrightarrow{\bigoplus_{i \in I} \alpha} \bigoplus_{i \in I} N$$

commutes, so $F \otimes_R \alpha$ is injective. Assume that $P \otimes_R \alpha$ had a non-trivial kernel. As

$$F \otimes_R \alpha = (P \oplus \tilde{P}) \otimes_R \alpha \cong (P \otimes_R \alpha) \oplus (\tilde{P} \otimes_R \alpha),$$

then $F \otimes_R \alpha$ also had one. $\square$

Definition I.3.14. An R-module M is *flat*, if $(-) \otimes_R M$ preserves short exact sequences.

Remark I.3.15. Dual to Proposition I.3.13 one can show that projective R-modules are flat.

The converse is not true in general:

Example I.3.16. The $\mathbb{Z}$-module $\mathbb{Q}$ is flat, but not projective: Let $f\colon M' \to M$ be injective and assume that $\sum_{i=1}^n m_i' \otimes \frac{a_i}{b_i}$ is in the kernel of $f \otimes_{\mathbb{Z}} \mathbb{Q}$, so

$$(f \otimes_{\mathbb{Z}} \mathbb{Q})(\sum_{i=1}^n m_i' \otimes \frac{a_i}{b_i}) = \sum_{i=1}^n f(m_i') \otimes \frac{a_i}{b_i} = \sum_{i=1}^n f(N_i m_i') \otimes \frac{1}{A} = 0$$

where the $0 \neq A = b_1 \cdot \ldots \cdot b_n$ arises from extending denominators. We can rewrite this as

$$m \otimes \frac{1}{A} \text{ for } m = f(\sum_{i=1}^n N_i m_i')$$

and this is trivial in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ if and only if m is a torsion element. But then $\sum_{i=1}^n N_i m_i'$ was also a torsion element, as f is injective, and then

$$\sum_{i=1}^n m_i' \otimes \frac{a_i}{b_i} = 0 \in M' \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Therefore $\mathbb{Q}$ is flat. It is not projective because otherwise there exists an abelian group $\tilde{P}$ with

$$\mathbb{Q} \oplus \tilde{P} \cong \bigoplus_{i \in I} \mathbb{Z}.$$

The elements in $\mathbb{Q}$ are divisible by any natural number, and so are the elements in $\mathbb{Q} \oplus \{0\} \subset \mathbb{Q} \oplus \tilde{P} \cong \bigoplus_{i \in I} \mathbb{Z}$. But as no element in $\mathbb{Z} \setminus \{0\}$ has this divisibility property, this can't happen.

The following result is dual to Proposition I.3.9 and therefore we will not prove it:

Proposition I.3.17. *Let I be an R-module. Then the following are equivalent.*

(a) *For every monomorphism $f\colon U \to M$ and every $j \in \mathsf{Hom}_R(U, I)$, there is a $\zeta\colon M \to I$ with $\zeta \circ f = j$:*

$$\begin{tikzcd} & & I \\ 0 \arrow[r] & U \arrow[ur, "j"] \arrow[r, "f"] & M \arrow[u, dotted, "\zeta"'] \end{tikzcd}$$

(b) *Every short exact sequence*

$$0 \longrightarrow I \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

splits.

(c) *For every short exact sequence*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

the induced sequence

$$0 \longrightarrow \mathsf{Hom}_R(M'', I) \xrightarrow{\mathsf{Hom}_R(\alpha, I)} \mathsf{Hom}_R(M, I) \xrightarrow{\mathsf{Hom}_R(\beta, I)} \mathsf{Hom}_R(M'', I) \longrightarrow 0$$

is a short exact sequence.

□

Definition I.3.18. An R-module I satisfying the requirements of Proposition I.3.17 is called *injective*.

It suffices to check on of the defining properies of injective modules on ideals:

Theorem I.3.19 (Baer)**.** *An R-module N is injective if and only if for all left ideals $J \subset R$ and all solid diagrams*

$$\begin{tikzcd} & & N \\ 0 \arrow[r] & J \arrow[ur, "j"] \arrow[r, "i"] & R \arrow[u, dotted, "\zeta"'] \end{tikzcd}$$

there is an extension ζ as above. Here, i is the inclusion map.

Proof. If N is injective, then it has the extension property as above in particular for $i\colon J \hookrightarrow R$.

For the converse, assume that $0 \longrightarrow M' \longrightarrow M$ is exact and let $f \in \mathsf{Hom}_R(M', N)$. Without loss of generality we may assume that $M' \subset M$.

We define
$$\mathcal{P} := \{(K, g), M' \subset K \subset M, g \in \mathsf{Hom}_R(K, M), g \text{ extends } f\}.$$
$$\begin{tikzcd} M' \arrow[r] \arrow[d, "f"'] & K \arrow[r] \arrow[dl, "g"] & M \\ N & & \end{tikzcd}$$
Then $\mathcal{P}$ is non-empty and is a poset if we define
$$(K_1, g_1) \leqslant (K_2, g_2) \Leftrightarrow K_1 \subset K_2, \quad g_2|_{K_1} = g_1.$$
The poset $\mathcal{P}$ satisfies the assumptions of Zorn's Lemma and thus we obtain that $\mathcal{P}$ has a maximal element $g_\infty\colon K_\infty \to N$. Assume that $K_\infty \subsetneq M$, so there is an $m \in M \setminus K_\infty$. Define
$$J := \{r \in R, rm \in K_\infty\}.$$
This is a left ideal of R and we consider the diagram
$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "h"'] & R \\ & N & \end{tikzcd}$$
where $h(r) = g_\infty(rm)$. By assumption there is an extension of h, $\zeta\colon R \to N$, *i.e.*, $\zeta|_J(r) = g_\infty(rm)$. But then we can consider the R-module $K_\infty + R\{m\}$ together with the map $\tilde{g}\colon K_\infty + R\{m\} \to N$, $\tilde{g}(x + rm) := g_\infty(x) + \zeta(r)$. Then the pair $(K_\infty + R\{m\}, \tilde{g})$ is in $\mathcal{P}$ and it would be larger than (K_∞, g_∞). That's a contradiction and hence $K_\infty = M$. □

You'll learn about injective $\mathbb{Z}$-modules in an exercise.

I.4. Categories

Examples of categories are the categories of sets and functions, groups and group homomorphisms, R-modules with R-linear maps, topological spaces with continuous maps and many more. So for a certain collection of objects you consider those kinds of maps, that preserve the structure that you have.

Our goal for this lecture course is to use a common language for recurring settings, so that we can transfer constructions and proofs to several contexts without having to repeat them. For us the main focus is on categories that are relevant for homological algebra; these will be mostly so called abelian categories. For more background see for instance [**9**].

Definition I.4.1. A *category* $\mathcal{C}$ consists of

(a) A class of objects, $\mathrm{Ob}\mathcal{C}$.
(b) For each pair of objects C_1 and C_2 of $\mathcal{C}$, there is a set $\mathcal{C}(C_1, C_2)$. We call the elements of $\mathcal{C}(C_1, C_2)$ the *morphisms from* C_1 *to* C_2 *in* $\mathcal{C}$.
(c) For each triple C_1, C_2, and C_3 of objects of $\mathcal{C}$, there is a composition law
$$\mathcal{C}(C_1, C_2) \times \mathcal{C}(C_2, C_3) \to \mathcal{C}(C_1, C_3).$$
We denote the composition of a pair (f, g) of morphisms by $g \circ f$.
(d) For every object C of $\mathcal{C}$ there is a morphism id_C, called the *identity morphism on* C.

The composition of morphisms is associative, that is, for morphisms $f \in \mathcal{C}(C_1, C_2)$, $g \in \mathcal{C}(C_2, C_3)$, and $h \in \mathcal{C}(C_3, C_4)$, we have
$$h \circ (g \circ f) = (h \circ g) \circ f,$$
and identity morphisms do not change morphisms under composition, that is, for all $f \in \mathcal{C}(C_1, C_2)$,
$$\mathrm{id}_{C_2} \circ f = f = f \circ \mathrm{id}_{C_1}.$$

For $f \in \mathcal{C}(C_1, C_2)$ we call C_1 the *source of* f, $s(f)$, and C_2 the *target of* f, $t(f)$.

Typical examples of categories that you are familiar with are the following.

Examples I.4.2.
- The category of sets and functions of sets, Sets. In this case the objects form a proper class.
- The category of groups and group homomorphisms, Gr.
- The category of abelian groups and group homomorphisms, Ab.
- For a ring R, the category of (left) R-modules and R-linear maps, R-mod.

Definition I.4.3. A category $\mathcal{C}$ is *small* if its objects form a set.

Example I.4.4. Let X be a partially ordered set (poset), that is, a nonempty set X together with a binary relation $\leqslant$ on X that satisfies reflexivity, transitivity and antisymmetry.

We consider such a poset as a category, and by abuse of notation, we call this category X. Its objects are the elements of X, and the set of morphisms $X(x, y)$ consists of exactly one element if $x \leqslant y$. Otherwise, this set is empty.

For instance if $X = \{a, b, c\}$ with $a \leqslant b$ and $c \leqslant b$, the corresponding category can be visualized as

$$\begin{tikzcd} & c \arrow[d] \\ a \arrow[r] & b \end{tikzcd}$$

In such diagrams one omits identity morphisms.

Lecture 6

There are several constructions that build new categories from old ones.

Definition I.4.5.
- We will need the *empty category*. It has no object and hence no morphism.
- If we have two categories $\mathcal{C}$ and $\mathcal{D}$, then we can build a third one by forming their *product* $\mathcal{C} \times \mathcal{D}$. As the notation suggests, the objects of $\mathcal{C} \times \mathcal{D}$ are pairs of objects (C, D), with C an object of $\mathcal{C}$ and D an object of $\mathcal{D}$. Morphisms are pairs of morphisms:
$$\mathcal{C} \times \mathcal{D}((C_1, D_1), (C_2, D_2)) = \mathcal{C}(C_1, C_2) \times \mathcal{D}(D_1, D_2),$$
and composition and identity morphisms are formed componentwise:
$$(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1), \quad \mathrm{id}_{(C,D)} = (\mathrm{id}_C, \mathrm{id}_D).$$
This is indeed a category.
- Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can also form their disjoint union, $\mathcal{C} \sqcup \mathcal{D}$. Its objects consist of the disjoint union of the objects of $\mathcal{C}$ and $\mathcal{D}$. One defines
$$(\mathcal{C} \sqcup \mathcal{D})(X, Y) := \begin{cases} \mathcal{C}(X, Y), & \text{if } X, Y \text{ are objects of } \mathcal{C}, \\ \mathcal{D}(X, Y), & \text{if } X, Y \text{ are objects of } \mathcal{D}, \\ \varnothing, & \text{otherwise.} \end{cases}$$
- Let $\mathcal{C}$ be an arbitrary category. Let $\mathcal{C}^{op}$ be the category whose objects are the same as the ones of $\mathcal{C}$ but where
$$\mathcal{C}^{op}(C, C') = \mathcal{C}(C', C).$$
We denote by f^{op} the morphism in $\mathcal{C}^{op}(C, C')$ corresponding to $f \in \mathcal{C}(C', C)$.
The composition of $f^{op} \in \mathcal{C}^{op}(C, C')$ and $g^{op} \in \mathcal{C}^{op}(C', C'')$ is defined as $g^{op} \circ f^{op} := (f \circ g)^{op}$. The category $\mathcal{C}^{op}$ is called the *opposite category of* $\mathcal{C}$ or the *dual category of* $\mathcal{C}$.

Definition I.4.6. A *functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$
- assigns to every object C of $\mathcal{C}$ an object $F(C)$ of $\mathcal{D}$.
- For each pair of objects C, C' of $\mathcal{C}$, there is a function of sets
$$F \colon \mathcal{C}(C, C') \to \mathcal{D}(F(C), F(C')),\ f \mapsto F(f).$$

- The following two axioms hold:
$$F(g \circ f) = F(g) \circ F(f) \quad \text{for all } f \in \mathcal{C}(C, C'), g \in \mathcal{C}(C', C''),$$
$$F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$$
for all objects C of $\mathcal{C}$.

Like for morphisms, we use the arrow notation $F\colon \mathcal{C} \to \mathcal{D}$ to indicate a functor.

Examples I.4.7.

- The identity map on objects and morphisms of a category $\mathcal{C}$ define the *identity functor*
$$\mathrm{Id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}.$$
- Let $(-)_{ab}\colon \mathsf{Gr} \to \mathsf{Ab}$ be the functor that assigns to a group G the factor group of G with respect to its commutator subgroup: $G/[G, G]$. The resulting group is abelian, and the functor is called the *abelianization*.
- Often, we will consider functors that forget part of some structure. These are called *forgetful functors*. For instance, if K is a field, then every K-vector space V has an underlying abelian group $U(V)$, and this gives rise to a forgetful functor
$$U\colon K\text{-}\mathsf{vect} \to \mathsf{Ab}.$$
Here, we used that K-linear maps are morphisms of abelian groups.
- To every set X we can assign the free R-module with basis X, $F(X) = \bigoplus_{x \in X} R$ and this defines a functor
$$F\colon \mathsf{Sets} \to R\text{-}\mathsf{mod}.$$
- The tensor product is a functor
$$(-) \otimes_R (-)\colon R^{op}\text{-}\mathsf{mod} \times R\text{-}\mathsf{mod} \to \mathsf{Ab}.$$

Definition I.4.8.

- Let $\mathcal{C}$ be a category and let $(C_i)_{i \in I}$ be a family of objects of $\mathcal{C}$. An object C of $\mathcal{C}$ is a *coproduct of* $(C_i)_{i \in I}$ if there are $i_j \in \mathcal{C}(C_i, C)$ for all $i \in I$ and for all objects E of $\mathcal{C}$ the map
$$\phi\colon \mathcal{C}(C, E) \to \prod_{i \in I} \mathcal{C}(C_i, E), \quad \phi(f) := (f \circ i_j)_{j \in I}$$
is a bijection of sets.
So for every family $(f_j\colon C_j \to E)_{j \in I}$ there exists a unique $f \in \mathcal{C}(C, E)$ with $f \circ i_j = f_j$. The object C is often denoted by $\bigsqcup_{i \in I} C_i$.
- Dually, an object P of $\mathcal{C}$ is a *product of* $(C_i)_{i \in I}$, if there are $\pi_j\colon P \to C_j$ for all $j \in I$ such that the map
$$\psi\colon \mathcal{C}(E, P) \to \prod_{i \in I} \mathcal{C}(E, C_i), \quad \psi(g) = (\pi_i \circ g)_{i \in I}$$
is a bijection of sets.
So for every family $(g_j\colon E \to C_j)_{j \in I}$ there exists a unique $g \in \mathcal{C}(E, P)$ with $\pi_j \circ g = g_j$. The object P is often denoted by $\prod_{i \in I} C_i$.

Beware that coproducts and product do not necessarily exist in a given category $\mathcal{C}$:

Examples I.4.9.

- Consider the category of fields and morphisms of fields. Then this category does *not* have products.
- Let $(X, \leqslant)$ be a poset. When does a family of objects, *i.e.*a family of elements of $(X, \leqslant)$ have a coproduct or a product?
- Let R be a ring. Then for a family of R-modules $(M_i)_{i \in I}$ the object $\bigoplus_{i \in I} M_i$ is the coproduct and $\prod_{i \in I} M_i$ is the product of the $(M_i)_{i \in I}$.

We will consider categories of R-modules, chain complexes and similar categories. These have additional features:

Definition I.4.10. A category $\mathcal{C}$ is *additive* if

(a) For all objects C_1, C_2 of $\mathcal{C}$ the set $\mathcal{C}(C_1, C_2)$ is an abelian group.
(b) The composition of morphisms is a bilinear map.
(c) The category $\mathcal{C}$ has all finite products and coproducts.

Remark I.4.11. The above conditions also guarantee the existence of coproducts and products in the case where the indexing set I is empty. So what are these objects?

Let us denote the coproduct over the empty set by 0 and the corresponding product $*$. The universal properties then say that for all objects C of $\mathcal{C}$ there is exactly one morphism in $\mathcal{C}(0, C)$ and in $\mathcal{C}(C, *)$.

In general, such objects are call initial (0) respectively terminal ($*$) objects. In our case, our morphism sets are abelian groups, so we obtain that

$$\mathcal{C}(0, C) = \{0\} = \mathcal{C}(C, *).$$

Hence we also have $\mathcal{C}(0, 0) = \{0\}$ and this implies that $\mathcal{C}(C, 0) = \{0\}$ as well because for any $f \in \mathcal{C}(C, 0)$ we have $\mathrm{id}_0 \circ f = f$, but as id_0 is the generator of the trivial group, thus bilinearity yields

$$f = \mathrm{id}_0 \circ f = (\mathrm{id}_0 + \mathrm{id}_0) \circ f = f + f$$

and thus $f = 0$. Therefore $0 = *$.

Beware that this is a special feature of additive categories. For the category of topological spaces, for instance, the initial object is the empty space and any space with one point is terminal, so here these objects are not isomorphic.

If the initial object has the same universal property as the terminal object, then these are called a *zero object*. All these objects are unique up to isomorphism.

Definition I.4.12. Let $\mathcal{C}$ be an additive category and $f \in \mathcal{C}(A, B)$.

- An $i \in \mathcal{C}(K, A)$ is a *kernel of* f, if $f \circ i = 0$ and for every $h \in \mathcal{C}(X, A)$ with $f \circ h = 0$ there is a unique $g \colon X \to K$ with $i \circ g = h$.
- Dually, $p \in \mathcal{C}(B, C)$ is a *cokernel of* f, if $p \circ f = 0$ and for every $r \in \mathcal{C}(B, D)$ with $r \circ f = 0$ there is a unique $t \colon C \to D$ with $t \circ p = r$.

$$\begin{tikzcd}
X \arrow[d, dotted, "g"'] \arrow[dr, "h"] \arrow[drr, bend left, "0"] & & & \\
K \arrow[r, "i"] & A \arrow[r, "f"] \arrow[drr, bend right, "0"'] & B \arrow[r, "p"] \arrow[dr, "r"] & C \arrow[d, dotted, "t"] \\
& & & D
\end{tikzcd}$$

Example I.4.13. For the category $\mathcal{C} = \mathsf{Ab}$ and a homomorphism $f \colon A \to B$ we can set $K := \{a \in A, f(a) = 0\}$ and define i as the inclusion map. Then i is the kernel of f. For $C := B/\mathsf{im}(f)$ and the canonical projection map $p \colon B \to C$, we get p as the usual definition of the cokernel.

Let's actually show that: Assume that we have an $r \colon B \to D$ with $r \circ f = 0$. Then $\mathsf{im}(f) \subset \ker(r)$ and hence there is a unique $\bar{r} \colon B/\mathsf{im}(f) \to D$ with $\bar{r} \circ p = r$, so this $\bar{r}$ correponds to the t in the definition.

In a general additive category, kernels and cokernel don't have to exist for every f. In the cases we are interested in, they do:

Definition I.4.14. An additive category $\mathcal{C}$ is *abelian* if every morphism has a kernel and a cokernel and if in addition

- If i is a monomorphism, then $i = \ker(\mathrm{coker}(i))$.
- If p is an epimorphism, then $p = \mathrm{coker}(\ker(p))$.

Here, i is a monomorphism, if $i \circ f_1 = i \circ f_2$ implies $f_1 = f_2$ for all f_1, f_2 whose targets are the source of i, and dually p is an epimorphism, if $f_1 \circ p = f_2 \circ p$ imples $f_1 = f_2$ for all f_1, f_2 whose sources are the target of p. In an additive category it suffices to check for $i \circ f = 0$ and for $f \circ p = 0$.

Example I.4.15. The category R-mod is abelian for every ring R, and therefore also the category of right R-modules.

Definition I.4.16. Let $\mathcal{C}$ be an arbitrary category and let $f \in \mathcal{C}(C_1, C_2)$. If f can be factored as $f = i \circ p$ where p is an epimorphism and i is a monomorphism, then i is called the image of f, $\mathsf{im}(f)$, and p is called the coimage of f.

If they exist, then $\ker(f)$, $\operatorname{coker}(f)$, $\mathsf{im}(f)$ and the coimage of f are unique up to isomorphism. So we can define exact sequence in any abelian category: A sequence of morphisms

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence*, if f is a monomorphism, g is an epimorphism and if $\mathsf{im}(f) = \ker(g)$.

Definition I.4.17.

(a) Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *additive* if for any two objects C_1, C_2 of $\mathcal{C}$, the map $F\colon \mathcal{C}(C_1, C_2) \to \mathcal{D}(F(C_1), F(C_2))$ is a group homomorphism.

(b) If $\mathcal{C}$ and $\mathcal{D}$ are abelian categories and F is an additive functor, then F is called

- *right exact*, if
$$F(C') \xrightarrow{F(f)} F(C) \xrightarrow{F(g)} F(C'') \longrightarrow 0$$
is exact, for every short exact sequence
$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$
in $\mathcal{C}$.
- *left exact*, if
$$0 \longrightarrow F(C') \xrightarrow{F(f)} F(C) \xrightarrow{F(g)} F(C'')$$
is exact, for every short exact sequence
$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$
in $\mathcal{C}$.
- *exact*, if
$$0 \longrightarrow F(C') \xrightarrow{F(f)} F(C) \xrightarrow{F(g)} F(C'') \longrightarrow 0$$
is exact for every short exact sequence
$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$
in $\mathcal{C}$.

Lecture 7

Examples I.4.18.

- Let M be an R-module, then the functor
$$(-) \otimes_R M \colon R^{op}\text{-}\mathsf{mod} \to \mathsf{Ab}$$
is right-exact. It is exact if and only if M is flat.
- The functor $\mathsf{Hom}_R(M, -)\colon R\text{-}\mathsf{mod} \to \mathsf{Ab}$ is left exact and it is exact if and only if M is projective.
- The functor $\mathsf{Hom}_R(-, N)\colon (R\text{-}\mathsf{mod})^{op} \to \mathsf{Ab}$ is left-exact and exact if and only if N is injective: If
$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$
is an exact sequence, then
$$0 \longrightarrow M'' \xrightarrow{g^{op}} M \xrightarrow{f^{op}} M' \longrightarrow 0$$
is exact in $(R\text{-}\mathsf{mod})^{op}$. This is the starting point for our functor, so left-exactness says that
$$0 \longrightarrow \mathsf{Hom}_R(M'', N) \xrightarrow{\mathsf{Hom}_R(g,N)} \mathsf{Hom}_R(M, N) \xrightarrow{\mathsf{Hom}_R(f,N)} \mathsf{Hom}_R(M', N)$$
is exact.

The last map is an epimorphism, if and only if N is injective. That follows directly from the definition of injectivity.

- The functors $\sqcup\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, $(C_1,C_2)\mapsto C_1\sqcup C_2$ and $\prod\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, $(C_1,C_2)\mapsto C_1\prod C_2 = C_1\times C_2$ are exact for $\mathcal{C}$ abelian. We give the argument for the coproduct.

  Assume that
  $$0 \longrightarrow C_i' \xrightarrow{f_i} C_i \xrightarrow{g_i} C_i'' \longrightarrow 0$$
  is short exact.

  So, f_1 and f_2 are monomorphisms, g_1, g_2 are epimorphisms, and f_i is the kernel of g_i. Note that the maps $f_1\sqcup f_2$ are determined by the universal property of the coproduct:

  Consider the maps
  $$C_1' \xrightarrow{f_1} C_1 \xrightarrow{i_1} C_1\sqcup C_2 \xleftarrow{i_2} C_2 \xleftarrow{f_2} C_2'.$$
  They induce the map $f_1\sqcup f_2$.

  This yields that the sequence
  $$0 \longrightarrow C_1'\sqcup C_2' \xrightarrow{f_1\sqcup f_2} C_1\sqcup C_2 \xrightarrow{g_1\sqcup g_2} C_1''\sqcup C_2'' \longrightarrow 0$$
  is exact, because the properties of its maps are inherited from the ones of their summands.

Remark I.4.19. In a general abelian category arbitrary products and coproducts don't have to be exact, but they are exact in the category of R-modules.

I.5. Projective and injective resolutions

We can define projective and injective objects in any category:

Definition I.5.1.

- An object P in a category $\mathcal{C}$ is called *projective* if for every epimorphism $f\colon M\to Q$ in $\mathcal{C}$ and every $p\colon P\to Q$, there is a $\xi\in\mathcal{C}(P,M)$ with $f\circ\xi = p$:

$$\begin{tikzcd} & P \arrow[dl, dotted, "\xi"'] \arrow[d, "p"] \\ M \arrow[r, "f"] & Q \end{tikzcd}$$

- Dually, an object I in a category $\mathcal{C}$ is called *injective* if for every monomorphism $f\colon U\to M$ in $\mathcal{C}$ and every $j\colon U\to I$, there is a $\zeta\in\mathcal{C}(M,I)$ with $\zeta\circ f = j$:

$$\begin{tikzcd} & I \\ U \arrow[ur, "j"] \arrow[r, "f"] & M \arrow[u, dotted, "\zeta"'] \end{tikzcd}$$

In order to do homological algebra, we need certain types of resolutions. For these, we need the following notions:

Definition I.5.2. An abelian category $\mathcal{C}$

(a) *has enough projectives*, if for every object M of $\mathcal{C}$ there is an epimorphism $\pi\colon P\to M$ with P projective.

(b) *has enough injectives*, if for every object M of $\mathcal{C}$ there is a monomorphism $i\colon M\to I$ with I injective.

Examples I.5.3.

- The category Ab_f of finite abelian groups is an abelian category, but it has neither enough projectives nor enough injectives. The abelian group $\mathbb{Z}$ is not an object of Ab_f and finite abelian groups are not divisible.
- Let $\mathcal{C}$ be the category of abelian torsion groups, so the objects are all abelian groups A such that for all $a\in A$ there is an $n\in\mathbb{N}$ with $na=0$. This category does not have enough projective objects, but enough injectives: Every object can be embedded into some $\prod_I \mathbb{Q}/\mathbb{Z}$.

- Let $\mathcal{C}$ be the category of finitely generated abelian groups, Ab_{fg}. Then this category has enough projectives but not enough injectives: Let A be an object in Ab_{fg} and assume that $\{s_1, \ldots, s_n\}$ is a finite set of generators for A. Then there is an epimorphism
$$\pi\colon \bigoplus_{i=1}^{n} \mathbb{Z} \to A.$$
Here π sends 1 in the ith component to s_i.
As non-trivial divisible groups are not finitely generated, we don't have interesting injective objects in this category.

Proposition I.5.4. *Let R be a ring. The category of R-modules, R-*mod*, has enough projectives and injectives.*

Note that the following proof implicitly uses the characteristic map $\chi_R\colon \mathbb{Z} \to R$.

Proof. Without loss of generality assume that $R \neq 0$.

For the first claim we consider an arbitrary R-module M and the epimorphism
$$\pi\colon \bigoplus_{m\in M} R \to M$$
that sends 1_R in component $m \in M$ to $m \in M$.

It is more involved to show the second claim: For an arbitrary R-module M we define
$$I(M) := \mathsf{Ab}(M, \mathbb{Q}/\mathbb{Z}).$$
This abelian group carries the structure of an R^{op}-module. For any homomorphisms $f\colon M \to \mathbb{Q}/\mathbb{Z}$ we set
$$(f.r)(m) := f(rm).$$

If M is free, so $M = \bigoplus_{i\in I} R$, then
$$I(M) = I(\bigoplus_{i\in I} R) = \mathsf{Ab}(\bigoplus_{i\in I} R, \mathbb{Q}/\mathbb{Z}) \cong \prod_{i\in I} \mathsf{Ab}(R, \mathbb{Q}/\mathbb{Z}).$$
Let N be any right R-module, then
$$R^{op}\text{-}\mathsf{mod}(N, \mathsf{Ab}(R, \mathbb{Q}/\mathbb{Z})) \cong \mathsf{Ab}(N \otimes_R R, \mathbb{Q}/\mathbb{Z}) \cong \mathsf{Ab}(N, \mathbb{Q}/\mathbb{Z}).$$
As $N \mapsto \mathsf{Ab}(N, \mathbb{Q}/\mathbb{Z})$ is an exact functor, so is
$$N \mapsto R^{op}\text{-}\mathsf{mod}(N, \mathsf{Ab}(R, \mathbb{Q}/\mathbb{Z}))$$
and therefore $\mathsf{Ab}(R, \mathbb{Q}/\mathbb{Z})$ is injective. Products of injective modules are injective and therefore $I(\bigoplus_{i\in I} R)$ is injective.

We now define
$$I_M := \prod_{f\in I(M)} I(R) = \prod_{f\in I(M)} \mathsf{Ab}(R, \mathbb{Q}/\mathbb{Z}).$$
As a product of injectives I_M is injective.

We define
$$i\colon M \to I_M, \quad m \mapsto (r \mapsto f(rm)).$$
By construction i is R-linear, because for any $\tilde{r} \in R$ we obtain
$$\begin{aligned} i(\tilde{r}m) &= (r \mapsto f(r\tilde{r}m)) \\ &= (r \mapsto g(rm)) \end{aligned}$$
with $g(m) = f(\tilde{r}m) = (f\tilde{r})(m)$.

We claim that i is injective: Assume that $m \neq 0$ and consider $g\colon \mathbb{Z} \to M$, $g(1) = m = 1_R.m$. Then
$$\bar{g}\colon \mathbb{Z}/\ker(g) \to M$$
is injective and we consider $\mathbb{Z}/\ker(g)$ as an abelian subgroup of M consisting of the $\mathbb{Z}$-multiples of m.

In the case where $\ker(g) = 0$ we get

$$\mathsf{Ab}(\mathbb{Z}/\ker(g), \mathbb{Q}/\mathbb{Z}) = \mathsf{Ab}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}.$$

So any choice of an non-trivial $a \in \mathbb{Q}/\mathbb{Z}$ gives a non-trivial component of $i(m)$.

If $\ker(g) \neq 0$, then $\mathbb{Z}/\ker(g) \cong \mathbb{Z}/n\mathbb{Z}$ for some $1 \neq n \in \mathbb{N}$. We can embed $\mathbb{Z}/n\mathbb{Z}$ into $\mathbb{Q}/\mathbb{Z}$ by sending $1 + n\mathbb{Z}$ to the class of $\frac{1}{n}$. As $\mathbb{Q}/\mathbb{Z}$ is injective, we can extend the map $h\colon \mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$ over M to a map $\zeta\colon M \to \mathbb{Q}/\mathbb{Z}$. Then ζ is not the zero map and $\zeta(m) = h(1) \neq 0$. Then $\zeta \in I(M)$ is a non-trivial component of $i(m)$. $\square$

Definition I.5.5. Let $\mathcal{C}$ be an abelian category and let M be an object of $\mathcal{C}$.

(a) A *projective resolution of M* is an exact sequence

$$\ldots \longrightarrow P_i \xrightarrow{f_i} P_{i-1} \xrightarrow{f_{i-1}} \ldots \xrightarrow{f_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

such that all P_is are projective, $f_i \in \mathcal{C}(P_i, P_{i-1})$, $\varepsilon \in \mathcal{C}(P_0, M)$.

(b) Dually, an *injective resolution of M* is an exact sequence

$$0 \longrightarrow M \xrightarrow{\eta} I^0 \xrightarrow{g_0} I^1 \xrightarrow{g_1} I^2 \xrightarrow{g_2} \ldots$$

such that all I^js are injective, $g_j \in \mathcal{C}(I^j, I^{j+1})$, $\eta \in \mathcal{C}(M, I^0)$.

Examples I.5.6. Let $\mathcal{C}$ be the category Ab. For every $2 \leqslant n \in \mathbb{N}$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is a projective resolution of $\mathbb{Z}/n\mathbb{Z}$ and

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is an injective resolution of $\mathbb{Z}$.

Lecture 8

Lemma I.5.7. *If an abelian category $\mathcal{C}$ has enough projectives, then every object M of $\mathcal{C}$ has a projective resolution. Dually, if $\mathcal{C}$ has enough injectives, then every M has an injective resolution.*

Proof. We prove the claim for projective resolutions. The other proof is dual.

We start by choosing an epimorphism $\varepsilon\colon P_0 \to M$ where P_0 is projective. This is possible because $\mathcal{C}$ has enough projectives. Consider the short exact sequence

$$0 \longrightarrow \ker(\varepsilon) \xrightarrow{i_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

In general, $\ker(\varepsilon)$ is not projective but we can choose an epimorphism $q_1\colon P_1 \to \ker(\varepsilon)$ and we define $f_1\colon P_1 \to P_0$ as $f_1 := i_1 \circ q_1$.

$$\begin{tikzcd} P_1 \arrow[rr, "f_1"] \arrow[dr, "q_1"'] & & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0. \\ & \ker(\varepsilon) \arrow[ur, "i_1"'] & & & \end{tikzcd}$$

By an iteration of this construction you get a projective resolution of M. $\square$

Example I.5.8. Let $\mathcal{C}$ be the category of $\mathbb{Z}/4\mathbb{Z}$-modules and let M be $\mathbb{Z}/2\mathbb{Z}$. Then following the construction in the proof above one gets a projective resolution of $\mathbb{Z}/2\mathbb{Z}$ of infinite length

$$\ldots \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}/2\mathbb{Z}.$$

Can there be a projective resolution of finite length in this case?

Definition I.5.9. Let $\mathcal{C}$ be an abelian category. The *category of chain complexes in* $\mathcal{C}$, $\mathsf{Ch}(\mathcal{C})$, has as objects sequences of objects $(C_n)_{n\in\mathbb{Z}}$ of $\mathcal{C}$ together with morphisms $d_n \in \mathcal{C}(C_n, C_{n-1})$ for $n \in \mathbb{Z}$, such that $d_{n-1} \circ d_n = 0$. We abbreviate such an object with (C_*, d) or just with C_*.

A morphism between two chain complexes C_* and D_* is called a *chain map* $f\colon C_* \to D_*$. It consists of a sequence of momorphisms $f_n \in \mathcal{C}(C_n, D_n)$ such that $d_n^D \circ f_n = f_{n-1} \circ d_n^C$ for all n, *i.e.*, the diagram

$$\begin{tikzcd} C_n \arrow[r, "d_n^C"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "d_n^D"] & D_{n-1} \end{tikzcd}$$

commutes for all n.

Definition I.5.10.
- The d_n are *differentials* or *boundary operators.*
- The *n-cycles of C_** are
$$Z_n(C_*) := \ker(d_n).$$
- The *n-boundaries* are
$$B_n(C_*) := \mathsf{im}(d_{n+1}).$$
- The *nth homology of C_** is defined via the short exact sequence
$$0 \longrightarrow B_n(C_*) \longrightarrow Z_n(C_*) \longrightarrow H_n(C_*) \longrightarrow 0.$$

Remark I.5.11. Note that chain maps f_* map cycles to cycles and boundaries to boundaries. We therefore obtain an induced map
$$H_n(f_*)\colon H_n(C_*) \to H_n(D_*).$$
In fact, $H_n\colon \mathsf{Ch}(\mathcal{C}) \to \mathcal{C}$ is a functor.

Examples I.5.12. We consider the category $\mathcal{C} = \mathsf{Ab}$.
- Consider
$$C_n = \begin{cases} \mathbb{Z}, & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$
and let d_1 be the multiplication with $N \in \mathbb{N}$, then
$$H_n(C) = \begin{cases} \mathbb{Z}/N\mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$
- Take $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ and
$$d_n = \begin{cases} \mathrm{id}_{\mathbb{Z}}, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$
What is the homology of this chain complex?
- Consider $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ again, but let all boundary maps be trivial. Then $H_n(C_*) = C_n$ for all n.

There is a chain map from the chain complex mentioned in the first example above to the chain complex D_* that is concentrated in degree zero and has $D_0 = \mathbb{Z}/N\mathbb{Z}$.

(I.5.1)
$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\cdot n"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/N\mathbb{Z} \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

Note, that $H_0(f)$ is an isomorphism and hence all $H_i(f)$ are isomorphisms.

Definition I.5.13. A chain map $f_*\colon (C_*, d) \to (D_*, d')$ is a *quasi-isomorphism* if f_* induces an isomorphism on homology.

Definition I.5.14. A *chain homotopy* H between two chain maps $f, g\colon C_* \to D_*$ is a sequence of homomorphisms $(H_n)_{n\in\mathbb{Z}}$ with $H_n\colon C_n \to D_{n+1}$ such that for all n

$$d^D_{n+1} \circ H_n + H_{n-1} \circ d^C_n = f_n - g_n.$$

$$\begin{tikzcd}
\cdots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[dl, "H_{n+1}"'] \arrow[d, bend right, "f_{n+1}"'] \arrow[d, bend left, "g_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[dl, "H_n"'] \arrow[d, bend right, "f_n"'] \arrow[d, bend left, "g_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[dl, "H_{n-1}"'] \arrow[d, bend right, "f_{n-1}"'] \arrow[d, bend left, "g_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \cdots
\end{tikzcd}$$

If such an H exists, then f and g are *(chain) homotopic*: $f \simeq g$.

Remark I.5.15. Being chain homotopic is an equivalence relation.

Definition I.5.16. Let $f\colon C_* \to D_*$ be a chain map. We call f a *chain homotopy equivalence*, if there is a chain map $g\colon D_* \to C_*$ such that $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$. The chain complexes C_* and D_* are then *chain homotopy equivalent*.

Note, that such chain complexes have isomorphic homology. However, chain complexes with isomorphic homology do not have to be chain homotopy equivalent. The chain map in (I.5.1) is an example of this phenomenon.

Proposition I.5.17. *If f_*, g_* are chain homotopic, then they induces the same map on homology. Every chain homotopy equivalence is a quasi-isomorphism.*

Proof. Let $(H_n)_{n\in\mathbb{Z}}$ be a chain homotopy from f_* to g_*, so $H_n\colon C_n \to D_{n+1}$ and $d_{n+1}\circ H_n + H_{n-1}\circ d_n = f_n - g_n$. We define

$$h_n := d_{n+1} \circ H_n + H_{n-1} \circ d_n.$$

Then h_n restricted to $\ker(d_n)$ gives $d_{n+1} \circ H_n$. Thus on the level of homology groups h_n induces the zero map and therefore $H_n(f_*) = H_n(g_*)$ for all n.

If f_* is a chain homotopy equivalence with homotopy inverse g_*, then $H_n(f_*) \circ H_n(g_*) = H_n(\mathrm{id}_{D_*})$ and $H_n(g_*) \circ H_n(f_*) = H_n(\mathrm{id}_{C_*})$ by the first part of the proof. □

Dual to chain complexes are cochain complexes.

Definition I.5.18. Let $\mathcal{C}$ be an abelian category. The *category of cochain complexes in $\mathcal{C}$* has as objects sequences of objects $(C^n)_{n\in\mathbb{Z}}$ of $\mathcal{C}$ together with morphisms $d^n \in \mathcal{C}(C^n, C^{n+1})$ for $n \in \mathbb{Z}$, such that $d^{n+1}\circ d^n = 0$.

A morphism between two cochain complexes C^* and D^* is called a *cochain map* $f\colon C^* \to D^*$. It consists of a sequence of momorphisms $f^n \in \mathcal{C}(C^n, D^n)$ such that $d^n \circ f^n = f^{n+1} \circ d^n$ for all n.

You can shift back and forth between the two notions. If C_* is a chain complex, then $C^i := C_{-i}$ is a cochain complex and vice versa.

CHAPTER II

Derived functors

Lecture 9

II.1. Definition of left and right derived functors

Homological algebra works because of the following crucial result:

Lemma II.1.1 (Fundamental Lemma of Homological Algebra)**.** *Let $\mathcal{C}$ be an abelian category and $f \in \mathcal{C}(M,N)$. Assume that*

$$\ldots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \xrightarrow{d_{i-1}} \ldots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon_M} M$$

is a chain complex such that the objects P_i are projective for all i and let

$$\ldots \longrightarrow N_i \xrightarrow{d'_i} N_{i-1} \xrightarrow{d'_{i-1}} \ldots \xrightarrow{d'_1} N_0 \xrightarrow{\varepsilon_N} N \longrightarrow 0$$

be an exact sequence.

(a) *Then there is a chain map f_*, extending f,* i.e., *the diagram*

$$\begin{tikzcd}
\ldots \arrow[r] & P_i \arrow[r, "d_i"] \arrow[d, "f_i"'] & P_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "f_{i-1}"'] & \ldots \arrow[r, "d_1"] & P_0 \arrow[r, "\varepsilon_M"] \arrow[d, "f_0"'] & M \arrow[d, "f"'] \\
\ldots \arrow[r] & N_i \arrow[r, "d'_i"] & N_{i-1} \arrow[r, "d'_{i-1}"] & \ldots \arrow[r, "d'_1"] & N_0 \arrow[r, "\varepsilon_N"] & N
\end{tikzcd}$$

commutes.

(b) *Any two such extensions f_*, g_* of f are chain homotopic.*

Remark II.1.2. There is, of course, a dual statement that ensures the existence of cochain maps from an exact cochain complex to a cochain complex of injectives.

Proof. Consider the diagram

$$\begin{tikzcd}
P_0 \arrow[r, "\varepsilon_M"] \arrow[dr, "f\circ\varepsilon_M"] & M \arrow[d, "f"] \\
N_0 \arrow[r, "\varepsilon_N"] & N.
\end{tikzcd}$$

As P_0 is projective, we can lift the map $f \circ \varepsilon_M$ to an $f_0 \colon P_0 \to N_0$ such that $\varepsilon_N \circ f_0 = f \circ \varepsilon_M$:

$$\begin{tikzcd}
P_0 \arrow[r, "\varepsilon_M"] \arrow[dr, "f\circ\varepsilon_M"] \arrow[d, dotted, "f_0"'] & M \arrow[d, "f"] \\
N_0 \arrow[r, "\varepsilon_N"] & N.
\end{tikzcd}$$

For the next step we consider the kernels

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\ker(\varepsilon_M)"] & P_0 \arrow[r, "\varepsilon_M"] \arrow[dr, "f\circ\varepsilon_M"] \arrow[d, dotted, "f_0"'] & M \arrow[d, "f"] \\
0 \arrow[r] & N' \arrow[r, "\ker(\varepsilon_N)"] & N_0 \arrow[r, "\varepsilon_N"] & N.
\end{tikzcd}$$

As

$$\varepsilon_N \circ f_0 \circ \ker(\varepsilon_M) = f \circ \varepsilon_M \circ \ker(\varepsilon_M) = 0,$$

the universal property of the kernel ensures that there is a morphism $h\colon M' \to N'$ with $f_0 \circ \ker(\varepsilon_M) = \ker(\varepsilon_N) \circ h$. As $\varepsilon_M \circ d_1 = 0$, the property of the kernel ensures that there is a $g_1 \in \mathcal{C}(P_1, M')$ that makes the following diagram commutative:

$$\begin{tikzcd}
& P_1 \arrow[d, "d_1"] \arrow[dl, dotted, "g_1"'] & \\
M' \arrow[r, "\ker(\varepsilon_M)"] & P_0 \arrow[r, "\varepsilon_M"] & M.
\end{tikzcd}$$

Similarly, as $\varepsilon_N \circ d_1' = 0$, there is an $h_1 \in \mathcal{C}(N_1, N')$. with $\ker(\varepsilon_N) \circ h_1 = d_1'$.

Consider the diagram

$$\begin{tikzcd}
P_1 \arrow[r, "p_1"] & M' \arrow[d, "h"] \\
N_1 \arrow[r, "h_1"] & N'.
\end{tikzcd}$$

As the image of d_1' is the kernel of ε_N, the map h_1 is an epimorphism and therefore there is an $f_1 \in \mathcal{C}(P_1, N_1)$ with

$$h_1 \circ f_1 = h \circ g_1.$$

As the diagram

$$\begin{tikzcd}
P_1 \arrow[rr, "d_1"] \arrow[dr, "g_1"'] \arrow[ddd, "f_1"'] & & P_0 \arrow[ddd, "f_0"] \\
& M' \arrow[ur, "\ker(\varepsilon_M)"'] \arrow[ddd, "h"] & \\
& & \\
N_1 \arrow[rr, "d_1'"] \arrow[dr, "h_1"'] & & N_0 \\
& N' \arrow[ur, "\ker(\varepsilon_N)"'] &
\end{tikzcd}$$

commutes, we can read off that

$$\begin{aligned}
d_1' \circ f_1 &= \ker(\varepsilon_N) \circ h_1 \circ f_1 \\
&= \ker(\varepsilon_N) \circ h \circ g_1 \\
&= f_0 \circ \ker(\varepsilon_M) \circ g_1 \\
&= f_0 \circ d_1.
\end{aligned}$$

This construction can be repeated for all n and this proves (a).

For (b) it suffices to show that every extension f_* of the zero map $0 \in \mathcal{C}(M, N)$ is chain homotopic to the trivial chain map, *i.e.*, we want $(H_n)_{n \geqslant 0}$ with

$$f_n = d_{n+1}' \circ H_n + H_{n-1} \circ d_n.$$

Consider

$$\begin{tikzcd}
& P_0 \arrow[r, "\varepsilon_M"] \arrow[d, "f_0"'] & M \arrow[d, "0"] \\
N_1 \arrow[r, "d_1'"] & N_0 \arrow[r, "\varepsilon_N"] & N.
\end{tikzcd}$$

As $\varepsilon_N \circ f_0 = 0$ and as the bottom row is exact the image of f_0 has target $\ker(\varepsilon_N) = \mathsf{im}(d_1')$ and we can lift f_0 to N_1, so there is an $H_0\colon P_0 \to N_1$ with

$$d_1' \circ H_0 = f_0.$$

We know that

$$d_1'(f_1 - H_0 \circ d_1) = f_0 \circ d_1 - d_1' \circ H_0 \circ d_1 = 0.$$

But as the complex $N_* \to N$ is exact, there is an $H_1 \in \mathcal{C}(P_1, N_2)$ with

$$d_2' \circ H_1 = f_1 - H_0 \circ d_1$$

and this yields the desired

$$f_1 = d_2' \circ H_1 + H_0 \circ d_1.$$

Again, an iteration proves the claim. □

Definition II.1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be additive.

(a) If F is right-exact and if $\mathcal{C}$ has enough projectives, then the *left-derived functors, L_nF, $n \geqslant 0$, of F* are defined as

$$(L_nF)(M) := H_n(F(P_*)),$$

where P_* is any projective resolution of M.

(b) Dually, if F is left-exact and if $\mathcal{C}$ has enough injectives, then the *right-derived functors, R^nF, $n \geqslant 0$, of F* are defined as

$$(R^nF)(M) := H^n(F(I^*)),$$

where I^* is any injective resolution of M.

Remark II.1.4.

- As F is additive, $F(P_*)$ is a chain complex, but not exact in general. Similarly, $F(I^*)$ is a cochain complex and not exact in general.
- Lemma II.1.1 implies that $(L_nF)(M)$ and $(R^nF)(M)$ are well-defined up to isomorphism: Assume that $\varepsilon_M\colon P_* \to M$ and $\varepsilon_M'\colon P_*' \to M$ are two projective resolutions of M, then there are chain maps $f_*\colon P_* \to P_*'$ and $g_*\colon P_*' \to P_*$ compatibel with ε_M and ε_M'. The composites $f_* \circ g_*$ and $g_* \circ f_*$ both extend the identity map of M and therefore

$$g_* \circ f_* \simeq \mathrm{id}_{P_*} \text{ and } f_* \circ g_* \simeq \mathrm{id}_{P_*'}.$$

As F is additive, this yields that $F(g_*) \circ F(f_*) \simeq \mathrm{id}_{F(P_*)}$ and $F(f_*) \circ F(g_*) \simeq \mathrm{id}_{F(P_*')}$. Therefore

$$H_nF(P_*) \cong H_nF(P_*') \text{ for all } n.$$

- L_nF and R^nF are actually functors. For an $f \in \mathcal{C}(M, N)$ and injective resolutions I^* of M and J^* of N we obtain a map of cochain complexes f^*

$$\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[d, "f"'] & I^0 \arrow[r] \arrow[d, "f^0"'] & I^1 \arrow[r] \arrow[d, "f^1"'] & \ldots \\
N \arrow[r, "\eta_N"] & J^0 \arrow[r] & J^1 \arrow[r] & \ldots
\end{tikzcd}$$

by Lemma II.1.1. As F is an additive functor the induced map $F(f^*)$ is again a map of cochain complexes in $\mathcal{D}$ and we obtain induced maps

$$H^nF(f^*)\colon (R^nF)(M) = H^nF(I^*) \to H^nF(J^*) = (R^nF)(N).$$

In degree zero, we do not loose any information:

Proposition II.1.5. *In the situation of Definition II.1.3: $L_0F \cong F$ if F is right-exact and $R^0F \cong F$ if F is left-exact.*

Proof. Let

$$0 \longrightarrow M \xrightarrow{\eta_M} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \dots$$

be an injective resolution of M. As F is left-exact, the sequence

$$0 \longrightarrow F(M) \xrightarrow{F(\eta_M)} F(I^0) \xrightarrow{F(d^0)} F(\mathsf{im}(d^0))$$

is exact. Therefore

$$F(\eta_M) = \ker(d^0) \cong H^0(F(I^*))$$

which we identify with $s(F(\eta_M)) = F(M)$.

In the dual case, the right-exactness of F gives that $H_0F(P_*) = F(P_0)/\mathsf{im}(F(d_1)) \cong F(M)$. □

Lecture 10

Two very important examples of derived functors are Tor- and Ext-functors.

Definition II.1.6. Let R be a ring.

(a) Let M be a right R-module and

$$F_M \colon R\text{-}\mathsf{mod} \to \mathsf{Ab}, \quad N \mapsto M \otimes_R N.$$

Then $\mathsf{Tor}_*^R(M, N)$ is defined as

$$\mathsf{Tor}_n^R(M, N) := (L_n F_M)(N).$$

(b) Let $\tilde{N}$ be an R-module and let $G_{\tilde{N}}$ be the functor

$$G_{\tilde{N}} \colon R\text{-}\mathsf{mod} \to \mathsf{Ab}, \quad N \mapsto \mathsf{Hom}_R(\tilde{N}, N).$$

Then $\mathsf{Ext}_R^*(\tilde{N}, N)$ is defined as

$$\mathsf{Ext}_R^n(\tilde{N}, N) := (R^n G_{\tilde{N}})(N).$$

Example II.1.7. You will determine $\mathsf{Tor}_*^{\mathbb{Z}}(A, B)$ for all finitely generated abelian groups and also the groups $\mathsf{Ext}_{\mathbb{Z}}^*(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for all natural numbers n and m in an exercise.

Example II.1.8. Let C_n be the cyclic group with n elements and we write $C_n = \langle t \rangle = \{1, t, \dots, t^{n-1}\}$. We can view $\mathbb{Z}$ as a module over the group ring $\mathbb{Z}[C_n]$ by defining $t^i.x = x$ for all i. Then a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[C_n]$-module is

$$\dots \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{1-t} \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{1-t} \mathbb{Z}[C_n] \xrightarrow{\varepsilon} \mathbb{Z}.$$

Here, $\varepsilon(t^i) = 1$ for all i and $N = \sum_{i=0}^{n-1} t^i$. So in this case we actually find a free resolution. In order to calculate $\mathsf{Tor}_*^{\mathbb{Z}[C_n]}(\mathbb{Z}, \mathbb{Z})$ we have to determine the homology groups of the complex

$$\dots \xrightarrow{\mathrm{id}\otimes(1-t)} \mathbb{Z} \otimes_{\mathbb{Z}[C_n]} \mathbb{Z}[C_n] \xrightarrow{\mathrm{id}\otimes N} \mathbb{Z} \otimes_{\mathbb{Z}[C_n]} \mathbb{Z}[C_n] \xrightarrow{\mathrm{id}\otimes(1-t)} \mathbb{Z} \otimes_{\mathbb{Z}[C_n]} \mathbb{Z}[C_n].$$

As $\mathbb{Z} \otimes_{\mathbb{Z}[C_n]} \mathbb{Z}[C_n] \cong \mathbb{Z}$, the above complex is isomorphic to the complex

$$\dots \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

and hence the homology groups are

$$\mathsf{Tor}_*^{\mathbb{Z}[C_n]}(\mathbb{Z}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & * = 0, \\ \mathbb{Z}/n\mathbb{Z}, & * \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

Thus in this example we have infinitely many non-trivial Tor-groups.

II.2. The long exact sequence for derived functors

Remark II.2.1. Working in the general context of abelian categories is cumbersome, because we cannot in general assume that objects have elements. But now we want to do several proofs that use the method of *diagram chase*, where you prove things by playing pin-ball with elements.

There is the famous *Freyd-Mitchell theorem*, saying that if $\mathcal{C}$ is a small abelian category, then there exists a ring R and a (full, faithful and) exact functor $F\colon \mathcal{C} \to R\text{-}\mathsf{mod}$. In this case we can think of the objects of $\mathcal{C}$ as modules and these have element. However, note the smallness assumption.

From now on we will assume that our abelian categories are of this type. As we will mostly consider categories of modules anyway, this assumption is not too absurd.

Definition II.2.2. If A_*, B_*, C_* are chain complexes in an abelian category $\mathcal{C}$ and $f_*\colon A_* \to B_*$, $g\colon B_* \to C_*$ are chain maps, then we call the sequence

$$A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_*$$

exact, if the image of f_n is the kernel of g_n for all $n \in \mathbb{Z}$.

Thus such an exact sequence of chain complexes is a commuting double ladder

$$\begin{tikzcd}
\vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] \\
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d"] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d"] & C_{n+1} \arrow[d, "d"] \\
A_n \arrow[r, "f_n"] \arrow[d, "d"] & B_n \arrow[r, "g_n"] \arrow[d, "d"] & C_n \arrow[d, "d"] \\
A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d"] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "d"] & C_{n+1} \arrow[d, "d"] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

in which every row is exact.

Example II.2.3. Let p be a prime, then

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, "p"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "p^2"] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "p"] \arrow[d, "\pi"] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\pi"] & \mathbb{Z}/p\mathbb{Z} \arrow[d, "\mathrm{id}"] \\
\mathbb{Z}/p\mathbb{Z} \arrow[r, "p"] \arrow[d] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r, "\pi"] \arrow[d] & \mathbb{Z}/p\mathbb{Z} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

has exact rows and columns, in particular it is an exact sequence of chain complexes. Here, π denotes varying canonical projection maps.

Construction II.2.4. Assume that $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ is a short exact sequence of chain complexes.

We define $\delta \in \mathcal{C}(H_n(C_*), H_{n-1}(A_*))$:

For a $c \in C_n$ with $d_n(c) = 0$, we choose a $b \in B_n$ with $g_n b = c$. This is possible because g_n is surjective. We know that $dg_n b = dc = 0 = g_{n-1}db$ thus db is in the kernel of g_{n-1}, hence it is in the image of f_{n-1}. Thus there is an $a \in A_{n-1}$ with $f_{n-1}a = db$. We have that $f_{n-2}da = df_{n-1}a = ddb = 0$ and as f_{n-2} is injective, this shows that a is a cycle.

We define $\delta[c] := [a]$.

$$B_n \ni b \xmapsto{g_n} c \in C_n$$

$$A_{n-1} \ni a \xmapsto{f_{n-1}} db \in B_{n-1}$$

The map δ is called the *connecting homomorphism.*

Lemma II.2.5. *The morphism δ is well-defined.*

Proof. Assume that there are b and b' with $g_n b = g_n b' = c$. Then $g_n(b - b') = 0$ and thus there is an $\tilde{a} \in A_n$ with $f_n\tilde{a} = b - b'$. Define a' as $a - d\tilde{a}$. Then

$$f_{n-1}a' = f_{n-1}a - f_{n-1}d\tilde{a} = db - db + db' = db'$$

because $f_{n-1}d\tilde{a} = db - db'$. As f_{n-1} is injective, we get that a' is uniquely determined with this property. As a is homologous to a' we get that $[a] = [a'] = \delta[c]$, thus the latter is independent of the choice of b.

In addition, we have to make sure that the value stays the same if we add a boundary term to c, *i.e.*, take $c' = c + d\tilde{c}$ for some $\tilde{c} \in C_{n+1}$. Choose preimages of $c, \tilde{c}$ under g_n and g_{n+1}, *i.e.*, b and $\tilde{b}$ with $g_n b = c$ and $g_{n+1}\tilde{b} = \tilde{c}$. Then the element $b' = b + d\tilde{b}$ has boundary $db' = db$ and thus both choices will result in the same a.

Therefore $\delta\colon H_n(C_*) \to H_{n-1}(A_*)$ is well-defined. $\square$

Proposition II.2.6. *The morphism δ is natural,* i.e., *if*

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "f"] \arrow[d, "\alpha"] & B_* \arrow[r, "g"] \arrow[d, "\beta"] & C_* \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "f'"] & B'_* \arrow[r, "g'"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of chain maps in which the rows are exact then $H_{n-1}(\alpha) \circ \delta = \delta \circ H_n(\gamma)$,

$$\begin{tikzcd}
H_n(C_*) \arrow[r, "\delta"] \arrow[d, "H_n(\gamma)"'] & H_{n-1}(A_*) \arrow[d, "H_{n-1}(\alpha)"] \\
H_n(C'_*) \arrow[r, "\delta"] & H_{n-1}(A'_*)
\end{tikzcd}$$

Proof. Let $c \in Z_n(C_*)$, then $\delta[c] = [a]$ for a $b \in B_n$ with $g_n b = c$ and an $a \in A_{n-1}$ with $f_{n-1}a = db$. Therefore, $H_{n-1}(\alpha)(\delta[c]) = [\alpha_{n-1}(a)]$.

On the other hand, we have

$$f'_{n-1}(\alpha_{n-1}a) = \beta_{n-1}(f_{n-1}a) = \beta_{n-1}(db) = d\beta_n b$$

and

$$g'_n(\beta_n b) = \gamma_n g_n b = \gamma_n c$$

and we can conclude that by the construction of δ

$$\delta[\gamma_n(c)] = [\alpha_{n-1}(a)]$$

and this shows $\delta \circ H_n(\gamma) = H_{n-1}(\alpha) \circ \delta$. $\square$

With this auxiliary result at hand we can now prove the main result in this section:

Proposition II.2.7. *For any short exact sequence*

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

of chain complexes we obtain a long exact sequence of homology groups

$$\ldots \xrightarrow{\delta} H_n(A_*) \xrightarrow{H_n(f)} H_n(B_*) \xrightarrow{H_n(g)} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{H_{n-1}(f)} \ldots$$

Lecture 11

Proof. a) Exactness at the spot $H_n(B_*)$:

We have $H_n(g) \circ H_n(f)[a] = [g_n(f_n(a))] = 0$ because the composition of g_n and f_n is zero. This proves that the image of $H_n(f)$ is contained in the kernel of $H_n(g)$.

For the converse, let $[b] \in H_n(B_*)$ with $[g_n b] = 0$. Then there is a $c \in C_{n+1}$ with $dc = g_n b$. As g_{n+1} is surjective, we find a $b' \in B_{n+1}$ with $g_{n+1}b' = c$. Hence

$$g_n(b - db') = g_n b - dg_{n+1}b' = dc - dc = 0.$$

Exactness gives an $a \in A_n$ with $f_n a = b - db'$ and $da = 0$ and therefore $f_n a$ is homologous to b and $H_n(f)[a] = [b]$ thus the kernel of $H_n(g)$ is contained in the image of $H_n(f)$.

b) Exactness at the spot $H_n(C_*)$:

Let $b \in H_n(B_*)$, then $\delta[g_n b] = 0$ because b is a cycle, so 0 is the only preimage under f_{n-1} of $db = 0$. Therefore the image of $H_n(g)$ is contained in the kernel of δ.

Now assume that $\delta[c] = 0$, thus in the construction of δ, the a is a boundary, $a = da'$. Then for a preimage of c under g_n, b, we have by the definition of a

$$d(b - f_n a') = db - df_n a' = db - f_{n-1}a = 0.$$

Thus $b - f_n a'$ is a cycle and $g_n(b - f_n a') = g_n b - g_n f_n a' = g_n b - 0 = g_n b = c$, so we found a preimage for $[c]$ and the kernel of δ is contained in the image of $H_n(g)$.

c) Exactness at $H_{n-1}(A_*)$:

Let c be a cycle in $Z_n(C_*)$. Again, we choose a preimage b of c under g_n and an a with $f_{n-1}(a) = db$. Then $H_{n-1}(f)\delta[c] = [f_{n-1}(a)] = [db] = 0$. Thus the image of δ is contained in the kernel of $H_{n-1}(f)$.

If $a \in Z_{n-1}(A_*)$ with $H_{n-1}(f)[a] = 0$. Then $f_{n-1}a = db$ for some $b \in B_n$. Take $c = g_n b$. Then by definition $\delta[c] = [a]$. ☐

With the help of the results above, we can deduce a long exact sequence for derived functors:

Theorem II.2.8. *Let $\mathcal{C}, \mathcal{D}$ be abelian categories, assume that $\mathcal{C}$ has enough projectives and injectives and let F be an additive functor. Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be a short exact sequence in $\mathcal{C}$.*

(a) *If F is right-exact, then the sequence*

$$\ldots \xrightarrow{(L_1F)(g)} (L_1F)(M'') \xrightarrow{\delta} (L_0F)(M') \xrightarrow{(L_0F)(f)} (L_0F)(M) \xrightarrow{(L_0F)(g)} (L_0F)(M'') \longrightarrow 0$$

is exact.

(b) *If F is left-exact, then the sequence*

$$0 \longrightarrow (R^0F)(M') \xrightarrow{(R^0F)(f)} (R^0F)(M) \xrightarrow{(R^0F)(g)} (R^0F)(M'') \xrightarrow{\delta} (R^1F)(M') \xrightarrow{(R^1F)(f)} \ldots$$

is exact.

Proof. We prove the claim in (a); the one for (b) is dual.

Let $P'_* \xrightarrow{\varepsilon_{M'}} M'$ be a projective resolution of M' and let $P''_* \xrightarrow{\varepsilon_{M''}} M''$ be one for M''. We set $P_0 := P'_0 \oplus P''_0$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & P'_0 \arrow[r] \arrow[d, "\varepsilon_{M'}"'] & P_0 \arrow[r] & P''_0 \arrow[r] \arrow[d, "\varepsilon_{M''}"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0. \end{tikzcd}$$

As P_0'' is projective, we obtain a $\xi \in \mathcal{C}(P_0'', M)$ with $g \circ \xi = \varepsilon_{M''}$. We define

$$\varepsilon_M \colon P_0 = P_0' \oplus P_0'' \to M$$

by setting $\varepsilon_M|_{P_0'} = f \circ \varepsilon_{M'}$ and $\varepsilon_M|_{P_0''} = \xi$. We view the resulting diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P_0' \arrow[r] \arrow[d, "\varepsilon_{M'}"'] & P_0 \arrow[r] \arrow[d, "\varepsilon_M"] & P_0'' \arrow[r] \arrow[d, "\varepsilon_{M''}"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

As a short exact sequence of chain complexes and thus we obtain a long exact sequence on homology groups

$$0 \longrightarrow \ker(\varepsilon_{M'}) \longrightarrow \ker(\varepsilon_M) \longrightarrow \ker(\varepsilon_{M''}) \longrightarrow \operatorname{coker}(\varepsilon_{M'}) = 0.$$

In the next step we set again $P_1 := P_1' \oplus P_1''$ and we define $d_1 \colon P_1 \to \ker(\varepsilon_M)$ by setting $d_1|_{P_1'} = d_1'$ and by using the projectivity of P_1'' to obtain a morphism in $\mathcal{C}(P_1'', \ker(\varepsilon_M))$ that yields $d_1|_{P_1''}$. An iteration of this argument for $P_n = P_n' \oplus P_n''$ for all n gives a projective resolution of M such that

$$0 \longrightarrow P_*' \longrightarrow P_* \longrightarrow P_*'' \longrightarrow 0$$

is a short exact sequence of chain complexes. Note that by the very construction of P_n as $P_n' \oplus P_n''$ we obtain a splitting in every degree

$$\begin{tikzcd}
0 \arrow[r] & P_n' \arrow[r] & P_n = P_n' \oplus P_n'' \arrow[r] & P_n'' \arrow[r] \arrow[l, bend left, "s_n"] & 0.
\end{tikzcd}$$

Applying F therefore gives a short exact sequence of chain complexes in $\mathcal{D}$ whose corresponding long exact sequence of homology groups gives the desired long exact sequence for the left derived functors of F. $\square$

Remark II.2.9.

- The long exact sequences in Theorem II.2.8 are natural with respect to morphisms of short exact sequences.
- For Tor we get for every short exact sequence $0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$ that

$$\begin{tikzcd}
 & & \mathsf{Tor}_n^R(M, N'') \arrow[dll, rounded corners] \\
\mathsf{Tor}_{n-1}^R(M, N') \arrow[r] & \mathsf{Tor}_{n-1}^R(M, N) \arrow[r] & \mathsf{Tor}_{n-1}^R(M, N'') \arrow[dll, rounded corners] \\
\dots & & \arrow[dll, rounded corners] \\
\mathsf{Tor}_1^R(M, N') \arrow[r] & \mathsf{Tor}_1^R(M, N) \arrow[r] & \mathsf{Tor}_1^R(M, N'') \arrow[dll, rounded corners] \\
M \otimes_R N' \arrow[r] & M \otimes_R N \arrow[r] & M \otimes_R N'' \to 0
\end{tikzcd}$$

is a long exact sequence of abelian groups for every right R-module M.

- Dually for Ext we obtain the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Hom}_R(\tilde{N}, N') \arrow[r] & \mathsf{Hom}_R(\tilde{N}, N) \arrow[r] & \mathsf{Hom}_R(\tilde{N}, N'') \arrow[dlll, rounded corners] \\ & \mathsf{Ext}^1_R(\tilde{N}, N') \arrow[r] & \dots \end{tikzcd}$$

Proposition II.2.10. *For all R-modules N we have $\mathsf{Tor}_n^R(M,N) = 0$ for all $n \geqslant 1$ if and only if M is flat.*

Proof. We know that M is flat if and only if $M \otimes_R (-)$ is an exact functor, so

$$0 \longrightarrow M \otimes_R N' \longrightarrow M \otimes_R N \longrightarrow M \otimes_R N'' \longrightarrow 0$$

is an exact sequence for all short exact sequences

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0.$$

Assume that $P_* \to N$ is a projective resolution. Then we claim that $H_n(M \otimes_R P_*) \cong 0$ for all $n \geqslant 1$, because

$$\begin{aligned} H_n(M \otimes_R P_*) &= \frac{\ker(\mathrm{id} \otimes d_n \colon M \otimes_R P_n \to M \otimes_R P_{n-1})}{\mathsf{im}(\mathrm{id} \otimes d_{n+1} M \otimes_R P_{n+1} \to M \otimes_R P_n)} \\ &\cong M \otimes_R \frac{\ker(d_n \colon P_n \to P_{n-1})}{\mathsf{im}(d_{n+1} P_{n+1} \to P_n)} \text{ as } M \text{ is flat} \\ &= M \otimes_R H_n(P_*) \\ &= \begin{cases} M \otimes_R N, & \text{for } n = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

□

In particular, $\mathsf{Tor}_*^R(M,N) = 0$ for $n \geqslant 1$ if M is a projective right R-module.

II.3. Balancing Tor and Ext

Our next goal is to show that we can also calculate Tor and Ext groups by resolving the left variable, so

$$\mathsf{Ext}^n_R(M,N) = R^n(N \mapsto \mathsf{Hom}_R(M,N)) \cong R^n(M \mapsto \mathsf{Hom}_R(M,N))$$

and

$$\mathsf{Tor}_n^R(M,N) = L_n(M \mapsto M \otimes_R N) \cong L_n(N \mapsto M \otimes_R N).$$

Note for the first claim that a projective object in $(R\text{-}\mathsf{mod})^{op}$ is an injective object in R-mod.

Definition II.3.1. Let $\mathcal{C}$ be an abelian category. A *double complex in* $\mathcal{C}$ is a $\mathbb{Z} \times \mathbb{Z}$-graded family of objects $(C_{ij})_{i,j \in \mathbb{Z}}$ of $\mathcal{C}$ together with $d^h \in \mathcal{C}(C_{ij}, C_{i-1,j})$, $d^v \in \mathcal{C}(C_{ij}, C_{i,j-1})$ for $i,j \in \mathbb{Z}$ such that

$$\begin{aligned} d^h \circ d^h &= 0 = d^v \circ d^v \\ d^h \circ d^v &= -d^v \circ d^h. \end{aligned}$$

Morphisms of double complexes are families $f_{ij} \in \mathcal{C}(C_{ij}, D_{ij})$ that commute with d^h and d^v.

$$
\begin{tikzcd}
 & \vdots \arrow[d, "d^v"] & \vdots \arrow[d, "d^v"] & \vdots \arrow[d, "d^v"] & \\
\dots & C_{00} \arrow[l, "d^h"'] \arrow[d, "d^v"] & C_{1,0} \arrow[l, "d^h"'] \arrow[d, "d^v"] & C_{2,0} \arrow[l, "d^h"'] \arrow[d, "d^v"] & \dots \arrow[l, "d^h"'] \\
 & C_{0,-1} \arrow[l, "d^h"'] \arrow[d, "d^v"] & C_{1,-1} \arrow[l, "d^h"'] \arrow[d, "d^v"] & C_{2,-1} \arrow[l, "d^h"'] \arrow[d, "d^v"] & \dots \arrow[l, "d^h"'] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Lecture 12

In the following we denote the coproduct by $\oplus$.

Lemma II.3.2. *If $(C_{*,*}, d^h, d^v)$ is a double complex in an abelian category $\mathcal{C}$ that possesses all products and coproducts, then the following $\mathbb{Z}$-graded objects of $\mathcal{C}$ are chain complexes:*

- $$\mathsf{Tot}(C_{*,*})_n := \mathsf{Tot}^{\oplus}(C_{*,*})_n := \bigoplus_{p+q=n} C_{p,q}$$

  and

- $$\mathsf{Tot}^{\Pi}(C_{*,*})_n := \prod_{p+q=n} C_{p,q}.$$

  In both cases the differential d is given by taking $d^h + d^v$ in every component.

□

This is actually a differential because of the defining properties of d^h and d^v from Definition II.3.1:

$$d^2 = (d^h + d^v) \circ (d^h + d^v) = d^h \circ d^h + d^h \circ d^v + d^v \circ d^h + d^v \circ d^v = 0.$$

The following two special cases are crucial for the properties of Tor and Ext:

Definition II.3.3. Let (C_*, d^C) be a complex of right R-modules and let (D_*, d^D) be a complex of left R-modules. Then $(C_* \otimes_R D_*, d_\otimes)$ is the chain complex $\mathsf{Tot}(E_{*,*})$ with $E_{p,q} = C_p \otimes_R D_q$ and for $x \in C_p$, $y \in D_q$ we have

$$d^h(x \otimes y) = d^C(x) \otimes y, \text{ and } d^v(x \otimes y) = (-1)^p x \otimes d^D(y).$$

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\dots & C_0 \otimes_R D_2 \arrow[l, "d^C\otimes\mathrm{id}"'] \arrow[d] & C_1 \otimes_R D_2 \arrow[l, "d^C\otimes\mathrm{id}"'] \arrow[d] & C_2 \otimes_R D_2 \arrow[l, "d^C\otimes\mathrm{id}"'] \arrow[d] & \dots \arrow[l, "d^C\otimes\mathrm{id}"'] \\
\dots & C_0 \otimes_R D_1 \arrow[l, "d^C\otimes\mathrm{id}"'] \arrow[d] & C_1 \otimes_R D_1 \arrow[l, "d^C\otimes\mathrm{id}"'] \arrow[d] & C_2 \otimes_R D_1 \arrow[l, "d^C\otimes\mathrm{id}"'] \arrow[d] & \dots \arrow[l, "d^C\otimes\mathrm{id}"'] \\
\dots & C_0 \otimes_R D_0 \arrow[l, "d^C\otimes\mathrm{id}"'] \arrow[d] & C_1 \otimes_R D_0 \arrow[l, "d^C\otimes\mathrm{id}"'] \arrow[d] & C_2 \otimes_R D_0 \arrow[l, "d^C\otimes\mathrm{id}"'] \arrow[d] & \dots \arrow[l, "d^C\otimes\mathrm{id}"'] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Thus, $(C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q$ and $d_\otimes(x \otimes y) = d^C(x) \otimes y + (-1)^p x \otimes d^D(y)$ for $x \in C_p$ and $y \in D_q$.

Definition II.3.4. If (C_*, d^C) is a chain complex and (I^*, δ_I) is a cochain complex, then $\mathcal{C}(C_*, I^*)$ is the double cochain complex with

$$\mathcal{C}(C_*, I^*)^{p,q} := \mathcal{C}(C_p, I^q)$$

and for $f \in \mathcal{C}(C_*, I^*)^{p,q}$ we define

$$d^h(f) := f \circ d_c, \quad d^v(f) = (-1)^{p+q+1}\delta_I \circ f.$$

We consider the total cochain complex $\mathsf{Tot}^{\prod}\mathcal{C}(C_*, I^*)$.

We will mostly consider the latter construction in the category of R-modules. In this case one denotes $\mathsf{Tot}^{\prod}\mathcal{C}(C_*, I^*)$ often by $\underline{\mathsf{Hom}}(C_*, I^*)$, so

$$\underline{\mathsf{Hom}}(C_*, I^*)^n := \prod_{p+q=n} \mathsf{Hom}_R(C_p, I^q).$$

One often also considers the chain complex of homomorphisms between two chain complexes. This is doable with our convention earlier: if (D_*, d^D) is a chain complex, then D_{-*} is a cochain complex. For this case we obtain

$$\underline{\mathsf{Hom}}(C_*, D_*)_n := \underline{\mathsf{Hom}}(C_*, D_*)^n = \prod_{p+q=n} \mathsf{Hom}_R(C_p, D^q) = \prod_{p+q=n} \mathsf{Hom}_R(C_p, D_{-q}) = \prod_p \mathsf{Hom}_R(C_p, D_{p-n}).$$

If you want a *chain* complex, you need to consider $\underline{\mathsf{Hom}}(C_*, D_*)_n = \prod_p \mathsf{Hom}_R(C_p, D_{p+n})$.

Lemma II.3.5. *Let $C_{*,*}$ be a double complex in an abelian category with products and coproducts and assume that for all $q \in \mathbb{Z}$ the complex $C_{*,q}$ is exact. If ther is an $N \in \mathbb{Z}$ with $C_{p,q} = 0$ for all $p < N$, then both chain complexes $\mathsf{Tot}^{\oplus}(C_{*,*})$ and $\mathsf{Tot}^{\prod}(C_{*,*})$ are also exact.*

Proof. We give the proof for $\mathsf{Tot}^{\oplus}$, but it should be clear how to adapt the proof in the other case.

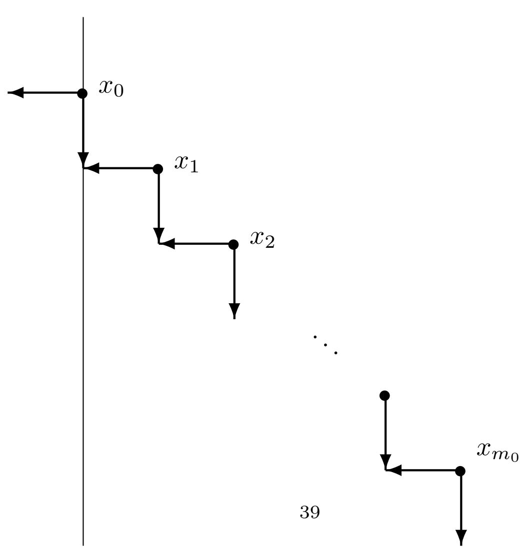
Without loss of generality we can assume $N = 0$ and it suffices to show that $H_0(\mathsf{Tot}(C_{*,*})) \cong 0$. Otherwise we can use the suspension functor that you know from the exercises.

By definition

$$\mathsf{Tot}(C_{*,*})_0 = \bigoplus_{p \in \mathbb{Z}} C_{p,-p}$$

and by our assumption this is equal to $\bigoplus_{p \in \mathbb{N}_0} C_{p,-p}$. If $x \in \mathsf{Tot}(C_{*,*})_0$ then we can write it as $x = (x_0, \ldots, x_{m_0})$ with $x_i \in C_{i,-i}$. Assume that $d(x) = 0$. We have to show that x is a boundary. The condition $d(x) = 0$ is equivalent to

$$d^h(x_0) = 0, \quad d^h(x_i) + d^v(x_{i-1}) = 0, \quad 0 \leqslant i \leqslant m_0, \text{ and } d^v(x_{m_0}) = 0.$$



As the rows are exact, we find an y_0 with $d^h y_0 = x_0$ and iteratively there exist y_is with $d^h y_i = x_i - d^v y_{i-1}$ because

$$\begin{aligned} d^h(x_i - d^v(y_{i-1})) &= d^h x_i - d^h d^v(y_{i-1}) \\ &= d^h x_i + d^v d^h(y_{i-1}) \end{aligned}$$

and by iteration this is equal to

$$d^h x_i + d^v(x_{i-1} + d^v(y_{i-2})) = d^h x_i + d^v x_{i-1} = 0.$$

Therefore $y := (y_0, \ldots, y_{m_0})$ satisfies $dy = x$. □

Construction II.3.6. Let M be a right R-module and N be a left R-module. Assume that P_* is a projective resolution of M and Q_* be a projective resolution of N. We set

$$\widetilde{\mathsf{Tor}}_n^R(M,N) := H_n\mathsf{Tot}(P_* \otimes_R Q_*).$$

Theorem II.3.7. *For all $n \geqslant 0$:*

$$\widetilde{\mathsf{Tor}}_n^R(M,N) \cong \mathsf{Tor}_n^R(M,N).$$

Proof. Let P_* be a projective resolution with $\varepsilon_M\colon P_0 \to M$. Define the chain complex X_* with

$$X_n = \begin{cases} P_n, & n \geqslant 0, \\ M, & n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Then X_* is an exact sequence. For a projective resolution Q_* of N each Q_j is projective and hence $X_* \otimes_R Q_j$ is also exact for all $j \geqslant 0$. By Lemma II.3.5 we obtain that $\mathsf{Tot}(X_* \otimes Q_*)$ is exact. The sequence

$$0 \longrightarrow \Sigma^{-1}M \longrightarrow X_* \longrightarrow P_* \longrightarrow 0$$

is a short exact sequence of chain complexes and it induces a short exact sequence of chain complexes

$$0 \longrightarrow \mathsf{Tot}(\Sigma^{-1}M \otimes_R Q_*) \longrightarrow \mathsf{Tot}(X_* \otimes_R Q_*) \longrightarrow \mathsf{Tot}(P_* \otimes_R Q_*) \longrightarrow 0$$

and a long exact sequence on the level of homology groups. As $H_n\mathsf{Tot}(X_* \otimes_R Q_*) \cong 0$ for all n, this implies that the connecting homomorphism is an isomorphism

$$\widetilde{\mathsf{Tor}}_n^R(M,N) = H_n(\mathsf{Tot}(P_* \otimes_R Q_*)) \cong H_{n-1}(\Sigma^{-1}M \otimes_R Q_*)$$

but

$$(\Sigma^{-1}M \otimes_R Q_*))_{n-1} = M \otimes Q_n$$

and hence

$$H_{n-1}(\Sigma^{-1}M \otimes_R Q_*) \cong \mathsf{Tor}_n^R(M,N).$$

□

Remark II.3.8.
- A similar argument shows that $\mathsf{Tor}_n^R(M,N) = H_n(P_* \otimes_R N)$, where P_* is a projective resolution of M, thus it doesn't matter whether you resolve M or N projectively.
- Using $\underline{\mathsf{Hom}}$ instead of $\otimes$ yields the analogous result for Ext: If P_* is a projective resolution of M and I^* is an injective resolution of N, then

  $$\mathsf{Ext}_R^n(M,N) \cong H^n(\mathsf{Hom}_R(M,I^*)) \cong H^n(\mathsf{Hom}_R(P_*,N)) \cong H^n(\underline{\mathsf{Hom}}_R(P_*,I^*)).$$

  So for Ext you can resolve M projectively or N injectively or both.

II.4. Ext and extensions

If we take $R = \mathbb{Z}$ then $\mathsf{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/m\mathbb{Z})$ detects the m-torsion in M because

$$\mathsf{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/m\mathbb{Z}) = H_1(\ldots \longrightarrow 0 \longrightarrow M \cong M \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{M \otimes_R (m\cdot)} M \otimes_{\mathbb{Z}} \mathbb{Z} \cong M \longrightarrow 0)$$

and this is the kernel of $\cdot m \colon M \to M$, so the Tor is related to torsion.

For Ext the origin of the name comes from the following concept.

Definition II.4.1. Let M and N be R-modules.

(a) An *extension of M by N* is an exact sequence of R-modules

$$0 \longrightarrow N \xrightarrow{i} X \xrightarrow{\pi} M \longrightarrow 0.$$

(b) Two extensions of M by N $0 \longrightarrow N \xrightarrow{i} X \xrightarrow{\pi} M \longrightarrow 0$ and $0 \longrightarrow N \xrightarrow{i'} X' \xrightarrow{\pi'} M \longrightarrow 0$ are *equivalent*, if there is an isomorphism $\phi \colon X \to X'$ such that the diagram

$$\begin{tikzcd} & & X \arrow[dd, "\phi"] \arrow[dr, "\pi"] & & \\ 0 \arrow[r] & N \arrow[ur, "i"] \arrow[dr, "i'"'] & & M \arrow[r] & 0 \\ & & X' \arrow[ur, "\pi'"'] & & \end{tikzcd}$$

commutes.

(c) An extension is *split* if it is equivalent to

$$0 \longrightarrow N \xrightarrow{i_1} N \oplus M \xrightarrow{\pi_2} M \longrightarrow 0.$$

Lecture 13

Examples II.4.2. Let p be a prime. The sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{i_1} \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \xrightarrow{\pi_2} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

is a split extension of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$ but

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

is not split.

Lemma II.4.3. *If $\mathsf{Ext}_R^1(M, N) \cong 0$, then every extension of M by N is split.*

Proof. Let

$$0 \longrightarrow N \xrightarrow{i} X \xrightarrow{\pi} M \longrightarrow 0. \tag{II.4.1}$$

be an arbitrary extension of M by N and assume that $\mathsf{Ext}_R^1(M, N) \cong 0$. As (II.4.1) is a short exact sequence of modules, it induces a long exact sequence when we apply $\mathsf{Ext}_R^*(M, -)$ by Theorem II.2.8, so in particular

$$\mathsf{Hom}_R(M, X) \to \mathsf{Hom}_R(M, M) \to \mathsf{Ext}_R^1(M, N)$$

is exact. As $\mathsf{Ext}_R^1(M, N) \cong 0$, the map $\mathsf{Hom}_R(M, X) \to \mathsf{Hom}_R(M, M)$ is surjective and we find a lift of id_M, $s \colon M \to X$. But then

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i"] & X \arrow[r, "\pi"] & M \arrow[l, bend right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

is a split exact sequence. $\square$

We want to strengthen the above result to an 'if and only if'. To this end we introduce another useful concept from category theory.

Definition II.4.4. Let $\mathcal{C}$ be a category.

(a) Let $f \in \mathcal{C}(A, C)$ and $g \in \mathcal{C}(B, C)$. An object P of $\mathcal{C}$ together with morphisms $\varrho_A \in \mathcal{C}(P, A)$ and $\varrho_B \in \mathcal{C}(P, B)$ with $f \circ \varrho_A = g \circ \varrho_B$ is a *pullback of f and g*, if for every object W of $\mathcal{C}$ with morphisms $\alpha_A \in \mathcal{C}(W, A)$ and $\alpha_B \imath \mathcal{C}(W, B)$ with $f \circ \alpha_A = g \circ \alpha_B$ there is a unique $\xi \in \mathcal{C}(W, P)$ with $\varrho_A \circ \xi = \alpha_A$ and $\varrho_B \circ \xi = \alpha_B$.

$$\begin{tikzcd} W \arrow[drr, bend left, "\alpha_B"] \arrow[ddr, bend right, "\alpha_A"'] \arrow[dr, dotted, "\exists!\xi"] & & \\ & P \arrow[r, "\varrho_B"] \arrow[d, "\varrho_A"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"] & C \end{tikzcd}$$

(b) Dually, for $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(A, C)$ an object P together with morphisms $i_B \in \mathcal{C}(B, P)$ and $i_C \in \mathcal{C}(C, P)$ satisfying $i_B \circ f = i_C \circ g$ is a *pushout of f and g* if for all objects D of $\mathcal{C}$ with morphisms $j_B \in \mathcal{C}(B, D)$ and $j_C \in \mathcal{C}(C, P)$ satisfying $j_B \circ f = j_C \circ g$ there is a unique $\zeta \colon P \to D$ with $\zeta \circ i_B = j_B$ and $\zeta \circ i_C = j_C$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "i_B"] \arrow[ddr, bend left, "j_B"] & \\ C \arrow[r, "i_C"] \arrow[drr, bend right, "j_C"'] & P \arrow[dr, dotted, "\exists!\zeta"] & \\ & & D \end{tikzcd}$$

Remark II.4.5. If they exist, then pullbacks and pushouts are unique up to isomorphisms. They are objects that are 'as close to the defining diagram as possible'.

These two concepts are ubiquitous in mathematics:

Examples II.4.6.

- If $\mathcal{C}$ is the category of sets, Sets, and $f \colon X \to Z$ and $g \colon Y \to Z$ are functions, then the pullback of f and g is the set
$$P = X \times_Z Y := \{(x, y) \in X \times Y, f(x) = g(y)\}$$
together with the projection maps
$$\varrho_X = \pi_X \colon X \times_Z Y \to X, \quad \varrho_Y = \pi_Y \colon X \times_Z Y \to Y.$$
$$\begin{tikzcd} P = X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z. \end{tikzcd}$$
This pullback always exists, but it can be empty. Take for instance two inclusions $X \hookrightarrow Z$ and $Y \hookrightarrow Z$ with disjoint image.
- In the same category the pushout of $f \colon X \to Y$ and $g \colon X \to Z$ is given by
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "i_Z"] \\ Y \arrow[r, "i_Y"] & Y \sqcup Z/(f(x) \sim g(x)). \end{tikzcd}$$
- In the category of R-modules, $R\text{-}\mathsf{mod}$, the pullback of two R-linear maps $f \colon A \to C$ and $g \colon B \to C$ is again $A \times_C B$ as in the category of sets.

- For two $f \in R\text{-}\mathsf{mod}(A,B)$, $g \in R\text{-}\mathsf{mod}(A,C)$ the pushout is the R-module $P := B \oplus C/U$ where U is the submodule generated by $(f(a), -g(a))$ for $a \in A$.

With these constructions at hand we can now state the main theorem of this section:

Theorem II.4.7. *Let M and N be two R-modules. There is a bijection between the set of equivalence classes of extension of M by N and $\mathsf{Ext}^1_R(M,N)$.*

In the following proof we omit certain justifications that maps are actually well-defined. Otherwise, the proof is complete.

Sketch of proof. We denote by $\mathcal{E}(M,N)$ the set of equivalence classes of extensions of M by N.
Let P_* be a projective resolution of M with $\varepsilon_M\colon P_0 \to M$ and let

$$0 \longrightarrow N \xrightarrow{i} X \xrightarrow{\pi} M \longrightarrow 0$$

be an extension of M by N. By the fundamental lemma (Lemma II.1.1) there are maps $\alpha_0\colon P_0 \to X$ and $\alpha_1\colon P_1 \to N$ making the diagram

$$\begin{tikzcd} P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "\alpha_1"'] & P_0 \arrow[r, "\varepsilon_M"] \arrow[d, "\alpha_0"] & M \arrow[d, equal] \\ 0 \arrow[r] & N \arrow[r, "i"] & X \arrow[r, "\pi"] & M \end{tikzcd}$$

commutative. The equation $\alpha_1 \circ d_2 = 0$ should be read as

$$\mathsf{Hom}_R(d_2, N)(\alpha_1) = 0$$

hence $\alpha_1 \in \mathsf{Hom}_R(P_1, N)$ is a 1-cocycle in the cochain complex whose cohomology is $\mathsf{Ext}^*_R(M,N)$.

We define $\psi\colon \mathcal{E}(M,N) \to \mathsf{Ext}^1_R(M,N)$ by setting

$$\psi[0 \longrightarrow N \xrightarrow{i} X \xrightarrow{\pi} M \longrightarrow 0] := [\alpha_1].$$

You check that ψ is well-defined, *i.e.*, it is independent of the choice of P_*, of the choice of α_* and of the representative of the equivalence class of the extensions.
For the inverse of ψ we observe that for a class $[\alpha_1] \in \mathsf{Ext}^1_R(M,N)$ the representative α_1 is a cocycle, so $\alpha_1 \circ d_2 = 0$. Therefore α_1 factors through

$$\overline{\alpha}_1\colon P_1/\mathsf{im}(d_2) \to N.$$

Let X be the pushout of

$$\begin{tikzcd} P_1/\mathsf{im}(d_2) \arrow[r, "\bar{d}_1"] \arrow[d, "\bar{\alpha}_1"'] & P_0 \arrow[d, dotted, "j"] \\ N \arrow[r, dotted, "i"] & X. \end{tikzcd}$$

We define $\phi\colon \mathsf{Ext}^1_R(M,N) \to \mathcal{E}(M,N)$ as

$$\phi[\alpha_1] := [0 \longrightarrow N \xrightarrow{i} X \xrightarrow{\pi} M \longrightarrow 0]$$

where $\pi\colon X \to M$ is the R-linear map that exists thanks to the universal property of the pushout:

$$\begin{tikzcd} P_1/\mathsf{im}(d_2) \arrow[r, "\bar{d}_1"] \arrow[d, "\bar{\alpha}_1"'] & P_0 \arrow[d, "j"] \arrow[ddr, bend left, "\varepsilon_M"] & \\ N \arrow[r, "i"] \arrow[drr, bend right, "0"'] & X \arrow[dr, dotted, "\pi"] & \\ & & M. \end{tikzcd}$$

We show that $0 \longrightarrow N \xrightarrow{i} X \xrightarrow{\pi} M \longrightarrow 0$ is an extension. You know that $X = N \oplus P_0/U$, so if $i(n) = [(n,0)] = [(0,0)]$, then this is exactly the case if $n = \bar{\alpha}_1(y)$ and $\bar{d}_1(y) = 0$, so some $y \in P_1/\mathsf{im}(d_2)$. But $\bar{d}_1$ is a monomorphism, so $y = 0$ and hence $n = 0$. Therefore i is a monomorphism.

It is clear from the construction of the extension that $\mathsf{im}(i) \subset \ker(\pi)$. Denote by φ the isomorphism $\varphi\colon \ker(\varepsilon_M) \cong P_1/\mathsf{im}(d_2)$.

Assume that $\pi[(n,p)] = 0$. Hence $\varepsilon_M(p) = 0$, so

$$p \in \ker(\varepsilon_M) \cong P_1/\mathsf{im}(d_2).$$

Then

$$i(n - \bar{\alpha}_1(\varphi(p)) = [(n - \bar{\alpha}_1(p), 0)] = [(n,p)],$$

hence, $[(n,p)] \in \mathsf{im}(i)$.

Last but not least we claim that π is an epimorphism, but as ε_M is an epimorphism, we find for every $m \in M$ a $p \in P_0$ with $\varepsilon_M(p) = m$. But then

$$\pi[(0,p)] = \varepsilon_M(p) = m.$$

You show that ϕ is well-defined. It is then not hard to see that ϕ is actually the inverse of ψ.

□

CHAPTER III

Homology of groups

Lecture 14

III.1. Definition of group homology

Let G be a group and let M be a $\mathbb{Z}[G]$-module. Then, often, M is called a G-module.

Examples III.1.1.

- Let M be an arbitrary abelian group. Then we can consider the *trivial G-action* on M, *i.e.*, $g.m = m$ for all $g \in G$ and all $m \in M$. We can rewrite this and say that the $\mathbb{Z}[G]$-module structure on M factors through the $\mathbb{Z}$-module structure on M via the augmentation map
$$\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}, \quad \varepsilon(g) = 1 \text{ for all } g \in G.$$
- The ring $\mathbb{Z}[G]$ is of course a $\mathbb{Z}[G]$-module.
- If X is a G-set, then the free abelian group with basis X, $\bigoplus_{x\in X} \mathbb{Z}$, is a G-module.

Definition III.1.2. Let M be a G-module.

(a) Let U be the sub $\mathbb{Z}[G]$-module of M generated by elements of the form $m - gm$ for $m \in M$ and $g \in G$. Then
$$M_G := M/U$$
are the *G-coinvariants of M*.

(b) The *G-invariants of M* are
$$M^G := \{m \in M, gm = m \text{ for all } g \in G\}.$$

Remark III.1.3. One can characterize M^G as the largest submodule of M on which G acts trivially and M_G is the largest quotient on which G acts trivially.

The following result gives us descriptions of invariants and coinvariants in terms of Hom and tensor functors.

Lemma III.1.4. *Let M be a G-module and denote by $\mathbb{Z}$ the trivial $\mathbb{Z}[G]$-module. Then*

(a) $M^G \cong \mathsf{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ *and*

(b) $M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$.

Proof. In (a) an $m \in M^G$ corresponds to $f_m\colon \mathbb{Z} \to M$ with $f_m(1) = m$. As $m \in M^G$ we get that $f_m(g.1) = f_m(1) = m$ and this is equal to $g.m = g.f_m(1)$ so f_m is $\mathbb{Z}[G]$-linear.

For (b) we consider the map
$$\varphi\colon M_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} M, \quad [m] \mapsto 1 \otimes m.$$
Then φ is well-defined, because
$$1 \otimes (m - gm) = 1 \otimes m - 1 \otimes gm = 1 \otimes m - 1.g \otimes m = 1 \otimes m - 1 \otimes m = 0.$$
We also define $\psi\colon \mathbb{Z} \otimes_{\mathbb{Z}[G]} M \to M_G$ as $\psi(x \otimes m) = [xm]$. Then $\psi = \varphi^{-1}$. □

Remark III.1.5. We therefore know that the functor
$$M \mapsto M^G \cong \mathsf{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

is additive and left-exact and that the functor

$$M \mapsto M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$$

is additive and right-exact, so we can feed them into the machinery of derived functors.

Definition III.1.6. Let G be a group and let M be a G-module.

(a) We define the *ith homology of the group G with coefficients in M* for $i \geqslant 0$ as

$$H_i(G; M) := \mathsf{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, M) = L_i(M \mapsto M_G).$$

(b) The *ith cohomology of the group G with coefficients in M* for $i \geqslant 0$ is

$$H^i(G; M) := \mathsf{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M) = R^i(M \mapsto M^G).$$

(c) If $M = \mathbb{Z}$, then we abbreviate $H_i(G; \mathbb{Z})$ by $H_i(G)$ and $H^i(G; \mathbb{Z})$ by $H^i(G)$.

Example III.1.7. We can calculate the homology and cohomology of every finite cyclic group:

Let $C_n = \langle t \rangle$ be a cyclic group of order n for $2 \leqslant n < \infty$. You have established that the sequence

$$\ldots \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{1-t} \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{1-t} \mathbb{Z}[C_n]$$

is a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[C_n]$-module and we have already calculated the homology groups of C_n with coefficients in $\mathbb{Z}$ earlier as

$$H_i(C_n) = \mathsf{Tor}_i^{\mathbb{Z}[C_n]}(\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z} = \mathbb{Z}_{C_n}, & i = 0, \\ \mathbb{Z}/n\mathbb{Z}, & i \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Similarly, we obtain

$$\mathsf{Ext}^i_{\mathbb{Z}[C_n]}(\mathbb{Z}, \mathbb{Z}) = H^i(C_n)$$

using this resolution. Note that $\mathsf{Hom}_{\mathbb{Z}[C_n]}(\mathbb{Z}[C_n], \mathbb{Z}) \cong \mathbb{Z}$, so we obtain that $H^i(C_n)$ is the ith cohomology of the cochain complex

$$(\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \ldots)$$

and this is

$$H^i(C_n) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}/n\mathbb{Z}, & i \text{ even}, \ i > 0, \\ 0, & \text{otherwise}. \end{cases}$$

For coefficients in an arbitrary G-module M we consider the elements $1 - t$ and $N := \sum_{i=0}^{n-1} t^i$ in $\mathbb{Z}[C_n]$. By definition

$$H_i(C_n; M) = H_i(\ \ldots \xrightarrow{(1-t)\otimes M} \mathbb{Z}[C_n] \otimes_{\mathbb{Z}[C_n]} M \xrightarrow{N \otimes M} \mathbb{Z}[C_n] \otimes_{\mathbb{Z}[C_n]} M \xrightarrow{(1-t)\otimes M} \mathbb{Z}[C_n] \otimes_{\mathbb{Z}[C_n]} M)$$

and this is isomorphic to

$$H_i(\ \ldots \xrightarrow{(1-t)} M \xrightarrow{N} M \xrightarrow{(1-t)} M).$$

Thus we obtain

$$H_i(C_n; M) \cong \begin{cases} M_{C_n} = M/\mathsf{im}(1-t), & i = 0, \\ \ker(1-t)/\mathsf{im}(N), & i \text{ odd}, \\ \ker(N)/\mathsf{im}(1-t), & i \text{ even}, \ i > 0. \end{cases}$$

As

$$(1-t)N = 0 = N(1-t),$$

we obtain an induced map $\bar{N} \colon M_{C_n} = M/\mathsf{im}(1-t) \to M^{C_n} = \ker(1-t)$. We can rewrite the above result as

$$H_i(C_n; M) \cong \begin{cases} M_{C_n} = M/\mathsf{im}(1-t), & i = 0, \\ \operatorname{coker}(\bar{N}), & i \text{ odd}, \\ \ker(\bar{N}), & i \text{ even}, \ i > 0. \end{cases}$$

For cohomology we obtain with the dual calculation

$$H^i(C_n; M) \cong \begin{cases} M^{C_n} = \ker(1-t), & i = 0, \\ \ker(\bar{N}), & i \text{ odd} \\ \operatorname{coker}(\bar{N}), & i \text{ even}, \ i > 0. \end{cases}$$

Remark III.1.8. For a fixed group and an $f \in \mathsf{Hom}_{\mathbb{Z}[G]}(M, N)$ there are induced morphisms

$$H_i(G; f)\colon H_i(G; M) \to H_i(G; N) \text{ and } H^i(G; f)\colon H^i(G; M) \to H^i(G; N).$$

Definition III.1.9. We consider the category of pairs (G, M) where G is a group and M is a $\mathbb{Z}[G]$-module. A morphism $(\alpha, f)\colon (G, M) \to (G', N)$ consists of a group homomorphism $\alpha \in \mathsf{Gr}(G, G')$ and $f \in \mathsf{Ab}(M, N)$ such that

$$f(gm) = \alpha(g) f(m)$$

for all $g \in G$ and $m \in M$.

Lemma III.1.10. *A morphism (α, f) as above induces a morphism*

$$H_i(\alpha, f)\colon H_i(G; M) \to H_i(G'; N).$$

Proof. Let P_* be a projective resolution of M as a $\mathbb{Z}[G]$-module and let Q_* be a projective resolution of N as a $\mathbb{Z}[G']$-module. We can view every Q_i as a $\mathbb{Z}[G]$-module by defining

$$g.q := \alpha(g).q$$

for $g \in G$ and $q \in Q_i$. We call the abelian group Q_i with this G-module structure $\alpha^*(Q_i)$ and therefore we obtain a chain complex $\alpha^*(Q_*)$. Note that $\alpha^*(Q_i)$ is not necessarily projective as a $\mathbb{Z}[G]$-module, but as we didn't change the underlying abelian group, we still get

$$H_*(\alpha^*(Q_*)) \cong \begin{cases} 0, & * > 0, \\ N, & * = 0. \end{cases}.$$

By the fundamental lemma we obtain a chain map $f_*\colon P_* \to \alpha^*(Q_*)$ extending f:

$$\begin{tikzcd} \dots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\varepsilon_M"] \arrow[d, "f_1"] & M \arrow[d, "f"] \\ \dots \arrow[r, "d_2"] & \alpha^*(Q_1) \arrow[r, "d_1"] & \alpha^*(Q_0) \arrow[r, "\varepsilon_N"] & N. \end{tikzcd}$$

Note that there is a morphism $\mathbb{Z} \otimes_{\mathbb{Z}[G]} \alpha^*(Q_i) \to \mathbb{Z} \otimes_{\mathbb{Z}[G']} Q_i$, so in total we obtain a morphism from $\mathbb{Z} \otimes_{\mathbb{Z}[G]} P_*$ to $\mathbb{Z} \otimes_{\mathbb{Z}[G']} Q_*$. □

Example III.1.11. If $G' = \{e\}$ is the trivial group, then N is just an abelian group and any $\alpha\colon G \to \{e\}$ is constant. The condition on f in this case is

$$f(gm) = \alpha(g) f(m) = e f(m) = f(m),$$

so f is constant on G-orbits. The induced map on homology $H_*(G; M) \to H_*(\{e\}; N)$ has as target $H_*(\{e\}; N)$ which is N for $* = 0$ and which is trivial for $* > 0$. So the induced map has to be trivial in positive degrees and is the map $M_G \to N$ induced by f in degree zero.

Example III.1.12. In the other extreme case when $G = \{e\}$ any $\alpha\colon \{e\} \to G'$ sends e to $e_{G'}$. Here, M is just an abelian group and f is just a homomorphism of abelian groups $f\colon M \to N$. Now, the source has trivial homology groups except in degree zero and there we obtain the map $M \to N_{G'}$ which is the composite $M \xrightarrow{f} N \longrightarrow N_{G'}$. Here, the second map is the canonical projection $N \to N_{G'}$.

Of course, if the groups involved are non-trivial, then maps as in Lemma III.1.10 can be highly non-trivial. We will see examples later in Section III.3.

Lecture 15

III.2. Bar resolution and homology groups in low degrees

By the very definition of group (co)homology we have that $H_0(G;M) = M_G$ and $H^0(G;M) = M^G$. Our aim is to get explicit descriptions of H_1, H^1 and H^2.

Definition III.2.1. Let R be a ring, let M be a right R-module and N be a left R-module. The *two-sided bar construction of M and N over R* is the chain complex $B_*(M,R,N)$ whose degree p part is

$$B_p(M,R,N) := M \otimes R^{\otimes p} \otimes N.$$

Here, the unadorned tensor products are over $\mathbb{Z}$. We define a boundary operator by defining face maps

$$d_i\colon B_p(M,R,N) \to B_{p-1}(M,R,N),$$

$$d_i(m \otimes r_1 \otimes \ldots \otimes r_p \otimes n) := \begin{cases} mr_1 \otimes r_2 \otimes \ldots \otimes r_p \otimes n), & i = 0, \\ m \otimes r_1 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_p \otimes n, & 0 < i < p, \\ m \otimes r_1 \otimes \ldots \otimes r_{p-1} \otimes r_p n, & i = p. \end{cases}$$

We let $d\colon B_p(M,R,N) \to B_{p-1}(M,R,N)$ be

$$d = \sum_{i=0}^{p} (-1)^i d_i.$$

Remark III.2.2. The above definition ensures that $d^2 = 0$, because $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$.

Traditionally elements in $B_p(M,R,N)$ are denoted by $m[r_1 \mid \ldots \mid r_p]n$ or $[m \mid r_1 \mid \ldots \mid r_p \mid n]$ and the bars $\mid$ give the construction its name.

For $M = R$, $B_*(R,R,N)$ is a chain complex of R-modules by setting

$$r.(r_0 \otimes r_1 \otimes \ldots \otimes r_p \otimes n) := (rr_0) \otimes r_1 \otimes \ldots \otimes r_p \otimes n.$$

Lemma III.2.3. *For all R-modules N, the complex $B_*(R,R,N)$ is a resolution of N. If R and N are free as abelian groups, then $\varepsilon_N\colon B(R,R,N) \to N$ is a free resolution of N as an R-module.*

Proof. For the second claim note that the tensor product of free abelian groups is free abelian and

$$R \otimes_{\mathbb{Z}} \bigoplus_{i \in I} \mathbb{Z} \cong \bigoplus_{i \in I} R.$$

For the first claim we construct a chain homotopy and set

$$H_p\colon B_p(R,R,N) \to B_{p+1}(R,R,N), \quad H_p(r_0 \otimes r_1 \otimes \ldots \otimes r_p \otimes n) := 1 \otimes r_0 \otimes r_1 \otimes \ldots \otimes r_p \otimes n.$$

Then we get $dH_p + H_{p-1}d = \mathrm{id}$. □

We apply the above result in the case where $R = \mathbb{Z}[G]$ and $N = \mathbb{Z}$ and abbreviate $B_*(\mathbb{Z}[G], \mathbb{Z}[G], \mathbb{Z})$ by $B_*(G)$. Here, the notation $[g_0 \mid g_1 \mid \ldots \mid g_p]$ for $g_0 \otimes g_1 \otimes \ldots \otimes g_p \otimes 1$ is common.

We obtain the following description of $H_1(G)$.

Proposition III.2.4. *For every group G*

$$H_1(G) \cong G/[G,G].$$

Proof. In low degrees the bar construction $B_*(G)$ looks as follows

$$\ldots B_2(G) \cong \mathbb{Z}[G]^{\otimes 3} \xrightarrow{d} B_1(G) \cong \mathbb{Z}[G]^{\otimes 2} \xrightarrow{d} B_0(G) = \mathbb{Z}[G].$$

Note that $\mathbb{Z}[G]^{\otimes m} \cong \mathbb{Z}[G^m]$.

For a $[g_0 \mid g_1] \in B_1(G)$ the boundary is

$$d[g_0 \mid g_1] = g_0 g_1 - g_0$$

and for a $[g_0 \mid g_1 \mid g_2] \in B_2(G)$ we obtain

$$d[g_0 \mid g_1 \mid g_2] = [g_0 g_1 \mid g_2] - [g_0 \mid g_1 g_2] + [g_0 \mid g_1].$$

Applying $\mathbb{Z} \otimes_{\mathbb{Z}[G]} (-)$ yields the chain complex

$$\dots \mathbb{Z}[G \times G] \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z}$$

and the boundary maps are

$$(g_1, g_2) \mapsto g_2 - g_1 g_2 + g_1, \quad g_1 \mapsto 0$$

and therefore

$$H_1(G) \cong \mathbb{Z}[G]/(g_2 - g_1 g_2 + g_1, g_1, g_2 \in G) =: \mathbb{Z}[G]/U.$$

We define $\varphi\colon G \to H_1(G)$ as $\varphi(g) = g + U$. As $H_1(G)$ is an abelian group, φ factors through the abelianization

$$\bar{\varphi}\colon G/[G,G] \to H_1(G)$$

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr] & & H_1(G) \\ & G/[G,G] \arrow[ur, dotted, "\bar{\varphi}"'] & \end{tikzcd}$$

Then $\bar{\varphi}$ is an isomorphism with inverse $\psi\colon H_1(G) \to G/[G,G]$, $\psi(g + U) := g + [G,G]$. $\square$

Proposition III.2.5. *For every group G and every trivial G-module M*

$$H^1(G; M) \cong \mathsf{Gr}(G, M).$$

Note that as M is abelian, $\mathsf{Gr}(G, M)$ is $\mathsf{Hom}(G/[G,G], M)$.

Proof. We know that $H^1(G; M)$ is the first homology group of the cochain complex $\mathsf{Hom}_{\mathbb{Z}[G]}(B_*(G), M)$ and this is the first cohomology group of

$$\mathsf{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], M) \longrightarrow \mathsf{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G \times G], M) \longrightarrow \mathsf{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^3], M) \longrightarrow \dots$$

We can identify this cochain complex with

$$M \longrightarrow \mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M) \longrightarrow \mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}[G^2], M) \longrightarrow \dots$$

where the coboundary map sends an $m \in M$ to $\delta(m)(g) = gm - m = m - m = 0$. An $\tilde{f} \in \mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M)$ corresponds to a map of sets $f\colon G \to M$. Here, δ sends f to $\delta(f)(g_1, g_2) = f(g_1) - f(g_1 g_2) + f(g_2)$. So f is a cocycle if and only if

$$f(g_1 g_2) = f(g_1) + f(g_2)$$

and this is equivalent to f being a homomorphism. $\square$

Next we want to identify $H^2(G; M)$ with a suitable set of equivalence classes of extensions. Here, we consider extensions of the form

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1. \tag{III.2.1}$$

So, G is a given group, M is the given G-module viewed as an abelian group and the sequence (III.2.1) is a short exact sequence in the category Gr. The sequence is called *split* if there is a $\sigma \in \mathsf{Gr}(G, E)$ with $\pi \circ \sigma = \mathrm{id}_G$. You know that in this case $E \cong M \rtimes G$.

Lemma III.2.6. *In any extension as in* (III.2.1) *the group G acts on M by conjugation in E.*

Proof. As i is a monomorphism, we can identify M with its image $\mathsf{im}(i) < E$ and as $\mathsf{im}(i) = \ker(\pi)$ we also know that $\mathsf{im}(i) \lhd E$. The group E acts on itself by conjugation, and as M is abelian, $\mathsf{im}(i)$ acts on itself trivially by conjugation, so the conjugation action by E on $\mathsf{im}(i)$ factors through

$$E/\mathsf{im}(i) \times \mathsf{im}(i) \to \mathsf{im}(i).$$

As $E/\mathsf{im}(i) \cong G$, the claim follows. $\square$

We can now state the main result of this section. The proof will take a while...

Theorem III.2.7. *Let G be a group and let M be a G-module. There is a bijection between $H^2(G; M)$ and equivalence classes of extensions as in* (III.2.1) *such that the induced G-action on M coincides with the given G-module structure.*

Here, we consider the analogous equivalence relation on extensions as in Definition II.4.1: Two extensions $0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ and $0 \longrightarrow M \xrightarrow{i'} E' \xrightarrow{\pi'} G \longrightarrow 1$ are *equivalent* if there is an isomorphism of group $\phi\colon E \to E'$ such that the diagram

$$\begin{tikzcd}
 & & E \arrow[dd, "\phi"] \arrow[dr, "\pi"] & & \\
0 \arrow[r] & M \arrow[ur, "i"] \arrow[dr, "i'"'] & & G \arrow[r] & 1 \\
 & & E' \arrow[ur, "\pi'"'] & &
\end{tikzcd}$$

commutes.

For every extension $0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 1$ we can find a set-level section $s\colon G \to E$ with $\pi \circ s = \mathrm{id}_G$ and we can also choose s such that $s(1_G) = 1_E$. But of course this section won't be a group homomorphism in general. We measure its deviation from being one as follows:

Lemma III.2.8. *The function*

$$[-,-]\colon G \times G \to E, \quad (g_1, g_2) \mapsto s(g_1)s(g_2)(s(g_1g_2))^{-1}$$

has values in $\mathsf{im}(i) \cong M$.

Proof. As π is a group homomorphism and as $\pi \circ s = \mathrm{id}_G$, we obtain

$$\pi(s(g_1)s(g_2)(s(g_1g_2))^{-1}) = g_1g_2(g_1g_2)^{-1} = 1_G.$$

$\square$

The notation $[-,-]$ goes back to Otto Schreier, 1901–1929.

Definition III.2.9. The function $[-,-]$ is called the *factor set*.

Beware that $[-,-]$ actually depends on the extension and a choice of s. This is suppressed in the notation.

Lemma III.2.10. *If two extensions* $0 \longrightarrow M \xrightarrow{i_j} E_j \xrightarrow{\pi_j} G \longrightarrow 1$ $(j = 1, 2)$ *with chosen set level sections* s_j *yield the same factor set, then the extensions are equivalent.*

Proof. Both sections give bijections of sets $E_1 \cong G \times M \cong E_2$ where we send $e \in E_j$ to $(\pi_j(e), e(s_j(\pi_j(e))^{-1}))$ respectively (g, m) to $i_j(m)s_j(g)$. Fixing the bijection defines a group structure on $G \times M$ by demanding that the bijection be an isomorphism. This yields the relations

(i) $(1, m)(1, n) = (1, m + n)$
(ii) $(1, m)(g, 0) = (g, m)$
(iii) $(g, 0)(1, m) = (g, gm)$

for $1, g \in G$, $0, m, n \in M$.

We prove (i) and leave the remaining relations as an exercise: The bijection above sends $((1, m), (1, n))$ to $(i_j(m)s_j(1), i_j(n)s_j(1))$. We have $s_j(1) = 1$, so if we multiply the values we obtain $i_j(m)i_j(n) = i_j(m+n)$ and this has as a unique preimage under the bijection the element $(1, m + n)$.

So the above three relations give a group structure that is independent of the chosen section. We have to know what $(g_1, 0)(g_2, 0)$ is in order to understand what the group structure is on E_1 and E_2.

The bijection sends $((g_1, 0), (g_2, 0))$ to

$$(i_j(0)s_j(g_1), i_j(0)s_j(g_2)) = (s_j(g_1), s_j(g_2))$$

and this is sent back to the element

$$(g_1g_2, s_j(g_1)s_j(g_2)(s_j(g_1g_2)^{-1})) = (g_1g_2, [g_1, g_2]_{s_j}).$$

By assumption $[g_1, g_2]_{s_1} = [g_1, g_2]_{s_2}$, so the induced group structure on $G \times M$ agrees and we obtain the desired equivalence of extensions. $\square$

Remark III.2.11. The proof above gives the explicit group structure for given E and chosen s on $G \times M$ as $(g_1, m_1)(g_2, m_2) = (g_1g_2, m_1 + g_1m_2 + [g_1, g_2])$ because we can unravel this as

$$\begin{aligned}(g_1, m_1)(g_2, m_2) &= (1, m_1)(g_1, 0)(1, m_2)(g_2, 0) \quad \text{by (ii)}\\ &= (1, m_1)(g_1, g_1m_2)(g_2, 0), \quad \text{by (iii)}\\ &= (1, m_1)(1, g_1m_2)(g_1, 0)(g_2, 0), \quad \text{by (ii)}\\ &= (1, m_1 + g_1m_2)(g_1g_2, [g_1, g_2]), \quad \text{by (i) and the argument above}\\ &= (1, m_1 + g_1m_2)(1, [g_1, g_2])(g_1g_2, 0), \quad \text{by (ii)}\\ &= (g_1g_2, m_1 + g_1m_2 + [g_1, g_2]), \quad \text{by (i) and (ii).}\end{aligned}$$

If s happens to be a group homomorphism, then $[g_1, g_2] = 1$ for all $g_i \in G$ and then $E \cong M \rtimes G$, so that the extension splits. In total we obtain that the extension splits if and only if $[-,-] \equiv 1$.

Proposition III.2.12. *Let M be a G-module. A function $[-,-]\colon G \times G \to M$ is a factor set if and only if it satisfies*

(a) $[1, g] = [g, 1] = 0$ *for all* $g \in G$.
(b) *For all* $g_1, g_2, g_3 \in G$*:*

$$g_1[g_2, g_3] - [g_1g_2, g_3] + [g_1, g_2g_3] - [g_1, g_2] = 0.$$

Functions that satisfy the two conditions above are called normalized 2-cocycles. Condition (b) just says that $[-,-] \in Z^2(G; M)$ and (a) is a normalization condition.

Proof. If we assume that $[-,-]$ is a factor set for a section s, then as $s(1) = 1$ we get

$$[1, g] = s(1)s(g)s(1 \cdot g)^{-1} = 1.$$

As we write M as an additive group, this gives (a). As the composition in E is associative, we get in particular that

$$((g_1, 0)(g_2, 0))(g_3, 0) = (g_1, 0)((g_2, 0)(g_3, 0))$$

and this gives the condition

$$(g_1g_2g_3, [g_1, g_2] + 0 + [g_1g_2, g_3]) = (g_1g_2g_3, 0 + g_1[g_2, g_3] + [g_1, g_2g_3])$$

and thus (b) holds as well.

Assume now that E is $G \times M$ as a set with multiplication

$$(g_1, m_1)(g_2, m_2) = (g_1g_2, m_1 + g_1m_2 + [g_1, g_2]).$$

Then we obtain that

$$(g, m)(1, 0) = (g, m) = (1, 0)(g, m),$$

so $(1, 0)$ is a neutral element. We calculate that for an arbitrary (g, m)

$$\begin{aligned}(g, m)(g^{-1}, -g^{-1}m - g^{-1}[g, g^{-1}]) &= (1, m - gg^{-1}m - gg^{-1}[g, g^{-1}] + [g, g^{-1}])\\ &= (1, 0),\end{aligned}$$

so we have an inverse for every (g, m). You check that the product is associative.

Thus E is a group and we can embed M into E as $\{1\} \times M$. Then $\{1\} \times M$ is normal in E and $E/\{1\} \times M \cong G$, so

$$0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1$$

is an extension. $\square$

Lemma III.2.13. *Two choices of s, s' make the two corresponding factor sets $[-,-]_s$ and $[-,-]_{s'}$ differ by a coboundary.*

Proof. We identify M again with $i(M)$. Two sections s, s' of π with $s(1) = 1 = s'(1)$ result in the same coset

$$s(g)i(M) = s'(g)i(M)$$

for all $g \in G$ because $\mathsf{im}(i) = \ker(\pi)$. So for every $g \in G$ we find an $m = m_g$ such that $s(g) = i(m_g)s'(g)$. This defines a function

$$\beta \colon G \to M, \quad g \mapsto m_g.$$

A calculation now gives that $[g,h]_{s'} = [g,h]_s + (\delta\beta)(g,h)$:

$$\begin{aligned}
[g,h]_{s'} &= s'(g)s'(h)((s'(gh))^{-1}) \\
&= \beta(g)s(g)\beta(h)s(h)((\beta(gh)s(gh))^{-1}) \\
&= \beta(g)s(g)\beta(h)s(g)^{-1}s(g)s(h)s(gh)^{-1}\beta(gh)^{-1}.
\end{aligned}$$

Here, $\beta(g)$, $s(g)\beta(h)s(g)^{-1}$, $s(g)s(h)s(gh)^{-1}$ and $\beta(gh)^{-1}$ are elements of the abelian group M and we can therefore express the above term as

$$\beta(g) + s(g)\beta(h)s(g)^{-1} + [g,h]_s - \beta(gh).$$

The G-action on an element $\beta(h)$ is given by conjugation, hence

$$s(g)\beta(h)s(g)^{-1} = g.\beta(h)$$

and in total we obtain

$$[g,h]_s + g.\beta(h) - \beta(gh) + \beta(g) = [g,h]_s + \delta(\beta)(g,h).$$

$\square$

The normalization condition does not change the cohomology (see [**12**, Theorem 8.3.8]) and so we obtain the claimed bijection between $H^2(G;M)$ and the set of equivalence classes of extensions.

III.3. Shapiro's Lemma and transfer

We now turn to the relationship between the (co)homology of a group G and the (co)homology of its subgroups $H < G$. Again, we need some background from category theory.

Definition III.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *adjunction between $\mathcal{C}$ and $\mathcal{D}$* is a pair of functors $L\colon \mathcal{C} \to \mathcal{D}$, $R\colon \mathcal{D} \to \mathcal{C}$, such that for each pair of objects C of $\mathcal{C}$ and D of $\mathcal{D}$, there is a bijection of sets

(III.3.1) $$\varphi_{C,D}\colon \mathcal{D}(L(C), D) \cong \mathcal{C}(C, R(D)),$$

which is natural in C and D.

The functor L is then *left adjoint to* R, and R is *right adjoint to* L. We call (L,R) an *adjoint pair of functors*.

The naturality condition on the bijections $\varphi_{C,D}$ can be spelled out explicitly as follows: For all morphisms $f\colon C_1 \to C_2$ in $\mathcal{C}$ and $g\colon D_1 \to D_2$ in $\mathcal{D}$, the diagram

$$\begin{tikzcd}
\mathcal{D}(L(C_2), D_1) \arrow[r, "\mathcal{D}(Lf, D_1)"] \arrow[d, "\varphi_{C_2,D_1}"'] & \mathcal{D}(L(C_1), D_1) \arrow[r, "\mathcal{D}(L(C_1), g)"] \arrow[d, "\varphi_{C_1,D_1}"] & \mathcal{D}(L(C_1), D_2) \arrow[d, "\varphi_{C_1,D_2}"] \\
\mathcal{C}(C_2, R(D_1)) \arrow[r, "\mathcal{C}(f, R(D_1))"] & \mathcal{C}(C_1, R(D_1)) \arrow[r, "\mathcal{C}(C, R(g))"] & \mathcal{C}(C_1, R(D_2))
\end{tikzcd}$$

commutes.

One often denotes adjunctions as $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ or as $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$.

Remark III.3.2. For every object C of $\mathcal{C}$ and D of $\mathcal{D}$ there are morphisms $\eta_C \in \mathcal{C}(C, RL(C))$ and $\varepsilon_D \in \mathcal{D}(LR(D), D)$ that correspond to id_{LC} and id_{RD} under the above bijection φ. One calls η the *unit of the adjunction* and ε the *counit of the adjunction.*

Definition III.3.3. Let F, G be two functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation* τ from F to G consists of a class of morphisms $\tau_C \in \mathcal{D}(F(C), G(C))$, the *components of* τ, such that for every morphism $f \in \mathcal{C}(C_1, C_2)$,

$$\tau_{C_2} \circ F(f) = G(f) \circ \tau_{C_1},$$

that is, the diagram

$$\begin{tikzcd} F(C_1) \arrow[r, "\tau_{C_1}"] \arrow[d, "F(f)"'] & G(C_1) \arrow[d, "G(f)"] \\ F(C_2) \arrow[r, "\tau_{C_2}"] & G(C_2) \end{tikzcd}$$

commutes.

For an adjunction (L, R) the morphisms η_C are the components of a natural transformation $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$ and the ε_Ds are the components of $\varepsilon\colon LR \Rightarrow \mathrm{Id}_{\mathcal{D}}$.

Examples III.3.4.

- Let U be the functor that maps an abelian group A to the underlying set of A and let F be the functor that sends a set S to the free abelian group with basis S, $\bigoplus_S \mathbb{Z}$. Then F is left adjoint to U: for any function f from S to the underlying set of an abelian group A, there is a unique morphism of abelian groups from $F(S)$ to A extending f that is determined by sending the basis element of the copy of $\mathbb{Z}$ in component $s \in S$ to $f(s)$.
- Let $I\colon \mathsf{Ab} \to \mathsf{Gr}$ be the functor that sends an abelian group A to A and an $f \in \mathsf{Ab}(A, B)$ to $I(f) = f$. Then I has a left adjoint, namely the functor that sends a group G to $G/[G, G]$ because you know from your lecture course in algebra that
$$\mathsf{Gr}(G, I(A)) = \mathsf{Gr}(G, A) \cong \mathsf{Ab}(G/[G, G], A).$$
- Let R_1, R_2 be rings and let f be a ring homomorphism $f\colon R_1 \to R_2$. Then f defines a functor $f^*\colon R_2\text{-}\mathsf{mod} \to R_1\text{-}\mathsf{mod}$ by sending M to $f^*(M)$ whose underlying abelian group is the same as M, but $f^*(M)$ carries an R_1-module structure via $r_1.m := f(r_1).m$. If $\alpha\colon M_1 \to M_2$ is R_2-linear, then α induces an R_1-linear map
$$f^*(\alpha)\colon f^*(M_1) = M_1 \to M_2 = f^*(M_2)$$
with
$$f^*(\alpha)(r_1.m) = f^*(\alpha)(f(r_1).m) = \alpha(f(r_1)m) = f(r_1)\alpha(m) = f(r_1)f^*(\alpha)(m).$$
The functor f^* is often called the *restriction of scalars*.

Lemma III.3.5. *Let $f\colon R_1 \to R_2$ be a ring map. Then f^* has a left adjoint L and a right adjoint R.*

Proof. We define $L\colon R_1\text{-}\mathsf{mod} \to R_2\text{-}\mathsf{mod}$ as $L(N) := R_2 \otimes_{R_1} N$, where we view R_2 as a right R_1-module via f, thus $r_2 \otimes r_1 n = r_2 f(r_1) \otimes n$ for $r_i \in R_i$ and $n \in N$. Then $R_2 \otimes_{R_1} N$ is a left R_2-module and

$$\mathsf{Hom}_{R_2}(R_2 \otimes_{R_1} N, M) \cong \mathsf{Hom}_{R_1}(N, f^*(M)):$$

A $g \in \mathsf{Hom}_{R_1}(N, f^*(M))$ is sent to $\tilde{g}\colon R_2 \otimes_{R_1} N \to M$ with

$$\tilde{g}(r_2 \otimes n) = r_2 g(n)$$

and we map an $h\colon R_2 \otimes_{R_1} N \to M$ to $\bar{h}$ with $\bar{h}(n) = h(1 \otimes n)$. This bijection is binatural.

For the right adjoint we define R as $R(N) := \mathsf{Hom}_{R_1}(R_2, N)$ where we view R_2 as an R_1-module via f. The R_2-module structure on $R(N)$ is given by the right R_2-module structure of R_2. Then

$$\mathsf{Hom}_{R_1}(f^*(M), N) \cong \mathsf{Hom}_{R_2}(M, \mathsf{Hom}_{R_1}(R_2, N))$$

and the bijection is given by sending a $g \in \mathsf{Hom}_{R_1}(f^*(M), N)$ to the map that sends an $m \in M$ to the morphism $r_2 \mapsto g(r_2 m)$. Conversely, an $h \in \mathsf{Hom}_{R_2}(M, \mathsf{Hom}_{R_1}(R_2, N))$ is mapped to

$$m \mapsto h(m)(1_{R_2}).$$

□

Example III.3.6. Every group homomorphism $\varphi\colon G_1 \to G_2$ induces a ring homomorphism $f = \mathbb{Z}[\varphi]\colon \mathbb{Z}[G_1] \to \mathbb{Z}[G_2]$. In particular, if $H < G$, then the inclusion map $i\colon H \hookrightarrow G$ induces a ring map $i\colon \mathbb{Z}[H] \to \mathbb{Z}[G]$.

Definition III.3.7. Let G be a group and $H < G$.

(a) For any $\mathbb{Z}[H]$-module N the $\mathbb{Z}[G]$-module $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ is the *induced module* and is denoted by $\mathsf{Ind}_H^G(N)$.
(b) The $\mathbb{Z}[G]$-module $\mathsf{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], N)$ is the *coinduced module* and is denoted by $\mathsf{Coind}_H^G(N)$.
(c) The inclusion $i\colon H \hookrightarrow G$ gives rise to a ring map $i\colon \mathbb{Z}[H] \to \mathbb{Z}[G]$ and for a $\mathbb{Z}[G]$-module M, $i^*(M)$ is the *restriction* and is denoted by $\mathsf{Res}_H^G(M)$.

Remark III.3.8. So we already know that

$$\mathsf{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N, M) \cong \mathsf{Hom}_{\mathbb{Z}[H]}(N, i^*M)(= \mathsf{Hom}_{\mathbb{Z}[H]}(N, M))$$

and

$$(\mathsf{Hom}_{\mathbb{Z}[H]}(M, N) =)\mathsf{Hom}_{\mathbb{Z}[H]}(i^*M, N) \cong \mathsf{Hom}_{\mathbb{Z}[G]}(M, \mathsf{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], N)).$$

Often, $i^*(-)$ is suppressed from the notation.

Note that $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-module because the decompositon of sets $G = \bigsqcup_{G/H} H$ gives

$$\mathbb{Z}[G] \cong \bigoplus_{G/H} \mathbb{Z}[H].$$

Theorem III.3.9 (Shapiro's Lemma)**.** *Let G be a group, H a subgroup of G and N an H-module. Then*

(a) $H_*(G; \mathsf{Ind}_H^G(N)) \cong H_*(H; N)$.
(b) $H^*(G; \mathsf{Coind}_H^G(N)) \cong H^*(H; N)$.

Proof. As $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-module, every projective resolution $P_* \to \mathbb{Z}$ by right $\mathbb{Z}[G]$-modules is also a projective resolution of $\mathbb{Z}$ by right $\mathbb{Z}[H]$-modules. The claim now follows from the identifications

$$H_*(G; \mathsf{Ind}_H^G(N)) = H_*(P_* \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N)) \cong H_*(P_* \otimes_{\mathbb{Z}[H]} N)$$

and

$$\begin{aligned} H^*(G; \mathsf{Coind}_H^G(N)) &= H^*(\mathsf{Hom}_{\mathbb{Z}[G]}(P_*, \mathsf{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], N))) \\ &\cong H^*(\mathsf{Hom}_{\mathbb{Z}[G]}(P_* \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G], N)) \\ &\cong H^*(\mathsf{Hom}_{\mathbb{Z}[H]}(P_*, N)) \\ &= H^*(H; N). \end{aligned}$$

□

Lemma III.3.10. *Assume that H is a subgroup of G and that $[G : H] < \infty$. Let N be an H-module. Then there is an isomorphism of $\mathbb{Z}[G]$-modules*

$$\mathsf{Ind}_H^G(N) \cong \mathsf{Coind}_H^G(N).$$

Proof. We define $\varphi\colon N \to \mathsf{Coind}_H^G(N)$ as

$$\varphi(n)(g) := \begin{cases} gn, & g \in H, \\ 0, & \text{otherwise.} \end{cases}$$

We have to show that $\varphi(n) \in \mathsf{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], N)$, but for every $h \in H$ we have that $hg \in H$ if and only if $g \in H$, so $\varphi(n)$ is $\mathbb{Z}[H]$-linear because in this case $\varphi(n)(hg) = hgn = h\varphi(n)(g)$.

We extend φ to a $\mathbb{Z}[G]$-linear map

$$\tilde{\varphi}\colon \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N \to \mathsf{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], N)$$

by Lemma III.3.5; this gives

$$\tilde{\varphi}(g' \otimes n)(g) = g'.\varphi(n)(g).$$

Conversely, we define

$$\psi\colon \mathsf{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], N) \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N, \quad \psi(\alpha) := \sum_{g \in X} g \otimes \alpha(g^{-1}),$$

where X is a set of representatives for the left cosets of H in G.

The composite $\psi \circ \tilde{\varphi}$ is the identity because

$$\begin{aligned}\psi(\tilde{\varphi}(g \otimes n)) &= \sum_{\tilde{g} \in X} \tilde{g} \otimes \tilde{\varphi}(g \otimes n)(\tilde{g}^{-1}) \\ &= \begin{cases} \tilde{g} \otimes \tilde{g}^{-1} g n, & \tilde{g}^{-1} g \in H, \\ 0, & \text{otherwise,} \end{cases} \\ &= g \otimes n.\end{aligned}$$

Here we use that if $\tilde{g}^{-1} g = h$ for some $h \in H$, then $\tilde{g} = gh^{-1}$, so then the value is $gh^{-1} \otimes hn = g \otimes n$.

For the other composite we get

$$\tilde{\varphi}(\psi(f)) = \tilde{\varphi}\left(\sum_{g \in X} g \otimes f(g^{-1})\right)$$

and

$$\tilde{\varphi}(\psi(f))(\tilde{g}) = \sum_{g \in X} \varphi(f(g^{-1}))(\tilde{g}g). \tag{III.3.2}$$

But there is only one $g \in X$ such that $\tilde{g}g \in H$ and again if $\tilde{g}g = h \in H$, then $g = \tilde{g}^{-1}h$ and $g^{-1} = h^{-1}\tilde{g}$, so that (III.3.2) is equal to $hf(h^{-1}\tilde{g}) = f(\tilde{g})$ because f is $\mathbb{Z}[H]$-linear by assumption. $\square$

Before we present some applications of Shapiro's lemma, we use it first for defining transfer maps.

Definition III.3.11. Let H be a subgroup of G with $[G : H] < \infty$ and let M be a G-module. We denote by i the inclusion of H into G.

(a) The *homological transfer* is the map $i^!\colon H_*(G; M) \to H_*(H; \mathsf{Res}^G_H(M))$ given by the composite

$$\begin{tikzcd} H_*(G; M) \arrow[r, dotted, "i^!"] \arrow[d, "H_*(G;\eta_M)"'] & H_*(H; \mathsf{Res}^G_H(M)) \\ H_*(G; \mathsf{Coind}^G_H(\mathsf{Res}^G_H(M))) \arrow[r, "\cong"] & H_*(G; \mathsf{Ind}^G_H(\mathsf{Res}^G_H(M))) \arrow[u, "\cong"'] \end{tikzcd}$$

where the isomorphism at the bottom is the one from Lemma III.3.10 and the vertical isomorphism comes from the Shapiro Lemma.

(b) Dually, the *cohomological transfer* is the map $i_!\colon H^*(H; \mathsf{Res}^G_H(M)) \to H^*(G; M)$ that is the composite

$$\begin{tikzcd} H^*(H; \mathsf{Res}^G_H(M)) \arrow[r, dotted, "i_!"] \arrow[d, "\cong"'] & H^*(G; M) \\ H^*(G; \mathsf{Coind}^G_H(\mathsf{Res}^G_H(M))) \arrow[r, "\cong"] & H^*(G; \mathsf{Ind}^G_H(\mathsf{Res}^G_H(M))) \arrow[u, "H^*(G;\varepsilon_M)"'] \end{tikzcd}$$

where the vertical isomorphism comes from the Shapiro Lemma and the horizontal isomorphism is again the one from Lemma III.3.10.

Remark III.3.12. Often $\mathsf{Res}^G_H(M)$ is just denoted by M and then the transfer maps look like $i^!\colon H_*(G; M) \to H_*(H; M)$ and $i_!\colon H^*(H; M) \to H^*(G; M)$, but this can be confusing.

The transfer maps are also often denoted by tr^G_H.

Lemma III.3.13. *If again $[G : H] < \infty$ and if M is a G-module, then the composite*

$$H_*(G; M) \xrightarrow{i^!} H_*(H; M) \xrightarrow{H_*(i;M)} H_*(G; M)$$

is the multiplication by $[G : H]$ and so is

$$H^*(G; M) \xrightarrow{H^*(i;M)} H^*(H; M) \xrightarrow{i_!} H^*(G; M).$$

Proof. We unravel the maps that are used in the definition. The composite

$$M \xrightarrow{\eta_M} \mathsf{Coind}_H^G(\mathsf{Res}_H^G(M)) \xrightarrow[\psi]{\cong} \mathsf{Ind}_H^G(\mathsf{Res}_H^G(M)) \xrightarrow{\varepsilon_M} M$$

induces the map $H_*(i; M) \circ i^!$ on homology. So we have to understand the unit and counit of the adjunction.

The unit

$$\eta_M : M \to \mathsf{Coind}_H^G(\mathsf{Res}_H^G(M)) = \mathsf{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], \mathsf{Res}_H^G(M))$$

is adjoint to the identity map on $\mathsf{Res}_H^G(M))$ and it sends an $m \in M$ to the map that sends a $g \in G$ to the identity evaluated on gm. We call this map α_m, so $\alpha_m(g) = gm$.

The isomorphism $\psi : \mathsf{Coind}_H^G(\mathsf{Res}_H^G(M)) \to \mathsf{Ind}_H^G(\mathsf{Res}_H^G(M))$ applied to α_m gives (compare the proof of Lemma III.3.10)

$$\psi(\alpha_m) = \sum_{g \in X} g \otimes \alpha_m(g^{-1}) = \sum_{g \in X} g \otimes g^{-1}m.$$

The counit of the adjunction $\varepsilon_M : \mathsf{Ind}_H^G(\mathsf{Res}_H^G(M)) \to M$ is adjoint to the identity map on $\mathsf{Res}_H^G(M)$ and it maps a generator $g \otimes m \in \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \mathsf{Res}_H^G(M)$ to gm. Therefore the composite $\varepsilon_M \circ \psi \circ \eta_M$ is

$$m \mapsto \sum_{g \in X} g \otimes g^{-1}m \mapsto \sum_{g \in X} gg^{-1}m = \sum_{g \in X} m = [G : H]m.$$

□

In the following we state some consequences and for simplicity we state them only in the case of homology groups. There are analogous statements for cohomology.

Corollary III.3.14. *Assume that $[G : H] < \infty$ and that the multiplication by the index is an isomorphism in a G-module M. Then $H_*(G; M)$ is a direct summand in $H_*(H; M)$.*

In particular, if p is a prime with $p \nmid |G|$, then $H_*(G; \mathbb{Z}/p\mathbb{Z}) = 0$ for all $* > 0$.

Proof. For the first claim we know that the composite

$$H_*(G; M) \xrightarrow{i^!} H_*(H; M) \xrightarrow{H_*(i;M)} H_*(G; M)$$

is the multiplication by $[G : H]$, so it is an isomorphism, hence the claim follows.

If p does not divide $|G|$, then the multiplication by $|G|$ is an isomorphism, but it is also the composite

$$H_*(G; M) \xrightarrow{i^!} H_*(\{e\}; M) \xrightarrow{H_*(i;M)} H_*(G; M).$$

But the trivial group does not have any non-trivial homology groups in positive degrees. □

Example III.3.15. The group Σ_3 has order 6 and $[\Sigma_3 : C_3] = 2$, $[\Sigma_3 : C_2] = 3$. Here, C_2 is any group generated by a transposition. Then we get that $H_*(\Sigma_3; \mathbb{F}_2)$ is a direct summand of $H_*(C_2; \mathbb{F}_2)$ and $H_*(\Sigma_3; \mathbb{F}_3)$ is a direct summand of $H_*(C_3, \mathbb{F}_3)$. We also get that $H_*(\Sigma_3; \mathbb{F}_p) = 0$ for $* > 0$ and p a prime with $p \geqslant 5$.

Lemma III.3.16. *If M is any abelian group, then $H_*(G; \mathbb{Z}[G] \otimes_{\mathbb{Z}} M) = 0$ for all $* > 0$.*

Proof. Apply the Shapiro Lemma to $H = \{1\}$. Then

$$H_*(G; \mathbb{Z}[G] \otimes_{\mathbb{Z}} M) = H_*(G; \mathsf{Ind}_{\{e\}}^G(M)) \cong H_*(\{e\}; M) = 0$$

for positive $*$. □

Example III.3.17. If $K \subset L$ is a finite Galois extension with Galois group G, then $H_*(G; L) \cong H^*(G; L) = 0$ for all $* > 0$ and of course $H^0(G; L) = L^G = K$, but also $K = L_G = H_0(G; L)$. We only show the claim for homology.

One can show that there is an $a \in L$ such that $(g(a))_{g \in G}$ is a basis of L over K (see for instance Jantzen-Schwermer, Algebra, VI.3) and as a K-vector space we can identify L and $K[G]$ by sending an element $\sum_{g \in G} \lambda_g g(a)$ to $\sum_{g \in G} \lambda_g g$. This is actually an isomorphism of G-modules and therefore

$$H_*(G; L) \cong H_*(G; K[G]) \cong H_*(G; \mathbb{Z}[G] \otimes_{\mathbb{Z}} K)$$

so we get the claimed result with Lemma III.3.16.

CHAPTER IV

Hochschild homology

Lecture 18

Hochschild homology is a homology theory for associative algebras. Again, we will only cover some basics. For more background and examples see [**5, 12**].

IV.1. Definition and basic examples

You start with a commutative ring with unit $k \neq 0$, a k-algebra A, and an A-bimodule M, so M has a compatible right and left A-action: $a_1(ma_2) = (a_1m)a_2$ for all $a_i \in A$ and $m \in M$ and restricted to k this module structure is symmetric. We denote the unit of the algebra A as $\eta_A \colon k \to A$ and we denote the multiplication in A by $\mu \colon A \otimes_k A \to A$ and abbreviate $\mu(a \otimes b)$ as usual to ab.

Definition IV.1.1. The *ith Hochschild homology group of A over k with coefficients in M*, $\mathsf{HH}_i^k(A;M)$ is defined as

$$H_i(C_*^k(A;M)) := H_i(\ \ldots \xrightarrow{\ b\ } M \otimes A^{\otimes 2} \xrightarrow{\ b\ } M \otimes A \xrightarrow{\ b\ } M\ ).$$

Here, the tensor products are over k and $b = \sum_{i=0}^{n}(-1)^i d_i$, where

$$d_i(a_0 \otimes \ldots \otimes a_n) = \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_n, & \text{for } i < n \text{ and} \\ a_n a_0 \otimes \ldots \otimes a_{n-1}, & \text{for } i = n. \end{cases} \tag{IV.1.1}$$

for $a_0 \in M$ and $a_i \in A$ for $0 < i < n$.

Again, we have $d_i \circ d_j = d_{j-1} \circ d_i$ for $i < j$ and this ensures that b is a boundary operator. The Hochschild homology groups carry the structure of k-modules. You will show that this can be extended to a module structure over the center of A, $Z(A)$.

A nice way to visualize this is to draw elements in the Hochschild complex in a cyclic manner:

$$\begin{array}{ccccc} & & a_1 & & \\ a_0 & \otimes & & \otimes & \cdot \\ & & & & \vdots \\ \otimes & & & & \\ a_n & & & & \\ & \otimes & \ldots & & \end{array}$$

Then the ith face map in the Hochschild complex just multiplies the elements a_i and a_{i+1} together, where now, the indices have to be read modulo $n+1$. If we take A as an A-module, then this gives rise to the important cyclic structure on the Hochschild complex [**5**]. For $M = A$ we shorten the notation to $\mathsf{HH}_*(A)$ and use $C_*^k(A)$ for the corresponding chain complex.

Remark IV.1.2. A morphism $f \colon M \to M'$ of A-bimodules induces a chain map $C_*^k(A;f) \colon C_*^k(A;M) \to C_*^k(A;M')$ and therefore an induced map $\mathsf{HH}_*^k(A;f) \colon \mathsf{HH}_*^k(A;M) \to \mathsf{HH}_*^k(A;M')$.

A map of k-algebras $g \colon A \to B$ gives rise to a morphism $\mathrm{id}_M \otimes g^{\otimes n} \colon M \otimes A^{\otimes n} \to M \otimes B^{\otimes n}$ which is compatible with the boundary maps for every B-bimodule M, so we obtain a chain map $C_*^k(g;M)$ and an induced map

$$\mathsf{HH}_*(g;M) \colon \mathsf{HH}_*(A;M) \to \mathsf{HH}_*(B;M)$$

where we view M as an A-bimodule via g.

Definition IV.1.3. Consider $A \otimes_k A^{op}$. This is a k-algebra with unit

$$k \cong k \otimes_k k \xrightarrow{\eta_A \otimes_k \eta_{A^{op}}} A \otimes_k A^{op}$$

and multiplication

$$(a_1 \otimes a_2)(a_3 \otimes a_4) := a_1 a_3 \otimes a_4 a_2 \text{ for } a_i \in A.$$

Then $A^e := A \otimes_k A^{op}$ is the *enveloping algebra of* A.

Lemma IV.1.4. *A left A^e-module structure on a symmetric k-module M is the same as an A-bimodule structure on M.*

Proof. The correspondence is given by

$$(a_1 \otimes a_2).m = a_1 m a_2.$$

□

This is also equivalent to the right A^e-structure on M given by

$$m.(a_1 \otimes a_2) = a_2 m a_1.$$

Proposition IV.1.5.

(a) *For every k-algebra A and every A-bimodule M we have*

$$\mathsf{HH}_0^k(A; M) = M/U$$

where U is the k-submodule of M generated by elements of the form $am - ma$ for $m \in M$ and $a \in A$. In particular, for $M = A$ we obtain

$$\mathsf{HH}_0(A) = A/[A, A],$$

where $[A, A]$ is the k-submodule of A generated by $ab - ba$ for $a, b \in A$.

(b)

$$\mathsf{HH}_*^k(k) \cong \begin{cases} k, & * = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. For (a) we consider $b\colon C_1^k(A; M) = M \otimes A \to C_0^k(A; M) = M$ and get

$$b(m \otimes a) = ma - am.$$

For (b) we use that $k^{\otimes_k n+1} \cong k$ for all $n \geqslant 0$, so the Hochschild chain complex in this case is isomorphic to

$$\ldots \xrightarrow{b} k \xrightarrow{b} k \xrightarrow{b} k \xrightarrow{b} k.$$

The face maps d_i correspond to identity maps under this isomorphism, so the Hochschild complex is isomorphic to the complex

$$\ldots \xrightarrow{\mathrm{id}} k \xrightarrow{0} k \xrightarrow{\mathrm{id}} k \xrightarrow{0} k$$

and hence the homology groups are as claimed. □

Example IV.1.6. If A is not flat over k, weird things can happen. Take for instance $A = \mathbb{Z}/n\mathbb{Z}$ for some $2 \leqslant n \in \mathbb{N}$. This is a ring, hence a $\mathbb{Z}$-algebra. But as $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$, the Hochschild complex $C_*^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to

$$\ldots \xrightarrow{\mathrm{id}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{0} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{0} \mathbb{Z}/n\mathbb{Z}$$

and

$$\mathsf{HH}_*^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z}, & * = 0, \\ 0, & \text{otherwise.} \end{cases}$$

One can avoid these pathologies by considering a derived version of Hochschild homology, called *Shukla homology.*

Example IV.1.7. For every group G, every commutative ring k and every $k[G]$-bimodule we can consider the Hochschild homology of $k[G]$ with coefficients in M, $\mathsf{HH}_*^k(k[G];M)$ but we can also view M as a G-module by defining the G-action on M as

$$g.m := gmg^{-1} \text{ for } g \in G, m \in M.$$

It is common to denote M with this G-action by M^c. You will show in an exercise that

$$\mathsf{HH}_*^k(k[G];M) \cong H_*(G;M^c),$$

so in particular for $M = k$ with the trivial G-action we obtain

$$\mathsf{HH}_*^k(k[G];k) \cong H_*(G).$$

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Remark IV.1.8. If A happens to be commutative, then $\mathsf{HH}_0^k(A) \cong A$. What about $\mathsf{HH}_1^k(A;M)$ in this case? The Hochschild complex in low degrees is

$$\ldots \xrightarrow{b} M \otimes A^{\otimes 2} \xrightarrow{b} M \otimes A \xrightarrow{b} M$$

and this yields

$$\mathsf{HH}_1^k(A;M) = \frac{\ker(b\colon M \otimes A \to M)}{\mathsf{im}(b\colon M \otimes A \otimes A \to M \otimes A)}.$$

Here, $b\colon M \otimes A \to M$ is zero, if M is a symmetric A-bimodule, *i.e.*, if $ma = am$ for all $a \in A$ and $m \in M$.

In this case, $\mathsf{HH}_1^k(A;M)$ is the quotient of $M \otimes A$ by the submodule generated by all $ma_1 \otimes a_2 - m \otimes a_1a_2 + a_2m \otimes a_1$.

Definition IV.1.9. Let A be a commutative k-algebra. The A-module of *Kähler differentials of A over k*, $\Omega^1_{A|k}$ is F/U, where F is the free A-module generated by symbols da for $a \in A$ and U is the A-submodule generated by

(a) $d(\lambda a + \mu b) - \lambda da - \mu db$, for $a, b \in A$, $\lambda, \mu \in k$.
(b) $d(ab) - adb - bda$ for $a, b \in A$.

Remark IV.1.10. Property (a) is the k-linearity of $d(-)$ and property (b) says that $d(-)$ is a *derivation*, so it satisfies the Leibniz rule that you know from differentiation of real functions: $(fg)' = f'g + fg'$.

There is also an analogue of the property that constant function have trivial derivative: If λ is an element of k, then we obtain

$$d(\lambda \cdot 1_A) = \lambda d(1_A) = \lambda d(1_A \cdot 1_A) = \lambda 1_A d(1_A) + \lambda 1_A d(1_A) = 2\lambda d(1_A),$$

and hence, $d(\lambda \cdot 1_A) = 0$.

Proposition IV.1.11.

(a) *If A is a commutative k-algebra, then*

$$\mathsf{HH}_1^k(A) \cong \Omega^1_{A|k}.$$

(b) *If in addition M is a symmetric A-bimodule, then*

$$\mathsf{HH}_1^k(A;M) \cong M \otimes_A \Omega^1_{A|k}.$$

Proof. We know that for (a) the boundary map $b\colon A \otimes A \to A$ is trivial because A is commutative and $b\colon A^{\otimes 3} \to A \otimes A$ is given by

$$b(a_0 \otimes a_1 \otimes a_2) = a_0a_1 \otimes a_2 - a_0 \otimes a_1a_2 + a_2a_0 \otimes a_1.$$

We define $\varphi\colon \mathsf{HH}_1^k(A) \to \Omega^1_{A|k}$ as

$$\varphi[a_0 \otimes a_1] := a_0 d(a_1).$$

This is well-defined, because $a_0a_1 \otimes a_2 - a_0 \otimes a_1a_2 + a_2a_0 \otimes a_1$ maps to

$$a_0a_1d(a_2) - a_0d(a_1a_2) + a_2a_0d(a_1) = a_0a_1d(a_2) - a_0a_1d(a_2) - a_0a_2d(a_1) + a_2a_0d(a_1) = 0.$$

The map $\psi\colon \Omega^1_{A|k} \to \mathsf{HH}^k_1(A)$, $\psi(adb) = [a \otimes b]$ is an inverse to φ. It is well-defined, because $a \otimes b$ is a cycle, tensors are bilinear, and the relation coming from the Leibniz rule gives rise to boundaries.

In the case of coefficients in M as in (b) note that $m \otimes a$ is a cycle for all $m \in M$ and $a \in A$ because M is a symmetric A-bimodule. Therefore $\varphi[m \otimes a] := m \otimes da$ gives a map from $\mathsf{HH}_1(A; M)$ to $M \otimes_A \Omega^1_{A|k}$. It is well-defined, because φ sends $ma_1 \otimes a_2 - m \otimes a_1 a_2 + a_2 m \otimes a_1$ to zero. The inverse of φ is $\psi(m \otimes ad(b)) = [ma \otimes b]$. As $\psi(ma \otimes b) = \psi(m \otimes ad(b))$, this map is well-defined. $\square$

Example IV.1.12. Let A be $k[x]$. For $\Omega^1_{k[x]|k}$ we have to understand d of a general polynomial. But as

$$d(c_n x^n + \ldots + c_1 x + c_0) = c_n d(x^n) + \ldots + c_1 d(x)$$

we only have to understand what $d(x^i)$ is for all $i \geqslant 1$. An induction shows $d(x^i) = ix^{i-1} d(x)$, so $\Omega^1_{k[x]|k}$ is generated as a $k[x]$-module by dx and as there are no relations for dx we obtain an isomorphism of $k[x]$-modules

$$\Omega^1_{k[x]|k} \cong k[x].$$

Example IV.1.13. Let A be $\mathbb{F}_p[x]/(x^p - x)$ for a prime p. Then again $\Omega^1_{\mathbb{F}_p[x]/(x^p-x)|\mathbb{F}_p}$ is generated by dx, but in this example the relation $x^p = x$ gives

$$d(x) = d(x^p) = px^{p-1} d(x) = 0$$

and hence $\Omega^1_{\mathbb{F}_p[x]/(x^p-x)|\mathbb{F}_p} = 0$.

IV.2. Hochschild homology as a derived functor

We consider an auxiliary complex:

Definition IV.2.1. For any associative k-algebra A we consider the *bar complex of A*, $C^{bar,k}_*(A)$, with $C^{bar,k}_n(A) = A^{\otimes n+2}$ and differential $b'\colon C^{bar,k}_n(A) = A^{\otimes n+2} \to C^{bar,k}_{n-1}(A) = A^{\otimes n+1}$, $b' := \sum_{i=0}^n (-1)^i d_i$:

$$\ldots \xrightarrow{b'} A^{\otimes 4} \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A \otimes A$$

Remark IV.2.2. Note that the last face map d_{n+1} does not occur!

There is an augmentation map $\varepsilon_A\colon C^{bar,k}_*(A) \to A$ given by

$$\varepsilon_A|_{C^{bar,k}_n(A)} = \begin{cases} \mu, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma IV.2.3. *For any associative k-algebra the complex $C^{bar,k}_*(A)$ is a resolution of A as a left A^e-module.*

Proof. We define the A^e-module structure on every $C^{bar,k}_n(A)$ as

$$(a \otimes a').(a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) := aa_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} a'.$$

As b' only involves the face maps d_0 up to d_n this module structure is compatible with b', and it is also compatible with the augmentation.

We define $s_n\colon C^{bar,k}_n(A) \to C^{bar,k}_{n+1}(A)$ as

$$s_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = 1 \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}.$$

Then $d_i s_n = s_{n-1} d_{i-1}$ holds for $1 \leqslant i \leqslant n+1$ and $d_0 s_n = \mathrm{id}$. Therefore $b' s_n + s_{n-1} b' = \mathrm{id}_{C^{bar,k}_n(A)}$. $\square$

Note the similarity to the proof of Lemma III.2.3. The chain homotopy $s = (s_n)$ if often called the *extra degeneracy.*

Theorem IV.2.4. *If A is an associative k-algebra whose underlying k-module is projective, then for every A-bimodule M*

$$\mathsf{HH}^k_*(A; M) \cong \mathsf{Tor}^{A^e}_*(M, A).$$

Proof. As A is projective over k, so is $A^{\otimes n}$ for all $n \geqslant 1$. Therefore, $C_*^{bar,k}(A)$ is a projective resolution of A as a A^e-module, so

$$\mathsf{Tor}_*^{A^e}(M, A) = H_*(M \otimes_{A^e} C_*^{bar,k}(A)).$$

We can identify

$$M \otimes_{A^e} C_n^{bar,k}(A) = M \otimes_{A^e} A^{\otimes n+2} \cong M \otimes A^{\otimes n},$$

because in $M \otimes_{A^e} C_n^{bar,k}(A)$ we have that

$$m \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} = a_{n+1} m a_0 \otimes 1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1$$

and we identify the latter with

$$a_{n+1} m a_0 \otimes a_1 \otimes \ldots \otimes a_n.$$

Under this isomorphism the boundary map $\mathrm{id}_M \otimes_{A^e} b'$ is identified with b: for the face maps $d_1, \ldots, d_{n-1}$ this is obvious.

The zeroth face map $d_0(m \otimes a_1 \otimes \ldots \otimes a_n)$ corresponds to $d_0(m \otimes 1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)$ in $M \otimes_{A^e} C_n^{bar,k}(A)$ and this is $m \otimes a_1 \otimes \ldots \otimes a_n \otimes 1$ which again corresponds to $ma_1 \otimes \ldots \otimes a_n$ and $d_n(m \otimes a_1 \otimes \ldots \otimes a_n)$ corresponds to $d_n(m \otimes 1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)$ in $M \otimes_{A^e} C_n^{bar,k}(A)$ and this is $m \otimes 1 \otimes a_1 \otimes \ldots \otimes a_n$ which is identified with $a_n m \otimes a_1 \otimes \ldots \otimes a_{n-1}$. □

Remark IV.2.5. For Theorem IV.2.4 to hold, it actually suffices to assume that A is flat as a k-module [**12**, Corollary 9.1.5]. Thus if A is flat over k, then $\mathsf{HH}_*^k(A; M)$ is the derived functor of $\mathsf{HH}_0^k(A; M) = M/(am - ma, a \in A, m \in M)$. You might think of $\mathsf{HH}_0^k(A; M)$ as the symmetrization of the A-bimodule structure on M.

Example IV.2.6. Let A be $k[x]$. Then this is even free as a k-module, so

$$\mathsf{HH}_*^k(k[x]) \cong \mathsf{Tor}_*^{k[x]^e}(k[x], k[x]).$$

But as $k[x]$ is commutative, we have

$$k[x]^e = k[x] \otimes_k k[x]^{op} = k[x] \otimes_k k[x] = k[x, y].$$

The $k[x, y]$-module structure on $k[x]$ is given by

$$k[x, y] \otimes_k k[x] \xrightarrow{f \otimes \mathrm{id}} k[x] \otimes_k k[x] \xrightarrow{\mu} k[x],$$

where $f(x) = x = f(y)$. We need a projective resolution of $k[x]$ as a $k[x, y]$-module and

$$k[x, y] \xrightarrow{x-y} k[x, y]$$

does the trick: Multiplication by $x - y$ is injective and $k[x, y]/(x - y) \cong k[x]$. Therefore

$$\mathsf{HH}_*^k(k[x]) = H_*(\ldots \longrightarrow 0 \longrightarrow k[x, y] \otimes_{k[x,y]} k[x] \xrightarrow{(x-y)\otimes \mathrm{id}} k[x, y] \otimes_{k[x,y]} k[x])$$

and this is the homology of the complex

$$\ldots \longrightarrow 0 \longrightarrow k[x] \xrightarrow{0} k[x].$$

Thus we obtain

$$\mathsf{HH}_*^k(k[x]) \cong \begin{cases} k[x], & * = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

We already new that $\mathsf{HH}_0^k(k[x]) \cong k[x]$ and $\mathsf{HH}_1^k(k[x]) \cong \Omega^1_{k[x]|k} \cong k[x]$, so the calculation above says that these are the only non-trivial Hochschild homology groups.

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Example IV.2.7. The above result can be generalized to quotients of the form $A = k[x]/\phi(x)$, where $\phi(x) \in k[x]$ is any monic polynomial [**2**]. Let $k \neq 0$ be again an arbitrary commutative ring and let A be $k[x]/(\phi(x))$ where ϕ is any monic polynomial. As A is commutative, we have $A^e = A \otimes_k A$ and this is a quotient of $k[x,y]$. We consider the complex of A^e-free modules F_* of A that is free of rank one over A^e with generator x_n in every degree $n \geqslant 0$. We define its boundary as

$$d(x_n) = \begin{cases} \frac{\phi(x)\otimes 1 - 1 \otimes \phi(y)}{x \otimes 1 - 1 \otimes y} x_{n-1}, & n \text{ even}, \\ (x \otimes 1 - 1 \otimes y)x_{n-1}, & n \text{ odd}. \end{cases}$$

In [**2**] it is shown that this is actually a resolution of A. Assume that M is a symmetric A-bimodule (for instance $M = A$). Then

$$\mathsf{HH}_*^k(k[x]/(\phi(x)); M) \cong H_*(\ldots \xrightarrow{\phi'(x)} M \xrightarrow{0} M \xrightarrow{\phi'(x)} M \xrightarrow{0} \ldots \xrightarrow{0} M).$$

For instance, if $k = M = \mathbb{F}_3$ and $A = \mathbb{F}_3[x]/(x^3)$ then the boundary maps in the above complex are trivial because $(x^3)' = 3x^2 \equiv 0$ and hence

$$\mathsf{HH}_*^{\mathbb{F}_3}(\mathbb{F}_3[x]/x^3; \mathbb{F}_3) \cong \mathbb{F}_3, \quad \text{for all } * \geqslant 0.$$

IV.3. Morita invariance

You might have seen the following important definition in other contexts:

Definition IV.3.1. Let A and B be two associative k-algebras. Then A and B are *Morita equivalent*, if there is an A-B-bimodule P and a B-A-bimodule Q together with isomorphisms

$$\varphi\colon P \otimes_B Q \cong A, \text{ and } \psi\colon Q \otimes_A P \cong B,$$

where φ is an isomorphism of A-bimodules and ψ is an isomorphism of B-bimodules.

Remark IV.3.2. You know from an exercise that in the above situation P is projective as an A-module and as a right B-module and dually, that Q is projective as a B-module and a right A-module.

Example IV.3.3. For all rings $\neq 0$ the ring R is Morita equivalent to the ring of $n \times n$-matrices over R, $M_n(R)$ for every $n \geqslant 1$. Here, one can take $P = R^n = Q$, but we view P as row vectors, $(R^n)^t$. Then

$$(R^n)^t \otimes_{M_n(R)} R^n \cong R, \text{ and } R^n \otimes_R (R^n)^t \cong M_n(R).$$

Hochschild homology is Morita invariant:

Theorem IV.3.4. *If A and B are two k-algebras that are Morita equivalent and if M is an A-bimodule, then*

$$\mathsf{HH}_*^k(A; M) \cong \mathsf{HH}_*^k(B; Q \otimes_A M \otimes_A P).$$

In particular, for $M = A$:

$$\mathsf{HH}_*^k(A) \cong \mathsf{HH}_*^k(B; Q \otimes_A A \otimes_A P) \cong \mathsf{HH}_*^k(B).$$

We prove a special case for now and defer the full proof of Theorem IV.3.4 to later (see Section V.4).

Theorem IV.3.5. *Let A be an associative k-algebra and let M be an A-bimodule, then*

$$\mathsf{HH}_*^k(M_r(A); M_r(M)) \cong \mathsf{HH}_*^k(A; M) \text{ for all } r \geqslant 1.$$

Proof. We define the generalized trace map

$$tr\colon M_r(M) \otimes M_r(A)^{\otimes n} \to M \otimes A^{\otimes n}$$

as

$$tr(B^0 \otimes B^1 \otimes \ldots \otimes B^n) := \sum_{1 \leqslant i_0, \ldots, i_n \leqslant r} b^0_{i_0,i_1} \otimes \ldots \otimes b^n_{i_n,i_0}.$$

(So if you multiply tensor factors in the summands you actually get the trace of the product of the matrices.) As $d_i \circ tr = tr \circ d_i$ for all $0 \leqslant i \leqslant n$, the generalized trace map induces a chain map

$$tr\colon C_*^k(M_r(A); M_r(M)) \to C_*^k(A; M).$$

Define $i\colon A \to M_r(A)$ and $i\colon M \to M_r(M)$ as

$$(i(x))_{ij} = \begin{cases} x, & i=j=1, \\ 0, & \text{otherwise.} \end{cases} = \begin{pmatrix} x & 0 & \dots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

We extend i to a map

$$i\colon M \otimes A^{\otimes n} \to M_r(M) \otimes M_r(A)^{\otimes n}, \quad i(m \otimes a_1 \otimes \ldots \otimes a_n) = i(m) \otimes i(a_1) \otimes \ldots \otimes i(a_n).$$

Again, $d_j \circ i = i \circ d_j$ for $0 \leqslant j \leqslant n$, so i induces a chain map.

The composite $tr \circ i$ applied to an element $m \otimes a_1 \otimes \ldots \otimes a_n$ is

$$tr(i(m) \otimes i(a_1) \otimes \ldots \otimes i(a_n))$$

and this actually has a single non-trivial summand, namely the one for $i_0 = \ldots = i_n = 1$ and this returns $m \otimes a_1 \otimes \ldots \otimes a_n$, so $tr \circ i = \mathrm{id}_{C^k_*(A;M)}$.

The other composite $i \circ tr$ places the result of the generalized trace map into the upper left corner of a matrix, so this is *not* the identity map, but we show that it is homotopic to the identity map.

To this end we define for $0 \leqslant j \leqslant n$

$$h_j(B^0 \otimes \ldots \otimes B^n) := \sum_{1 \leqslant i_0, \ldots, i_{j+1} \leqslant r} E_{i_0,1}(b^0_{i_0,i_1}) \otimes E_{1,1}(b^1_{i_1,i_2}) \otimes \ldots \otimes E_{1,1}(b^j_{i_j i_{j+1}}) \otimes E_{1,i_{j+1}}(1) \otimes B^{j+1} \otimes \ldots \otimes B^n.$$

Here, $E_{ij}(x)$ is the matrix that has x in spot (i,j) and is trivial otherwise, and $i_{n+1} = i_0$ for h_n.

- We obtain $d_0 h_0 = \mathrm{id}$ because

$$\begin{aligned} d_0 h_0(B^0 \otimes \ldots \otimes B^n) &= d_0(\sum_{i_0,i_1} E_{i_0,1}(b^0_{i_0,i_1}) \otimes E_{1,i_1}(1) \otimes B^1 \otimes \ldots \otimes B^n) \\ &= \sum_{i_0,i_1} E_{i_0,i_1}(b^0_{i_0,i_1}) \otimes B^1 \otimes \ldots \otimes B^n \\ &= B^0 \otimes B^1 \otimes \ldots \otimes B^n. \end{aligned}$$

- For $d_{n+1} \circ h_n(B^0 \otimes \ldots \otimes B^n)$ we obtain

$$d_{n+1}\left(\sum_{1 \leqslant i_0, \ldots, i_n \leqslant r} E_{i_0,1}(b^0_{i_0,i_1}) \otimes E_{1,1}(b^1_{i_1,i_2}) \otimes \ldots \otimes E_{1,1}(b^n_{i_n,i_0}) \otimes E_{1,i_0}(1)\right)$$

  but as

$$E_{1,i_0}(1) E_{i_0,1}(b^0_{i_0,i_1}) = E_{11}(b^0_{i_0,i_1}) = i(b^0_{i_0,i_1})$$

  we obtain $d_{n+1} h_n = i \circ tr$.
- These maps also satisfy the identities

$$\begin{aligned} d_i h_j &= h_{j-1} d_i \text{ for } i < j, \\ d_i h_i &= d_i h_{i-1} \text{ for } 0 < i \leqslant n, \\ d_i h_j &= h_j d_{i-1} \text{ for } i > j+1. \end{aligned}$$

This ensures that the map $H := \sum_{j=0}^n (-1)^j h_j$ satisfies

$$bH + Hb = \mathrm{id} - i \circ tr,$$

to that $i \circ tr$ is chain homotopic to the identity. □

Corollary IV.3.6. *For all $r \geqslant 2$ the trace map induces an isomorphism*

$$M_r(A)/[M_r(A), M_r(A)] \to A/[A,A].$$

Proof. This follows from the case $n=0$ from above: The trace $tr\colon M_r(A)\to A$ can be prolonged to

$$M_r(A)\xrightarrow{tr}A\longrightarrow A/[A,A]=\mathsf{HH}_0^k(A)$$

and this map in turn factors through $\mathsf{HH}_0^k(M_r(A))$

$$\begin{tikzcd} M_r(A) \arrow[rr, "tr"] \arrow[d] & & A \arrow[r] & A/[A,A]=\mathsf{HH}_0^k(A) \\ \mathsf{HH}_0^k(M_r(A))=M_r(A)/[M_r(A),M_r(A)] \arrow[urrr, dotted, "\overline{tr}"'] & & & \end{tikzcd}$$

and the proof of Theorem IV.3.5 shows that $\overline{tr}$ is an isomorphism. □

IV.4. Hochschild cohomology

Definition IV.4.1. Let A be a k-algebra and let M be an A-bimodule. We set

$$C_k^*(A;M):=\mathsf{Hom}_{A^e}(C_*^{bar,k}(A),M)$$

and define the *Hochschild cohomology of A with coefficients in M* as

$$\mathsf{HH}_k^*(A;M):=H^*(C_k^*(A;M)).$$

Remark IV.4.2.

- The definition above implies directly that $\mathsf{HH}_k^n(A;M)\cong\mathsf{Ext}_{A^e}^n(A;M)$ if A is projective as a k-module.
- We can simplify the description of $C_k^*(A;M)$. If g is an A^e-linear map

$$g\colon A\otimes A^{\otimes n}\otimes A\to M$$

then we can identify it with the k-linear map $f\colon A^{\otimes n}\to M$ where $f(a_1\otimes\ldots\otimes a_n)=g(1\otimes a_1\otimes\ldots\otimes a_n\otimes 1)$. The isomorphism

$$\mathsf{Hom}_{A^e}(A\otimes A^{\otimes n}\otimes A,M)\cong\mathsf{Hom}_k(A^{\otimes n},M)$$

also yields an identification of the coboundary operator which is given on an $f\in\mathsf{Hom}_k(A^{\otimes n},M)$ as

$$\delta(f)(a_1\otimes\ldots\otimes a_{n+1})=a_1f(a_2\otimes\ldots\otimes a_{n+1})+\sum_{i=1}^n(-1)^if(a_1\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_{n+1})+(-1)^{n+1}f(a_1\otimes\ldots\otimes a_n)a_{n+1}.$$

The next result follows directly from the definition of Hochschild cohomology and the above identifications.

Proposition IV.4.3. *For any k-algebra A and any A-bimodule M we obtain*

(a)

$$\mathsf{HH}_k^0(A;M)=\{m\in M, am=ma \text{ for all } a\in A\},$$

and in particular

$$\mathsf{HH}_k^0(A)=\{a\in A, ab=ba \text{ for all } b\in A\}=Z(A),$$

where $Z(A)$ denotes the center of A.

(b) $\mathsf{HH}_k^1(A;M)$ *is the quotient of the k-module of derivations from A to M modulo the k-submodule of inner derivations. Here a k-linear map $D\colon A\to M$ is a* derivation, *if $D(ab)=aD(b)+D(a)b$ for all $a,b\in A$. A derivation D is an* inner derivation *if there is an $m\in M$ such that $D(a)=ma-am=:[m,a]$.*

Lecture 21

Remark IV.4.4.

- Check that inner derivations are actually derivations!
- Compare this notion of derivations with the one for Kähler differentials. Beware that here the Leibniz rule is different: $D(a)$ and $D(b)$ are elements of M and for $aD(b)$ we use the left A-module structure on M whereas for $D(a)b$ we use the right A-module structure.

- $\mathsf{HH}^1_k(A)$ is actually a Lie-algebra with Lie bracket
$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1,$$
where D_1, D_2 are (classes of) derivations.

As for groups, we can relate the second Hochschild cohomology groups to equivalence classes of suitable extensions. We consider short exact sequences

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} A \longrightarrow 0$$

that are split as sequences of k-modules. The objects M, E, A are algebras where M is without unit and carries the trivial multiplication: $m \cdot m' = 0$ for all $m, m' \in M$.

Let $s\colon A \to E$ be a k-linear section. Hence $E \cong A \oplus M$ as k-modules. Then M can be viewed as a submodule of E and carries an induced A-bimodule structure via

$$\nu\colon A \otimes E \otimes A \to A, \quad \nu(a \otimes e \otimes b) := s(a) \cdot e \cdot s(b).$$

As $\pi(s(a) \cdot e \cdot s(b)) = a\pi(e)b = 0$ for $e \in i(M)$, this restricts to an A-bimodule structure on M.

Proposition IV.4.5. *For every k-algebra A and every A-bimodule M there is a bijection between $\mathsf{HH}^2_k(A; M)$ and the set of equivalence classes of extensions as above where the induced A-bimodule structure on M is the given one.*

Sketch of proof. Assume that $f\colon A^{\otimes 2} \to M$ is a Hochschild cocycle. Then we define a multiplication on the k-module $E(f) := A \oplus M$ as:

$$(a_1, m_1)(a_2, m_2) := (a_1a_2, a_1m_2 + m_1a_2 + f(a_1 \otimes a_2)).$$

The cocycle condition for f ensures that this multiplication is associative.

If $f_1 = f_2 + \delta(g)$ for some $g \in \mathsf{Hom}_k(A, M)$, then the corresponding extensions are equivalent, *i.e.*, there is an isomorphism of k-algebras $\phi\colon E(f_1) \to E(f_2)$ such that

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i_1"] \arrow[d, equal] & E(f_1) \arrow[r, "\pi_1"] \arrow[d, "\phi"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "i_2"] & E(f_2) \arrow[r, "\pi_2"] & A \arrow[r] & 0
\end{tikzcd}$$

commutes. Explicitly, ϕ is given by $\phi(a, m) := (a, m + g(a))$.

Vice versa, given an extension, we define $f\colon A^{\otimes 2} \to M$ by the requirement that

$$(a_1, 0)(a_2, 0) = (a_1a_2, f(a_1 \otimes a_2)).$$

Then the fact that the multiplication is associative implies the cocycle condition. Isomorphisms of extensions are always of the form $\phi(a, m) = (a, g(a) + m)$ and this implies that equivalent extensions give rise to two cocycles that differ by a coboundary. □

IV.5. Additional structure

Hochschild (co)homology has some rich structure that is similar to features of singular (co)homology. In the following we just present a few basics.

The first feature exploits the duality between homology and cohomology.

Definition IV.5.1. Let A be a k-algebra and let M_1, M_2 be A-bimodules.

The map
$$\langle -, - \rangle\colon C^n_k(A; M_1) \otimes C^k_n(A; M_2) \to M_1 \otimes_{A^e} M_2$$
that sends an $f \in \mathsf{Hom}_k(A^{\otimes n}, M_1)$ and an $m_2 \otimes a_1 \otimes \ldots \otimes a_n$ to
$$\langle f, m_2 \otimes a_1 \otimes \ldots \otimes a_n \rangle := f(a_1 \otimes \ldots \otimes a_n) \otimes m_2$$
is the *Kronecker pairing.*

Remark IV.5.2. A direct calculation shows that

$$\langle \delta(f), m_2 \otimes a_1 \otimes \ldots \otimes a_{n+1} \rangle = \langle f, b(m_2 \otimes a_1 \otimes \ldots \otimes a_{n+1}) \rangle,$$

so that we get an induced map

$$\langle -,- \rangle \colon \mathsf{HH}^n_k(A; M_1) \otimes \mathsf{HH}_n^k(A; M_2) \to M_1 \otimes_{A^e} M_2.$$

Definition IV.5.3. Let A be a k-algebra and let $f \in C^n_k(A)$, $g \in C^m_k(A)$. The *cup-product of f and g* is the cochain $f \cup g \in C^{n+m}_k(A)$ with

$$(f \cup g)(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} \otimes \ldots \otimes a_{n+m}) := f(a_1 \otimes \ldots \otimes a_n) \cdot g(a_{n+1} \otimes \ldots \otimes a_{n+m}).$$

Lemma IV.5.4. *The cup-product satisfies a Leibniz rule:*

$$\delta(f \cup g) = \delta(f) \cup g + (-1)^n f \cup \delta(g).$$

Proof. Let us write down, what $\delta(f \cup g)$ is:

$$\begin{aligned}
\delta(f \cup g)&(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} \otimes \ldots \otimes a_{n+m+1}) \\
=&a_1 f(a_2 \otimes \ldots \otimes a_{n+1}) \cdot g(a_{n+2} \otimes \ldots \otimes a_{n+m+1}) \\
&- f(a_1 a_2 \otimes \ldots \otimes a_{n+1}) \cdot g(a_{n+2} \otimes \ldots \otimes a_{n+m+1}) \\
&\pm \ldots + (-1)^n f(a_1 \otimes \ldots \otimes a_n a_{n+1}) \cdot g(a_{n+2} \otimes \ldots \otimes a_{n+m+1}) \\
&+ (-1)^{n+1} f(a_1 \otimes \ldots \otimes a_n) g(a_{n+1} a_{n+2} \otimes \ldots \otimes a_{n+m+1}) \\
&\pm \ldots + (-1)^{n+m+1} f(a_1 \otimes \ldots \otimes a_n) g(a_{n+1} \otimes \ldots \otimes a_{n+m}) a_{n+m+1}.
\end{aligned}$$

In the sum $(\delta(f) \cup g + (-1)^n f \cup \delta(g))(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} \otimes \ldots \otimes a_{n+m+1})$ we get the additional two summands $(-1)^{n+1}(f(a_1 \otimes \ldots \otimes a_n)a_{n+1})g(a_{n+2} \otimes \ldots \otimes a_{n+m+1})$ as the last coface map of $\delta(f) \cup g$ and $(-1)^n f(a_1 \otimes \ldots \otimes a_n)(a_{n+1} g(a_{n+2} \otimes \ldots \otimes a_{n+m+1}))$ from the zeroth coface map in $(-1)^n f \cup \delta(g)$, but their sum cancels. $\square$

Definition IV.5.5. Let p and q be natural numbers. A (p,q)-*shuffle* is a $\sigma \in \Sigma_n$ with $n = p+q$ such that

$$\sigma(1) < \ldots < \sigma(p) \text{ and } \sigma(p+1) < \ldots < \sigma(p+q).$$

We denote the set of (p,q)-shuffles by $\mathsf{Sh}(p,q)$.

So think of such a shuffle as shuffeling a deck of p cards and a deck of q cards.

Example IV.5.6. For $n = 3$ the $(1,2)$-shuffles are all permutations with $\sigma(2) < \sigma(3)$, so the identity, $\sigma = (1,2)$ and $(1,3,2)$.

Remark IV.5.7. Note that $|\mathsf{Sh}(p,q)|$ is just the binomial coefficient $\binom{p+q}{p} = \binom{p+q}{q}$ because a shuffle is completely determined by the set $\{\sigma(1), \ldots, \sigma(p)\}$. So you know that there are 6 $(2,2)$-shuffles in Σ_4. It's a good exercise to draw them.

We define shuffle maps on the level of Hochschild chains:

Definition IV.5.8. Let

$$sh_{p,q} \colon C^k_p(A; M) \otimes C^k_q(A'; M') \to C^k_{p+q}(A \otimes A'; M \otimes M')$$

be defined as

$$\begin{aligned}
sh_{p,q}&((m \otimes a_1 \otimes \ldots \otimes a_p) \otimes (m' \otimes a'_{p+1} \otimes \ldots \otimes a'_{p+q})) := \\
&m \otimes m' \otimes \left(\sum_{\sigma \in \mathsf{Sh}(p,q)} \operatorname{sign}(\sigma) \sigma.(a_1 \otimes 1) \otimes \ldots \otimes (a_p \otimes 1) \otimes (1 \otimes a'_{p+1}) \otimes \ldots \otimes (1 \otimes a'_{p+q}) \right).
\end{aligned}$$

Here, $\sigma.(c_1 \otimes \ldots \otimes c_{p+q}) = c_{\sigma^{-1}(1)} \otimes \ldots \otimes c_{\sigma^{-1}(p+q)}$.

So for instance $sh_{1,2}((m\otimes a_1)\otimes(m'\otimes a_2'\otimes a_3'))$ has three summands, namely
$(m\otimes m')\otimes((a_1\otimes 1)\otimes(1\otimes a_2')\otimes(1\otimes a_3')-(1\otimes a_2')\otimes(a_1\otimes 1)\otimes(1\otimes a_3')+(1\otimes a_2')\otimes(1\otimes a_3')\otimes(a_1\otimes 1))$.

A proof of the following Proposition can be found in [**5**][ref].

Proposition IV.5.9. *For all $x\in C_p^k(A;M)$ and $y\in C_q^k(A';M')$:*

$$b(sh_{p,q}(x\otimes y))=sh_{p-1,q}(b(x)\otimes y)+(-1)^p sh_{p,q-1}(x\otimes b(y)).\quad\square$$

Remark IV.5.10. If A is a commutative k-algebra, then the multiplication $\mu\colon A\otimes_k A\to A$ is a morphism of k-algebras, so in this case we can prolong the above shuffle map with μ.

Definition IV.5.11. Let A be a commutative k-algebra, the we define

$$*\colon C_p^k(A)\otimes C_q^k(A)\to C_{p+q}^k(A)$$

as the composite $C_{p+q}^k(\mu)\circ sh_{p,q}$:

$$C_p^k(A)\otimes C_q^k(A)\xrightarrow{sh_{p,q}}C_{p+q}^k(A\otimes A)\xrightarrow{C_{p+q}^k(\mu)}C_{p+q}^k(A).$$

Corollary IV.5.12. *If A is a commutative k-algebra, then the shuffle map induces a graded commutative product on $\mathsf{HH}_*^k(A)$:*

$$*\colon \mathsf{HH}_p^k(A)\otimes\mathsf{HH}_q^k(A)\to\mathsf{HH}_{p+q}^k(A).$$

This map actually factors through the tensor product over A:

$$*\colon \mathsf{HH}_p^k(A)\otimes_A\mathsf{HH}_q^k(A)\to\mathsf{HH}_{p+q}^k(A).$$

Remark IV.5.13. Why is this graded commutative and not just commutative? Consider the permutation $\chi\in\Sigma_{p+q}$ that is given by

$$\chi(i)=\begin{cases}p+i, & 1\leqslant i\leqslant q,\\ q-i, & q+1\leqslant i\leqslant q+p.\end{cases}$$

Then χ has $\mathrm{sign}(\chi)=(-1)^{pq}$ and a σ is in $\mathsf{Sh}(p,q)$ if and only if $\sigma\circ\chi\in\mathsf{Sh}(q,p)$. So χ induces a bijection on the indexing set for the shuffle product and therefore

$$(a_0\otimes a_1\otimes\ldots\otimes a_p)*(a_0'\otimes a_{p+1}\otimes\ldots\otimes a_{p+q})=(-1)^{pq}(a_0'\otimes a_{p+1}\otimes\ldots\otimes a_{p+q})*(a_0\otimes a_1\otimes\ldots\otimes a_p).$$

We saw that for commutative k-algebras A the module of Kähler differentials describes $\mathsf{HH}_1^k(A)$.

Definition IV.5.14. Let n be a natural number and let A be a commutative k-algebra. The *module of differential n-forms on A over k* is

$$\Omega^n_{A|k}:=\Lambda^n_A\Omega^1_{A|k}.$$

Elements in $\Omega^n_{A|k}$ are of the form $a_0da_1\wedge\ldots\wedge da_n$ and for every $\sigma\in\Sigma_n$ we have the relation

$$a_0da_{\sigma^{-1}(1)}\wedge\ldots\wedge da_{\sigma^{-1}(n)}=\mathrm{sign}(\sigma)a_0da_1\wedge\ldots\wedge da_n.$$

Example IV.5.15. For A commutative we know that $\mathsf{HH}_1^k(A)\cong\Omega^1_{A|k}$. Let $\varphi\colon\Omega^1_{A|k}\otimes\Omega^1_{A|k}\to\mathsf{HH}_2^k(A)$ be the map

$$\Omega^1_{A|k}\otimes\Omega^1_{A|k}\xrightarrow{\cong}\mathsf{HH}_1^k(A)\otimes\mathsf{HH}_1^k(A)\xrightarrow{*}\mathsf{HH}_2^k(A).$$

The graded-commutativity of $*$ implies that φ factors through $\Omega^2_{A|k}$.

Remark IV.5.16. We know that $\mathsf{HH}_*^k(k[x])\cong\Omega^*_{k[x]|k}$ if we agree to set $\Omega^0_{k[x]|k}=k[x]$. For $k[x_1,\ldots,x_n]$ one can similarly show that

$$\mathsf{HH}_*^k(k[x_1,\ldots,x_n])\cong\Omega^*_{k[x_1,\ldots,x_n]|k}.$$

By the Theorem of Hochschild-Kostant-Rosenberg [**5**, Theorem 3.4.4] this can be generalize to smooth k-algebras: If A is smooth over k, then

$$\mathsf{HH}_*^k(A)\cong\Omega^*_{A|k}.$$

Lecture 22

Remark IV.5.17. The Hochschild cochain complex of a k-algebra has a very rich structure which is structural similar to that of a double based loop space: If X is a pointed topological space, then $\Omega^2 X$ is the space of basepoint preserving maps from the 2-sphere to X. Pierre Deligne conjectured that a structure that Murray Gerstenhaber established on $\mathsf{HH}^*_k(A)$ [**4**] refines to such a structure on the level of cochains. Jim McClure and Jeff Smith proved that conjecture in 1999 [**8**].

You will investigate the Gerstenhaber algebra structure on $\mathsf{HH}^*_k(A)$ in an exercise.

IV.6. Cyclic homology

Let A be a k-algebra. If we take the Hochschild chain complex with coefficients then we already noticed the cyclic symmetriy of $C^k_*(A)$. Alain Connes was the first to exploit that symmetry: In 1981 he noted that the quotients $A^{\otimes n+1}/(1-t)$ still give rise to a chain complex. Here $\langle t \rangle = \mathbb{Z}/(n+1)\mathbb{Z}$. This chain complex has good properties if $\mathbb{Q} \subset k$. There is a definition of cyclic homology that does not need this restrictive assumption, but that works in any characteristic, for any commutative ring k.

Definition IV.6.1. Let t_n be a generator of $\mathbb{Z}/(n+1)\mathbb{Z} < \Sigma_{n+1} = \Sigma_{\{0,\ldots,n\}}$, $t_n = (0, 1, \ldots, n)$. We define

$$\mathbb{Z}/(n+1)\mathbb{Z} \times C^k_n(A) \to C^k_n(A)$$

by

$$t_n(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}.$$

Note that $\mathrm{sign}(t_n) = (-1)^n$, so this definition is consistent with the earlier action of the symmetric groups on the Hochschild complex, because

$$a_n \otimes a_0 \otimes \ldots \otimes a_{n-1} = a_{t_n^{-1}(0)} \otimes \ldots \otimes a_{t_n^{-1}(n)}.$$

Recall our earlier notation. We had

$$N = \sum_{i=0}^{n} t_n^i, \quad b = \sum_{i=0}^{n} (-1)^i d_i \text{ and } b' = \sum_{i=0}^{n-1} (-1)^i d_i.$$

These maps are compatible in the following sense:

Lemma IV.6.2.

$$(1-t)b' = b(1-t)$$
$$b'N = Nb.$$

Proof. The claims follow from the relations $d_i t_n = -t_{n-1} d_{i-1}$ for $0 < i \leqslant n$ and $d_0 t_n = (-1)^n d_n$, and these are checked by a direct computation. $\square$

Definition IV.6.3. Let A be a k-algebra.

(a) The *cyclic bicomplex of A over k* is $CC^k_{*,*}(A)$ defined as $CC^k_{p,q}(A) = A^{\otimes q+1}$ and

$$\begin{tikzcd}
\vdots \arrow[d, "b"'] & \vdots \arrow[d, "-b'"'] & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "-b'"'] & \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "\mathrm{id}-t_1"'] \arrow[d, "-b'"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "\mathrm{id}-t_1"'] \arrow[d, "-b'"'] & \ldots \arrow[l, "N"'] \\
A & A \arrow[l, "\mathrm{id}-t_0=0"'] & A \arrow[l, "N=\mathrm{id}"'] & A \arrow[l, "\mathrm{id}-t_0=0"'] & \ldots \arrow[l, "N=\mathrm{id}"']
\end{tikzcd}$$

(b) The *nth cyclic homology group of A over k*, $\mathsf{HC}^k_n(A)$, is

$$\mathsf{HC}^k_n(A) = H_n(\mathsf{Tot} CC^k_{*,*}(A)).$$

Note that

$$(\mathsf{Tot} CC^k_{*,*}(A))_n = \bigoplus_{p+q=n} CC^k_{p,q}(A) = \bigoplus_{i=1}^{n+1} A^{\otimes i}.$$

Example IV.6.4. As Σ_1 is the trivial group, we get that $\mathrm{id} - t_0 = 0$, so
$$\mathsf{HC}^k_0(A) = \mathsf{HH}^k_0(A) = A/[A, A].$$

If you stare at the zeroth column of the cyclic bicomplex, you see the Hochschild complex of A, and the same repeats in all even columns to the right. This periodicity gives rise to the following important exact sequence, that allows us to relate cyclic homology to Hochschild homology.

Theorem IV.6.5 (Connes' Periodicity Sequence)**.** *For any k-algebra A there is a long exact sequence*

$$\begin{tikzcd}
 & & \mathsf{HC}^k_{n-1}(A) \arrow[dll, rounded corners, "B"'] \\
\mathsf{HH}^k_n(A) \arrow[r, "I"] & \mathsf{HC}^k_n(A) \arrow[r, "S"] & \mathsf{HC}^k_{n-2}(A) \arrow[dll, rounded corners, "B"'] \\
\mathsf{HH}^k_{n-1}(A) \arrow[r, "I"] & \dots &
\end{tikzcd}$$

This is often called the BIS-sequence.

Proof. Consider the bicomplex $CC^{[2]}_{*,*}(A)$ that has as non-trivial entries only the first two columns of the bicomplex $CC^k_{*,*}(A)$, so

$$CC^{[2]}_{*,*}(A) = \begin{tikzcd}
\vdots \arrow[d, "b"'] & \vdots \arrow[d, "-b'"'] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "\mathrm{id}-t_1"'] \arrow[d, "-b'"'] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \dots \arrow[l] \\
A & A \arrow[l, "\mathrm{id}-t_0=0"'] & 0 \arrow[l] & 0 \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

We know that the b'-complex is contractible and you can adept the chain homotopy to this bicomplex. Therefore
$$H_n(\mathsf{Tot}(CC^{[2]}_{*,*}(A))) \cong \mathsf{HH}^k_n(A).$$

Denote by $I\colon CC^{[2]}_{*,*}(A) \hookrightarrow CC^k_{*,*}(A)$ the inclusion of the bicomplex $CC^{[2]}_{*,*}(A)$ into $CC^k_{*,*}(A)$. We get a short exact sequence of bicomplexes
$$0 \longrightarrow CC^{[2]}_{*,*}(A) \xrightarrow{\;I\;} CC^k_{*,*}(A) \xrightarrow{\;S\;} \mathrm{coker}(I) \longrightarrow 0$$
and a resulting short exact sequence of chain complexes after applying Tot.

Note that
$$\mathrm{coker}(I)_{p,q} \cong \begin{cases} CC^k_{p-2,q}(A), & p \geqslant 2, \\ 0, & p < 2. \end{cases}$$
Hence $H_n(\mathsf{Tot}(\mathrm{coker}(I)_{*,*}) \cong \mathsf{HC}_{n-2}(A)$. If we abuse notation and use I and S also for the maps induced on homology and if we denote by B the connecting homomorphism, then the result follows. $\square$

In many cases Hochschild homology is easier to calculate than cyclic homology, so the above sequence helps in these cases.

In low degrees we recover the result that $\mathsf{HH}^k_0(A) \cong \mathsf{HC}^k_0(A)$.

Example IV.6.6. We saw in Proposition IV.1.5, that $\mathsf{HH}^k_*(k)$ is only non-trivial for $* = 0$ where it is k. The Periodicity Sequence then gives us immediately that
$$\mathsf{HC}^k_*(k) \cong \begin{cases} k, & * \geqslant 0 \text{ and even}, \\ 0, & \text{otherwise}. \end{cases}$$

Example IV.6.7. If A is commutative, then we know that $\mathsf{HH}_0^k(A) = A = \mathsf{HC}_0^k(A)$ and $\mathsf{HH}_1^k(A) \cong \Omega^1_{A|k}$. You can check that the map

$$B\colon \mathsf{HC}_0^k(A) = A \to \mathsf{HH}_1^k(A) \cong \Omega^1_{A|k}$$

sends an $a \in A$ to da. This implies that $\mathsf{HC}_1^k(A)$ is isomorphic to $\Omega^1_{A|k}/U$ where U is the k-submodule of $\Omega^1_{A|k}$ generated by the elements da for $a \in A$.

Remark IV.6.8. If cyclic homology is harder to calculate than Hochschild homology, why should we bother? One reason is, that it is used as an approximation to algebraic K-theory.

The algebraic K-groups of a ring R, $K_*(R)$, contain arithmetic information about the ring, such as the units of R, and if R is commutative, then the Picard group of R and its Brauer group. The groups $K_*(R)$ are really hard to compute. For instance $K_*(\mathbb{Z})$ is not completely known.

The zeroth group, $K_0(R)$, is the so-called group completion of the monoid of isomorphism classes of finitely generated projective R-modules. If R is a field or a PID, then $K_0(R) \cong \mathbb{Z}$.

The next one, $K_1(R)$ is

$$K_1(R) = GL(R)/[GL(R), GL(R)]$$

where $GL(R) = \bigcup_{n\geqslant 1} GL_n(R)$ and $GL_n(R)$ is viewed as a subgroup of $GL_{n+1}(R)$ by identifying an $A \in GL_n(R)$ with the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(R)$. You know by Proposition III.2.4 that you can also describe $K_1(R)$ as $H_1(GL(R);\mathbb{Z})$. In $K_1(R)$ you find the units of R. For example $K_1(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

The group $K_2(R)$ can be expressed in terms of group homology as

$$K_2(R) \cong H_2(E(R);\mathbb{Z})$$

where $E(R)$ is the normal subgroup of $GL(R)$ generated by elementary matrices, *i.e.*, by quadratic matrices $e_{ij}(r)$ for $r \in R$ and $i \neq j$ with

$$(e_{ij}(r))_{p,q} = \begin{cases} r, & (p,q) = (i,j), \\ 1, & p = q, \\ 0, & \text{otherwise}. \end{cases}$$

For higher n, $K_n(R)$, still contains important information, but it is defined as the higher homotopy groups of a rather complicated space. For instance, it was shown in 2019 that $K_8(\mathbb{Z})$ is trivial [**3**].

There are 'trace maps' (called the Dennis trace map to Hochschild homology and the Chern character to negative cyclic homology)

$$\begin{tikzcd} & K_*(R) \arrow[dl] \arrow[dr] & \\ \mathsf{HC}_*^-(R) & & \mathsf{HH}_*^{\mathbb{Z}}(R) \end{tikzcd}$$

In this sense Hochschild homology and $HC_*^-(R)$, the negative cyclic homology of R, a slight variant of $\mathsf{HC}_*(R)$, can be thought of as approximations of algebraic K-theory. For more about algebraic K-theory and its relation to Hochschild and cyclic homology see [**10**].

Lecture 23

CHAPTER V

Spectral sequences

Spectral sequences were first studied starting from 1946 in many different contexts, for instance by Hochschild, Koszul, Leray, Lyndon, Serre and many more. They are an important tool for calculations, but they can also be used for proofs in homological algebra, algebraic topology, algebraic geometry and many other areas of mathematics. We will only cover an extremely concise introduction to spectral sequences. For more details and examples see [**12, 7, 6**].

So what are they? A typical statement about a spectral sequence is of the following type:

If M is a G-module for some groups G, if N is a normal subgroup of G, then there is a spectral sequence

$$E^2_{p,q} = H_p(G/N; H_q(N; M)) \Rightarrow H_{p+q}(G; M).$$

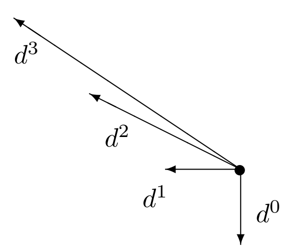
So our goals are to understand what the above statement actually means, and then to do calculations with the help of spectral sequences. In the above example you might want to calculate $H_*(G; M)$ and you do that by calculating the homology groups of G/N with coefficients in $H_q(N; M)$. So one has to understand for instance how G/N acts on $H_q(N; M)$, but then, of course, we need to know what the symbols $E^2_{p,q}$ and $\Rightarrow$ mean.

V.1. Definitions

Definition V.1.1.

- A *$\mathbb{Z}$-bigraded R-module* is a family $E = (E_{p,q})_{p,q\in\mathbb{Z}}$ of R-modules $E_{p,q}$.
- A *differential d^r of bidegree $(-r, r-1)$ on E* for $r \geqslant 0$ is a family $d^r\colon E_{p,q} \to E_{p-r,q+r-1}$ of R-linear maps with $d^r \circ d^r = 0$. We denote the bidegree of d^r by $||d^r|| = (-r, r-1)$.

  So we have $||d^0|| = (0, -1)$, $||d^1|| = (-1, 0)$, $||d^2|| = (-2, 1)$, $||d^3|| = (-3, 2)$ and so on:



- The *homology of E with respect to d^r* is the $\mathbb{Z}$-graded module
$$H(E, d^r)_{p,q} := \frac{\ker(d^r\colon E_{p,q} \to E_{p-r,q+r-1})}{\mathsf{im}(d^r\colon E_{p+r,q-r+1} \to E_{p,q})}.$$
- A *spectral sequence* $E = (E^r, d^r)$ is a sequence of $\mathbb{Z}$-bigraded modules E^r (for $r \geqslant 0$, or 1 or 2) together with a differential d^r of bidegree $||d^r|| = (-r, r-1)$ such that
$$E^{r+1} \cong H(E^r, d^r).$$
- If $E = (E^r, d^r)$ and $\tilde{E} = (\tilde{E}^r, \tilde{d}^r)$ are two spectral sequences, then a *morphism of spectral sequences* $f\colon E \to \tilde{E}$ is a family
$$f^r_{p,q}\colon E^r_{p,q} \to \tilde{E}^r_{p,q}$$
of R-linear maps such that
$$d^r \circ f^r = f^r \circ \tilde{d}^r \text{ for all } r$$
and such that f^{r+1} is induced by f^r.

Remark V.1.2.

(a) The definition above describes a spectral sequence *of homological type*. The ones of cohomomological type have differentials d_r of bidegree $||d_r|| = (r, 1-r)$ and the definition has to be modified accordingly.

(b) Think of a spectral sequence $E^r_{p,q}$ as an infinite stack of sheets of papers. You start with E^0 (or E^1 or E^2) and then you have to calculate E^{r+1} from E^r and d^r. This is a priori an infinite process...

(c) As we have $E^{r+1} = H(E^r, d^r)$ as bigraded objects, we can think of a spectral sequence in terms of nested subobjects. For instance,

$$E^3 = H(E^2, d^2) = Z^2/B^2$$

where we use Z^2 for the $\mathbb{Z}$-bigraded module of d^2-cycles and B^2 for the $\mathbb{Z}$-bigraded module of d^2-boundaries. Say we start with E^2, then we get

$$0 = B^1 \subset B^2 \subset B^3 \subset \dots \qquad\qquad \dots \subset Z^3 \subset Z^2 \subset Z^1 = E^2.$$

The boundary map $d^{r+1}\colon Z^r/B^r \to Z^r/B^r$ has kernel

$$\ker(d^{r+1}) = Z^{r+1}/B^r$$

and image

$$\mathsf{im}(d^{r+1}) = B^{r+1}/B^r.$$

How do we talk about this?

We call the elements of Z^r, the elements that survive until stage $r+1$,

The elements of B^r are the ones that are boundaries in stage r, so they 'die at stage r'.

We denote by Z^∞

$$Z^\infty := \bigcap_r Z^r$$

and call these the 'surviving cycles'.

We set B^∞ as

$$B^\infty = \bigcup_r B^r$$

and call its elements the ones that eventually become boundaries. These are the elements that 'die' in a spectral sequence.

Note that $B^\infty \subset Z^\infty$, so we can form

$$E^\infty_{p,q} := Z^\infty_{p,q}/B^\infty_{p,q}.$$

The E^r approximate E^∞ and the E^∞-term is called the *abutment of the spectral sequence.*

But what have we actually calculated with E^∞?

Definition V.1.3. A spectral sequence (E^r, d^r) is *first quadrant* if $E^r_{p,q} = 0$ for all negative p, q.

Remark V.1.4.

- If you fix a bidegree (p, q) in a first quadrant spectral sequence, then there are only finitely many non-trivial differentials that start in the spot (p, q), but there are also only finitely many non-trivial differentials that end in the spot (p, q). So for a fixed (p, q) you get at $E^\infty_{p,q}$ in finitely many steps!
- The elements in $E^r_{p,0}$ are special in such a spectral sequence: They can never be hit by a differential. Therefore

$$E^{r+1}_{p,0} = \ker(d^r\colon E^r_{p,0} \to E^r_{p-r,r-1}) \subset E^r_{p,0} \tag{V.1.1}$$

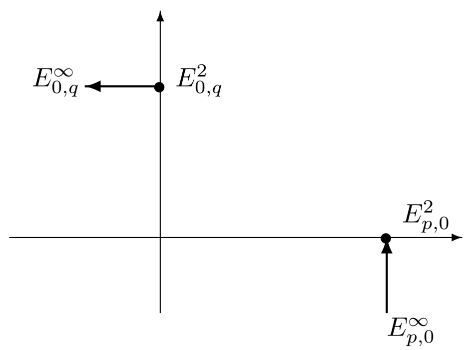
and hence

$$E^\infty_{p,0} = E^{p+1}_{p,0} \subset \dots \subset E^3_{p,0} \subset E^2_{p,0}.$$

- Dually, the elements $x \in E^r_{0,q}$ are always cycles, so $d^r x = 0$. Therefore

$$E^2_{0,q} \twoheadrightarrow E^3_{0,q} \twoheadrightarrow \dots \twoheadrightarrow E^\infty_{0,q}. \tag{V.1.2}$$

Definition V.1.5. The composite morphisms in (V.1.1) and (V.1.2) are called *edge homomorphisms*.



Definition V.1.6. If (E^r, d^r) is a spectral sequence, then we say that it *collapses at the* E^s*-term*, if $d^r = 0$ for all $r \geqslant s$.

If that happens, then $E^s = E^\infty$ because all elements on the E^s-term are cycles and there are only trivial boundaries.

How do spectral sequences arise?

V.2. Filtered complexes

Definition V.2.1. Let M be an R-module.

(a) An *increasing filtration* $(F_pM)_{p\in\mathbb{Z}}$ *of* M is a family of submodules $F_pM \subset M$ such that

$$\ldots \subset F_{p-1}M \subset F_pM \subset F_{p+1}M \subset \ldots$$

(b) The *associated graded module of* $(F_pM)_{p\in\mathbb{Z}}$ is grM with

$$gr_pM := F_pM/F_{p-1}M.$$

(c) Let $(F_pM)_{p\in\mathbb{Z}}$ and $(F_pN)_{p\in\mathbb{Z}}$ be two filtered R-modules. A *morphism of filtered modules* is an R-linear map $f\colon M \to N$ such that $f(F_pM) \subset F_pN$ for all $p \in \mathbb{Z}$.

If we reach the trivial submodule 0 to the left or the full submodule M to the right, we have reached the constant part of the filtration. Beware that there might be infinite filtrations that never become stationary in either direction.

Examples V.2.2.

(a)

$$\ldots 0 = 0 \subset \mathbb{Z}/4\mathbb{Z} \cong 4\mathbb{Z}/16\mathbb{Z} \subset \mathbb{Z}/16\mathbb{Z} = \mathbb{Z}/16\mathbb{Z} = \mathbb{Z}/16\mathbb{Z} = \ldots$$

and

$$\ldots 0 \subset F_1(\mathbb{Z}/16) = 8\mathbb{Z}/16\mathbb{Z} \subset F_2(\mathbb{Z}/16) = 4\mathbb{Z}/16\mathbb{Z} \subset F_3(\mathbb{Z}/16) = 2\mathbb{Z}/16\mathbb{Z} \subset F_4(\mathbb{Z}/16) = \mathbb{Z}/16\mathbb{Z} \ldots$$

are two filtrations of $\mathbb{Z}/16\mathbb{Z}$.

The associated graded of the first one is trivial but for two degrees where we obtain $\mathbb{Z}/4\mathbb{Z}$ whereas the second one has four non-trivial terms:

$$gr_p(\mathbb{Z}/16\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \quad 1 \leqslant p \leqslant 4.$$

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(b) Let M be $\mathbb{Z}/4\mathbb{Z}$ and $N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with filtrations

$$\ldots 0 \subset F_1M = 2\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z} = F_2M \ldots$$

and

$$\ldots 0 \subset F_1N = \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = F_2N \ldots$$

Then M and N have isomorphic associated graded modules, but of course M and N are not isomorphic.

(c) Also the filtrations

$$\ldots 0 \subset 2\mathbb{Z} \subset \mathbb{Z} \ldots$$

and

$$\ldots 0 \subset \mathbb{Z} \cong \mathbb{Z} \times \{0\} \subset \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ldots$$

have isomorphic associated graded modules, but again $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Remark V.2.3. We saw above that it can happen that the associated graded modules gr_*M and gr_*N are isomorphic despite the fact that M and N are not. If we want to reconstruct a filtered module M from its associated graded, we have to solve extension problems and you know that extensions are detected by Ext^1-terms.

Definition V.2.4. Let C_* be a chain complex of R-modules. An *increasing filtration of* C_* is a family of subcomplexes $F_pC_* \subset C_*$ such that

$$\ldots \subset F_{p-1}C_* \subset F_pC_* \subset F_{p+1}C_* \subset \ldots$$

Remark V.2.5.

- For all p the diagram

$$\begin{tikzcd} F_{p-1}C_n \arrow[r] \arrow[d, "d_n|_{F_{p-1}}"'] & F_pC_n \arrow[d, "d_n|_{F_p}"] \\ F_{p-1}C_{n-1} \arrow[r] & F_pC_{n-1}. \end{tikzcd}$$

  commutes. Therefore, we get an *induced filtration on the homology groups of* C_*:

$$F_pH_n(C_*) = \mathsf{im}(H_nF_p(C_*) \to H_n(C_*)).$$

  You know that inclusions of subcomplexes don't necessarily induce monomorphisms on the level of homology groups. Thus we have to take the image.
- A filtration on a chain complex C_* as above induces a filtration F_pC_n of each C_n, so in total we obtain a $\mathbb{Z}$-bigraded module.

Definition V.2.6. Let F_pC_* be a filtered chain complex. Then we define

$$(FC_*)_{p,q} := F_pC_{p+q}$$

and we call p the *filtration degree* and q the *internal degree.*

Example V.2.7. Let $C_{*,*}$ be a double complex, so we have R-modules $C_{p,q}$ together with horizontal and vertical differentials d^h, d^v, such that

$$d^hd^h = 0 = d^vd^v = d^hd^v + d^vd^h.$$

There are two common filtrations on such a bicomplex. We define

$$F'_p(C_{*,*})_n := \bigoplus_{r \leqslant p} C_{r,n-r}.$$

This gives rise to a filtration of the total complex $\mathsf{Tot}C_{*,*}$, where you consider all summands up to column number p.

The second filtration counts rows:

$$F''_p(C_{*,*})_n := \bigoplus_{r \leqslant p} C_{n-r,r}.$$

So on the total complex you consider all summands up to row p.

Definition V.2.8. A filtration F_pC_* of a chain complex C_* is *bounded* if for all n there is an $s = s(n) < t = t(n)$ such that $F_sC_n = 0$ and $F_tC_n = C_n$.

Thus in this case every module C_n has a finite filtration

$$0 = F_sC_n \subset F_{s+1}C_n \subset \ldots \subset F_{t-1}C_n \subset F_tC_n = C_n.$$

Definition V.2.9. A spectral sequence (E^r, d^r) *converges* to a graded module $H = \bigoplus H_n$ if there is a filtration F_pH on H and for every p there is an isomorphism $E_p^\infty \cong F_pH/F_{p-1}H$, *i.e.*,

$$E^\infty_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}.$$

In this case one uses the notation

$$E^2_{p,q} \Rightarrow H_{p+q}.$$

An important special case is the following.

Definition V.2.10. Let C_* be a chain complex with $C_i = 0$ for $i < 0$. Then a filtration $(F_pC_*)_p$ is *canonically bounded*, if $F_{-1}C_* = 0$ and if for all $n \geqslant 0$:

$$0 \subset F_0C_n \subset \ldots \subset F_nC_n = C_n.$$

Remark V.2.11. In the situation above the homology of C_* has an induced filtration of the form

$$0 = F_{-1}H_nC_* \subset F_0H_nC_* \subset \ldots \subset F_nH_nC_* = H_nC_*.$$

So in order to reconstruct H_nC_* from the E^∞-term, one has to solve extension problems because we want to know H_nC* from

$$F_nH_nC_*/F_{n-1}H_nC_*, F_{n-1}H_nC_*/F_{n-2}H_nC_*, \ldots, F_1H_nC_*/F_0H_nC_*, F_0H_nC_*/0 = F_0H_nC_*.$$

Hence you can read off $F_0H_nC_*$, but not $F_1H_nC_*$ up to $F_nH_nC_*$.

Theorem V.2.12. *Let F_pC_* be a filtered chain complex. Then F_pC_* determines a spectral sequence (E^r, d^r), $r \geqslant 1$ with*

$$E^1_{p,q} \cong H_{p+q}(F_pC_*/F_{p-1}C_*).$$

If the filtration is bounded, then there are natural isomorphisms

$$E^\infty_{p,q} \cong F_p(H_{p+q}(C_*))/F_{p-1}(H_{p+q}(C_*)).$$

Sketch of proof of Theorem V.2.12. We consider subobjects of the associated graded module and define

$$Z^r_{p,q} := \{[c] \in gr_pC_{p+q}, dc \in F_{p-r}C_{p+q-1}\}.$$

So these are elements whose boundary isn't necessarily zero, but is in lower filtration. Thus one can think of these elements as being cycles modulo F_{p-r}.

The differential of C_* restricts to morphisms of the form

$$d\colon Z^r_{p,q} \to Z^r_{p-r,q+r-1}$$

because the differential applied to a representative of $[c] \in Z^r_{p,q}$ has $dc \in F_{p-r}C_{p+q-1}$ and this gives rise to a class in

$$Z^r_{p-r,p+q-1-p+r} = Z^r_{p-r,q+r-1}.$$

Note that $d(d(c)) = 0$, because d was a differential.

We denote $d(Z^r_{p+r,q-r+1})$ by $B^r_{p,q}$ and by definition $d(Z^r_{p+r,q-r+1}) \subset Z^r_{p,q}$. For $r \geqslant 1$ we set

$$E^r_p := Z^r_p/B^r_p.$$

It is consistent to set

$$Z^0_{p,q} := F_pC_{p+q}/F_{p-1}C_{p+q} = E^0_{p,q} \text{ and } Z^\infty_{p,q} := \{c \in F_pC_{p+q}, dc = 0\}/F_{p-1}C_{p+q}$$

so we start with the associated graded and end up with the cycles in it.

We claim that

$$Z^{r+1}_{p,q} = \ker(d\colon Z^r_{p,q} \to Z^r_{p-r,q+r-1}).$$

An element $c \in F_p C_{p+q}$ represents an element in $Z^r_{p,q}$ if $dc \in F_{p-r}C_{p+q-1}$ and it represents an element in $Z^{r+1}_{p,q}$ if the stronger condition $dc \in F_{p-r-1}C_{p+q-1} \subset F_{p-r}C_{p+q-1}$ holds. But the second condition says precisely that dc represents zero in the quotient $F_{p-r}C_{p+q-1}/F_{p-r-1}C_{p+q-1}$. But this is equivalent to dc being zero in $Z^r_{p-r,q+r-1} \subset F_{p-r}C_{p+q-1}/F_{p-r-1}C_{p+q-1}$.

We obtain a tower of subquotients

$$0 =: B^0_{p,q} \subset \ldots \subset B^r_{p,q} \subset \ldots \subset Z^r_{p,q} \subset \ldots Z^1_{p,q} \subset Z^0_{p,q} = E^0_{p,q}.$$

We know that the differential of C_* induces a differential on Z^r_{p+q} and it induces one on $E^r_{p,q}$ which we call d^r, so

$$d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

For the fact that

$$E^{r+1}_{p,q} \cong \frac{\ker(d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1})}{\mathsf{im}(d^r\colon E^r_{p+r,q-r+1} \to E^r_{p,q})}$$

and for the identification of the E^∞-term, we refer to [**6**, Proof of Theorem 3.1 in Chapter IX]. Please beware the non-standard notation in [**6**]. $\square$

As an application we prove that Connes' definition of cyclic homology agrees with the one given as the homology of the cyclic bicomplex in characteristic zero:

Example V.2.13. Fix a k-algebra A. We defined the cyclic bicomplex $CC^k_{*,*}(A)$ of A as

$$\begin{tikzcd}
\vdots \arrow[d, "b"] & \vdots \arrow[d, "-b'"] & \vdots \arrow[d, "b"] & \vdots \arrow[d, "-b'"] & \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "\mathrm{id}-t_1"'] \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "\mathrm{id}-t_1"'] \arrow[d, "-b'"] & \cdots \arrow[l, "N"'] \\
A & A \arrow[l, "\mathrm{id}-t_0=0"'] & A \arrow[l, "N=\mathrm{id}"'] & A \arrow[l, "\mathrm{id}-t_0=0"'] & \cdots \arrow[l, "N=\mathrm{id}"']
\end{tikzcd}$$

We consider the second filtration on the associated total complex, so

$$F''_p \mathsf{Tot}(CC^k_{*,*}(A))_n = \bigoplus_{r \leqslant p} CC^k_{n-r,r}.$$

The filtration quotient

$$F''_p \mathsf{Tot}(CC^k_{*,*}(A))_* / F''_{p-1} \mathsf{Tot}(CC^k_{*,*}(A))_*$$

is isomorphic to $CC^k_{*,p}$ and this is precisely the pth row of the bicomplex $CC^k_{*,*}$. Therefore

$$E^1_{p,q} \cong H_q(CC^k_{*,p}) = \frac{\ker(d^h\colon CC^k_{q,p}(A) \to CC^k_{q-1,p}(A))}{\mathsf{im}(d^h\colon CC^k_{q+1,p}(A) \to CC^k_{q,p}(A))}.$$

The horizontal chain complex is

$$A^{\otimes p+1} \xleftarrow{\mathrm{id}-t_p} A^{\otimes p+1} \xleftarrow{N} A^{\otimes p+1} \xleftarrow{\mathrm{id}-t_p} \ldots$$

with $\langle t_p \rangle = \mathbb{Z}/(p+1)\mathbb{Z} =: C_{p+1} < \Sigma_{p+1}$ and therefore

$$E^1_{p,q} = H_q(C_{p+1}; A^{\otimes p+1}).$$

If $\mathbb{Q} \subset k$, then the group order of C_{p+1} is invertible in k and hence

$$H_q(C_{p+1}; A^{\otimes p+1}) \cong \begin{cases} (A^{\otimes p+1})/(\mathrm{id} - t_p), & q = 0, \\ 0, & \text{otherwise.} \end{cases}$$

by Corollary III.3.14.

So the spectral sequence is concentrated in one line. The next differential is induced by the b-differential, say $\bar{b}$, so we obtain the homology of the coinvariants with respect to this differential in bidegree $(p, 0)$: $H_p(A^{\otimes *+1}/(\mathrm{id} - t_*), \bar{b})$.

All higher differentials are trivial and there are no extension issues. We therefore obtain that if $\mathbb{Q} \subset k$, then

$$\mathsf{HC}_n^k(A) \cong H_n(A^{\otimes *+1}/(\mathrm{id} - t_*), \bar{b}),$$

and hence we obtain the agreement of the homology groups of Connes' chain complex and $\mathsf{HC}_*^k(A)$.

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V.3. Exact couples

Exact couples are a means for constructing spectral sequences.

Definition V.3.1. Let E and D be two R-modules for $R \neq 0$.

(a) Assume that we have R-linear maps $i\colon D \to D$, $j\colon D \to E$ and $k\colon E \to D$. We call

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

exact, if $\mathsf{im}(i) = \ker(j)$, $\mathsf{im}(j) = \ker(k)$ and $\mathsf{im}(k) = \ker(i)$. Then (D, E, i, j, k) is an *exact couple.*

(b) We set $D' := \mathsf{im}(i) \subset D$ and $E' := \frac{\ker d}{\mathsf{im} d}$ for $d := j \circ k$.
We define $i'\colon D' \to D'$ as $i'(i(x)) := i(i(x))$, $j'(i(x)) := [j(x)] = j(x) + \mathsf{im} d$ and $k'[y] := k(y)$. Then (D', E', i', j', k') is the *derived couple.*

You showed that the maps are well-defined and that the derived couple is again and exact couple.

Exact couples give rise to spectral sequences with explicit control over d^r. Given an exact couple as above we define

$$E^1 := E, d^1 = d, \quad E^2 = E', d^2 = j' \circ k'$$

and $E^r := E^{(r-1)}$, $d^r = j^{(r-1)} \circ k^{(r-1)}$, where $(D^{(r-1)}, E^{(r-1)}, i^{(r-1)}, j^{(r-1)}, k^{(r-1)})$ is the $(r-1)$-fold derived couple of (D, E, i, j, k).

Lemma V.3.2. *Setting $D^{(r)} := \mathsf{im}(i^{\circ r})$ and $E^{(r)} := k^{-1}(i^{\circ r}(D))/j(\ker(i^{\circ r}))$ gives the rth derived couple of (D, E, i, j, k) where i_r is the restriction $i|_{D^{(r)}}$, j_r is $j|_{(i^{\circ r})^{-1}(D)}$, so you take a preimage under the r-fold composite of i and then apply j. The map k_r is induced by the map k.*

Proof. Exactness is clear as long as $E^{(r+1)} = H_*(E^{(r)}, j_r \circ k_r)$. We show that

(a) $\ker(j_r \circ k_r) = k^{-1}(i^{\circ(r+1)}(D))/B$

(b) $\mathsf{im}(j_r \circ k_r) = j(\ker(i^{\circ(r+1)}))/B$

with $E^{(r)} = C/B$ where $C = k^{-1}(i^{\circ r}(D))$ and $B = j(\ker(i^{\circ r}))$.

For (a) assume that $(j_r \circ k_r)(c + B) = 0$ for $c + B \in E^{(r)}$. Then $k(c) = i^{\circ r}(y)$ for some $y \in D$, but this implies

$$j_r \circ k_r(c + B) = j_r(k(c)) = j(y) + B.$$

Thus if $j_r \circ k_r(c + B) = 0$, then $j(y) \in B$, hence there is an $x \in \ker(i^{\circ r})$ with $j(y) = j(x)$. Then $j(y - x) = 0$ and by exactness $y - x = i(z)$ for some $z \in D$. But this implies that

$$k(c) = i^{\circ r}(y) = i^{\circ r}(z) = i^{\circ(r+1)}(z)$$

and $c \in k^{-1}(i^{\circ(r+1)}(D))$.

For (b) we write $j_r \circ k_r(c + B)$ again as $j(y) + B$ as above with $k(c) = i^{\circ r}(y)$. But then

$$i^{\circ(r+1)}(y) = ik(c) = 0$$

because i and k are consecutive maps in an exact couple. This shows one subset relation.

Assume conversely that $i^{\circ(r+1)}(y) = 0$. Then by exactness there is a c with $k(c) = i^{\circ r}(y)$. But then $j(y) + B = j_r k(c) = j_r(k_r(c + B))$. $\square$

The techniques and results above transfer to the $\mathbb{Z}$-bigraded setting as follows:

Corollary V.3.3. *Let (D, E, i, j, k) be an exact couple of $\mathbb{Z}$-bigraded R-modules with*

$$||i|| = (1, -1), \quad ||j|| = (0, 0) \text{ and } ||k|| = (-1, 0).$$

Then (D, E, i, j, k) determines a spectral sequence (E^r, d^r) of homological type with $E^r = E^{(r-1)}$ and $d^r = j_{r-1} \circ k_{r-1}$.

Proof. It remains to check the bidegree of d^r. As i_{r-1} is just a restriction of i, it still has bidegree $(1, -1)$. As j_{r-1} takes a preimage under $i^{\circ(r-1)}$ and then applies j to it, we get a bidegree

$$||j_r|| = (-r + 1, r - 1).$$

The map k_{r-1} is again just a restriction of k, so it still has bidegree $(-1, 0)$.

Thus we obtain for $d^r = j_{r-1} \circ k_{r-1}$:

$$||d^r|| = ||j_{r-1} \circ k_{r-1}|| = (-r + 1, r - 1) + (-1, 0) = (-r, r - 1).$$

□

We can now define an exact couple associated to a filtered chain complex C_* with filtration $(F_pC_*)_{p\in\mathbb{Z}}$. Then for each p we get a short exact sequence of chain complexes

$$0 \longrightarrow F_{p-1}C_* \longrightarrow F_pC_* \twoheadrightarrow F_pC_*/F_{p-1}C_* \longrightarrow 0$$

and an induced long exact sequence on homology groups

$$\begin{tikzcd} & & H_n(F_pC_*/F_{p-1}C_*) \arrow[dll, "\delta=k"] \\ H_{n-1}(F_{p-1}C_*) \arrow[r, "i"] & H_{n-1}(F_pC_*) \arrow[r, "j"] & H_{n-1}(F_pC_*/F_{p-1}C_*) \arrow[dll, "\delta=k"] \\ H_{n-2}(F_{p-1}C_*) \arrow[r, "i"] & \dots \end{tikzcd}$$

Definition V.3.4. In the situation above we define

$$D_{p,q} = H_{p+q}(F_pC_*),$$
$$E_{p,q} = H_{p+q}(F_pC_*/F_{p-1}C_*)$$

Then $i\colon H_{p+q}(F_{p-1}C_*) = D_{p-1,q+1} \to H_{p+q}(F_pC_*) = D_{p,q}$ has bidegree $(1, -1)$, the map $j\colon H_{p+q}(F_pC_*) \to H_{p+q}(F_pC_*/F_{p-1}C_*)$ has bidegree $(0, 0)$ and $k = \delta\colon H_{p+q}(F_pC_*/F_{p-1}C_*) = E_{p,q} \to H_{p+q-1}(F_{p-1}C_*) = D_{p-1,q}$ has bidegree $(-1, 0)$.

This is an exact couple associated to the filtration F_pC_* and we get a spectral sequence as above. This spectral sequence is isomorphic to the one in Theorem V.2.12.

A prototypical situation is that you have a long exact sequence on homology that you can reinterpret as an exact couple.

Examples V.3.5.

- Let C_* be a chain complex of abelian groups such that every C_n is torsionfree (for instance the singular chains on a topological space). Consider the short exact sequence

  $$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

  This gives a short exact sequence

  $$0 \longrightarrow C_* \xrightarrow{n} C_* \xrightarrow{\pi} C_* \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

of chain complexes and hence a long exact sequence on homology groups

$$\begin{tikzcd}
 & & & H_m(C_* \otimes \mathbb{Z}/n\mathbb{Z}) \arrow[dlll, rounded corners, "\delta"'] \\
H_{m-1}(C_*) \arrow[r, "n"] & H_{m-1}(C_*) \arrow[r, "H_{m-1}(\pi)"] & H_{m-1}(C_* \otimes \mathbb{Z}/n\mathbb{Z}) \arrow[dll, rounded corners, "\delta"'] & \\
H_{m-2}(C_*) \arrow[r, "n"] & \ldots & &
\end{tikzcd}$$

In this case we can define D as the homology of C_*, E as the homology of $C_* \otimes \mathbb{Z}/n\mathbb{Z}$, the map i is the one that's induced by the multiplication by n, j is $H_*(\pi)$ and k is again the connecting homomorphism. The associated spectral sequence is the *Bockstein spectral sequence*.

- Let X be a topological space with a filtration

$$\varnothing \subset X_0 \subset \ldots \subset X_p \subset X_{p+1} \subset \ldots \subset X.$$

For every p we obtain a long exact sequence of singular homology groups

$$\begin{tikzcd}
 & & & H_{p+q+1}(X_p, X_{p-1}) \arrow[dlll, rounded corners, "\delta"'] \\
H_{p+q}(X_{p-1}) \arrow[r] & H_{p+q}(X_p) \arrow[r] & H_{p+q}(X_p, X_{p-1}) \arrow[dll, rounded corners, "\delta"'] & \\
H_{p+q-1}(X_{p-1}) \arrow[r] & \ldots & &
\end{tikzcd}$$

that gives rise to the exact couple and to a spectral sequence. What do you obtain when X is a CW complex and X_p is the p-skeleton?

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V.4. The two spectral sequences associated to a bicomplex

We have already seen a non-trivial example of one of these spectral sequences in Example V.2.13. How do they look like in general?

We start with a bicomplex $(C_{*,*}, d^h, d^v)$ and consider the filtrations of $\mathsf{Tot}(C_{*,*})$

$$(F'_p C_{*,*})_n = \bigoplus_{r \leqslant p} C_{r,n-r} \text{ and } (F''_p C_{*,*})_n = \bigoplus_{r \leqslant p} C_{n-r,r}$$

so in the first case we take everything in $\mathsf{Tot}(C_{*,*})$ up to column p and in the second case we throw away everything in $\mathsf{Tot}(C_{*,*})$ beyond row p.

For the first filtration we obtain ${}^I E^0_{p,q} := C_{p,q}$ and $d^0 = d^v$. Therefore the E^1-term in this case is

$${}^I E^1_{p,q} = H^v_q(C_{p,*}).$$

The d^1-differential on the E^1-page is induced by the horizontal differential, so

$$d^1 \colon H^v_q(C_{p,*}) \to H^v_q(C_{p-1,*}), \quad [c] \mapsto [d^h(c)].$$

It is common and suggestive to denote the resulting E^2-term as

$${}^I E^2_{p,q} = H^h_p(H^v_q(C_{*,*})).$$

If $C_{*,*}$ is for instance concentrated in the first quadrant we get

$${}^I E^\infty_{pq} = F'_p(H_{p+q}\mathsf{Tot}(C_{*,*}))/F'_{p-1}(H_{p+q}\mathsf{Tot}(C_{*,*}))$$

and this is denoted by

$${}^I E^2_{p,q} = H^h_p(H^v_q(C_{*,*})) \Rightarrow H_{p+q}\mathsf{Tot}(C_{*,*})$$

but beware that the ${}^{I}E^{\infty}_{pq}$-term only contains filtration quotients, so we only know the associated graded of $H_{p+q}\mathsf{Tot}(C_{*,*})$ and might need to solve extension problems to get the final result.

Dually, the second filtration has

$$^{II}E^0_{p,q} = C_{q,p}, \quad d^0 = d^h$$

and

$$^{II}E^1_{p,q} = H^h_q(C_{*,p})$$

and d^1 is induced by d^v:

$$d^1 \colon H^h_q(C_{*,p}) \to H^h_q(C_{*,p-1}), [c] \mapsto [d^v(c)].$$

The resulting E^2-page is

$$^{II}E^2_{p,q} = H^v_p H^h_q(C_{*,*})$$

and in good cases we get

$$^{II}E^2_{p,q} = H^v_p H^h_q(C_{*,*}) \Rightarrow H_{p+q}\mathsf{Tot}(C_{*,*}).$$

As a non-trivial example of a proof that uses these spectral sequences we prove the general version of Morita invariance of Hochschild homology from Theorem IV.3.4:

If A and B are two k-algebras that are Morita equivalent and if M is an A-bimodule, then we show that

$$\mathsf{HH}^k_*(A; M) \cong \mathsf{HH}^k_*(B; Q \otimes_A M \otimes_A P).$$

In particular, for $M = A$ this yields.

$$\mathsf{HH}^k_*(A) \cong \mathsf{HH}^k_*(B; Q \otimes_A A \otimes_A P) \cong \mathsf{HH}^k_*(B).$$

Before we start the actual proof, we need a lemma:

Lemma V.4.1. *Let A be a k-algebra and let P and M' be symmetric k-modules. If M' is a right A-module and if P is projective as an A-module, then $\mathsf{HH}^k_*(A; P \otimes_k M') \cong 0$ for $* > 0$ and $\mathsf{HH}^k_0(A; P \otimes_k M') \cong M' \otimes_A P$.*

Dually, if Q is projective as a right A-module and M'' is an A-module, then $\mathsf{HH}^k_*(A; M'' \otimes_k Q) \cong 0$ for $* > 0$ and $\mathsf{HH}^k_0(A; M'' \otimes_k Q) \cong Q \otimes_A M''$.

Proof. The Hochschild complex has

$$C^k_p(A; P \otimes_k M') = P \otimes M' \otimes A^{\otimes p} \cong M' \otimes A^{\otimes p} \otimes P$$

and the complex is isomorphic to the two-sided bar construction $B(M', A, P)$. If P is free as an A-module then we know that this complex is a resolution of $M' \otimes_A P$ by an adaption of the argument in the proof of Lemma III.2.3. The general case follows by finding a free R-module F with $P \oplus P' = F$. $\square$

We consider the bicomplex $C_{*,*}$ with

$$C_{p,q} = B^{\otimes p} \otimes N \otimes A^{\otimes q} \otimes P$$

and with $N := Q \otimes_A M$.

We define the horizontal differential

$$d^h \colon C_{p,q} \cong (N \otimes A^{\otimes q} \otimes P) \otimes B^{\otimes p} \to (N \otimes A^{\otimes q} \otimes P) \otimes B^{\otimes p-1} = C_{p-1,q}$$

as the Hochschild differential b and the vertical differential

$$d^v \colon C_{p,q} \cong (P \otimes B^{\otimes p} \otimes N) \otimes A^{\otimes q} \to (P \otimes B^{\otimes p} \otimes N) \otimes A^{\otimes q-1}$$

as $d^v = -b$.

We know that P is projective as an A-module and as a right B-module.

In a first step we calculate first the horizontal and then the vertical homology. The projectivity of P as a right B-module gives with Lemma V.4.1 that $H^h_p C_{*,q}$ is trivial for $p \neq 0$. For $p = 0$ we get an isomorphism to

$$P \otimes_B (Q \otimes_A M \otimes A^{\otimes p}) \cong (P \otimes_B Q) \otimes_A M \otimes A^{\otimes p} \cong A \otimes_A M \otimes A^{\otimes p} \cong M \otimes A^{\otimes p}$$

and the vertical differential can be identified with the negative of the Hochschild boundary for $C^k_*(A; M)$. Therefore the E^2-page is concentrated in the $(p = 0)$-line with terms $\mathsf{HH}^k_*(A; M)$.

Taking first the vertical homology and using that P is projective as an A-module gives something that is concentrated in the $(q=0)$-line isomorphic to

$$N \otimes_A (P \otimes B^{\otimes p}) \cong (Q \otimes_A M \otimes_A P) \otimes B^{\otimes p}$$

and the horizontal differential is the Hochschild boundary of $C_*^k(B; Q \otimes_A M \otimes_A P)$, so the E^2-page is

$$E^2_{p,0} \cong \mathsf{HH}_p^k(B; Q \otimes_A M \otimes_A P).$$

and this proves Theorem IV.3.4.

V.5. The Lyndon-Hochschild-Serre spectral sequence

This is actually the example from the very beginning of this chapter. Let G be a group with normal subgroup N and factor group G/N. We want to calculate $H_*(G)$ from $H_*(N)$ and $H_*(G/N)$.

Lemma V.5.1. *If M is a G-module, then G/N acts on $H_*(N; M)$.*

Proof. Let F_* be a $\mathbb{Z}[G]$-free resolution of $\mathbb{Z}$, for instance the bar resolution. Then F_* is also a $\mathbb{Z}[N]$-free resolution of $\mathbb{Z}$ because $\mathbb{Z}[G] \cong \bigoplus_{G/N} \mathbb{Z}[N]$. Let $g \in G$ and define

$$g_* \colon F_* \otimes_{\mathbb{Z}[N]} M \to F_* \otimes_{\mathbb{Z}[N]} M, \quad g_*(x \otimes m) := xg^{-1} \otimes gm.$$

As F_* is a chain complex of right $\mathbb{Z}[G]$-modules, g_* is a chain map. If $n \in N$, then $n_* = \mathrm{id}$, because $xn^{-1} \otimes nm = x \otimes n^{-1}nm = x \otimes m$. Therefore the G-action via g_* factors through G/N. $\square$

On the zeroth homology we get the correct result:

Lemma V.5.2. *For any G-module M and any normal subgroup N in G:*

$$(M_N)_{G/N} \cong M_G.$$

Proof. Recall that $M_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$. We obtain directly that

$$\mathbb{Z} \otimes_{\mathbb{Z}[G]} M \cong \mathbb{Z} \otimes_{\mathbb{Z}[G/N]} \mathbb{Z}[G/N] \otimes_{\mathbb{Z}[G]} M.$$

We claim that $\mathbb{Z}[G/N] \otimes_{\mathbb{Z}[G]} M \cong M_N$ and this proves the lemma.

We define $\varphi \colon M_N \to \mathbb{Z}[G/N] \otimes_{\mathbb{Z}[G]} M$ as $\varphi(m+U) := 1 \otimes m$, where U is the submodule generated by

$$\{nm - m, n \in N, m \in M\}.$$

Note that $1 = 1_{\mathbb{Z}[G/N]=N} = N$. This is well-defined: If $n \in N$, then

$$\varphi(nm + U) = 1 \otimes nm = 1 \otimes m.$$

Conversely let $\psi \colon \mathbb{Z}[G/N] \otimes_{\mathbb{Z}[G]} M \to M_N$ be $\psi(gN \otimes m) := gm + U$. Then

$$\psi \circ \varphi(m+U) = \psi(1 \otimes m) = m + U$$

and $\varphi \circ \psi(gN \otimes m) = \varphi(gm + U) = 1 \otimes gm = gN \otimes m$. $\square$

Remark V.5.3. If F_* is a free $\mathbb{Z}[G]$ resolution of $\mathbb{Z}$, then

$$(F_* \otimes_{\mathbb{Z}} M)_G \cong F_* \otimes_{\mathbb{Z}[G]} M$$

if the G-action on the left hand side is defined as $g.(x \otimes m) = xg^{-1} \otimes gm$.

With Lemma V.5.2 one therefore obtains on the level of resolutions

$$F_* \otimes_{\mathbb{Z}[G]} M \cong (F_* \otimes_{\mathbb{Z}} M)_G \cong ((F_* \otimes_{\mathbb{Z}} M)_N)_{G/N}.$$

Theorem V.5.4. *For every group G with a normal subgroup N and every G-module M there is a spectral sequence*

$$E^2_{p,q} = H_p(G/N; H_q(N; M)) \Rightarrow H_{p+q}(G; M).$$

This is the *Lyndon-Hochschild-Serre (LHS) spectral sequence.* On the p-axis you have the groups $H_p(G/N; M_N)$ whereas on the q-axis you get $(H_q(N; M))_{G/N}$.

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Proof. Let F_* be a $\mathbb{Z}[G]$-free resolution of $\mathbb{Z}$ and let $\bar{F}_*$ be a $\mathbb{Z}[G/N]$-free resolution of $\mathbb{Z}$.

We consider the bicomplex

$$C_{p,q} := \bar{F}_p \otimes_{\mathbb{Z}[G/N]} ((F_q \otimes M)_N).$$

Here we take as the horizontal differential $d_{\bar{F}_*} \otimes \mathrm{id}$ and adjust the sign for the vertical differential which is $d^v = \pm \mathrm{id} \otimes d_{F_*} \otimes \mathrm{id}$ so that we actually obtain a bicomplex with $d^h d^v + d^v d^h = 0$.

As each F_q is free over $\mathbb{Z}[G]$ we get

$$F_q = \bigoplus_{I_q} \mathbb{Z}[G] \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} X_q$$

with $X_q = \bigoplus_{I_q} \mathbb{Z}$.

We get a similar description of $\bar{F}_p$ as

$$\bar{F}_p \cong \mathbb{Z}[G/N] \otimes_{\mathbb{Z}} Y_p.$$

We can also express $(F_q \otimes M)_N$ as

$$(F_q \otimes M)_N = \mathbb{Z} \otimes_{\mathbb{Z}[N]} (F_q \otimes M) \cong \mathbb{Z} \otimes_{\mathbb{Z}[N]} \mathbb{Z}[G] \otimes_{\mathbb{Z}} X_q \otimes M \cong \mathbb{Z}[G/N] \otimes X_q \otimes M$$

and

$$C_{p,q} \cong \tilde{F}_p \otimes_{\mathbb{Z}[G/N]} \mathbb{Z}[G/N] \otimes X_q \otimes M.$$

But as $\mathbb{Z}[G/N] \otimes X_q$ is free over $\mathbb{Z}[G/N]$ Lemma III.3.16 yields

$$H_p^h(C_{*,q}) \cong \begin{cases} 0, & p \neq 0, \\ ((F_q \otimes M)_N)_{G/N} \cong (F_q \otimes M)_G, & p = 0. \end{cases}$$

If we then take the vertical homology we obtain the terms $H_q(G; M)$ concentrated in the $(p = 0)$-line. Hence the spectral sequence has no more non-trivial differentials and there are no extension problems.

If we take vertical homology first for fixed horizontal degree p, this gives the homology of the complex

$$Y_p \otimes (F_* \otimes_{\mathbb{Z}[N]} M)$$

with $d^v = \mathrm{id} \otimes d \otimes \mathrm{id}$ and thus

$$H^v_{p,q} \cong Y_p \otimes H_q(N; M) \cong \bar{F}_p \otimes_{\mathbb{Z}[G/N]} H_q(N; M).$$

Taking horizontal homology therefore yields

$$E^2_{p,q} = H_p(G/N; H_q(N; M)).$$

□

A typical application of the Lyndon-Hochschild-Serre spectral sequence is the following result.

Proposition V.5.5. *Let G be a finite group and assume that a prime p divides $|G|$. Assume that G has a normal p-Sylow subgroup S. Then*

$$H_n(G; \mathbb{F}_p) \cong (H_n(S; \mathbb{F}_p))_{G/S}.$$

Proof. We have the LHS spectral sequence with

$$E^2_{r.s} = H_r(G/S; H_s(S; \mathbb{F}_p)) \Rightarrow H_{r+s}(G; \mathbb{F}_p).$$

The groups $H_s(S; \mathbb{F}_p)$ are all $\mathbb{F}_p$-vector spaces. You can actually deduce from the bar resolution that each $H_s(S; \mathbb{F}_p)$ is finite-dimensional. We know that $|G/S|$ is prime to p, hence $|G/S|$ is invertible in $M = H_s(S; \mathbb{F}_p)$. Therefore the spectral sequence is concentrated in the $(r = 0)$-line with

$$H_r(G/S; H_s(S; \mathbb{F}_p)) \cong \begin{cases} 0, & r > 0, \\ (H_s(S; \mathbb{F}_p))_{G/S}, & r = 0. \end{cases}$$

There are no further non-trivial differentials and no extension problems. □

Example V.5.6. Take $G = \Sigma_3$. Its 3-Sylow subgroup is the alternating group $A_3 \cong C_3$ and this is a normal subgroup. Applying the above result gives

$$H_n(\Sigma_3; \mathbb{F}_3) \cong H_n(C_3; \mathbb{F}_3)_{\{\pm 1\}}.$$

With the calculation from Example III.1.7 we get that $H_n(C_3; \mathbb{F}_3) \cong \mathbb{F}_3$ for all $n \geqslant 0$. But we still have to take coinvariants and to this end we have to understand the $\{\pm 1\}$-action on $H_n(C_3; \mathbb{F}_3) \cong \mathbb{F}_3$. This is done in the following example.

In the following we apply the LHS in order to calculate group homology in a case that we couldn't do before.

Example V.5.7. We want to determine $H_*(\Sigma_3; \mathbb{Z})$. We view Σ_3 as the dihedral group with 6 elements, D_6. This is the symmetry group of a regular triangle, so we have a normal rotation subgroup isomorphic to C_3 and a subgroup generated by a reflection and this is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

We have to understand the LHS spectral sequence

$$E^2_{p,q} = H_p(\mathbb{Z}/2\mathbb{Z}; H_q(C_3; \mathbb{Z})) \Rightarrow H_{p+q}(D_6; \mathbb{Z}).$$

We know by Example III.1.7 that

$$H_q(C_3; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0, \\ \mathbb{Z}/3\mathbb{Z}, & q \text{ odd}, \\ 0, & \text{otherwise.} \end{cases}$$

so we have to calculate $H_p(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z})$ and $H_p(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/3\mathbb{Z})$. As the group order of $\mathbb{Z}/2\mathbb{Z}$ is invertible in $\mathbb{Z}/3\mathbb{Z}$, we obtain that the groups $H_p(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/3\mathbb{Z})$ are trivial for $p > 0$, but we need to understand the $\mathbb{Z}/2\mathbb{Z}$-action on $H_{2n-1}(C_3; \mathbb{Z})$ in order to understand the coinvariants for $p = 0$.

The group $\{\pm 1\}$ acts by conjugation on C_3 and this is easier to understand if we switch to the Σ_3-picture and write

$$C_3 = \langle (1, 2, 3) \rangle, \quad \{\pm 1\} \cong \langle \tau \rangle$$

and we choose $\tau = (1, 2)$. All the other transpositions are conjugate to $(1, 2)$ and conjugation induces the trivial action on homology.

As $(1, 2)(1, 2, 3)(1, 2) = (1, 3, 2)$, the action is non-trivial. We denote $(1, 2, 3)$ by t in the following. We take the standard resolution

$$P_* = \ldots \xrightarrow{\mathrm{id}-t} \mathbb{Z}[C_3] \xrightarrow{N} \mathbb{Z}[C_3] \xrightarrow{\mathrm{id}-t} \mathbb{Z}[C_3].$$

We need a chain map from P_* to $(\tau(-)\tau^{-1})^*(P_*)$. As $\tau \circ t \circ \tau^{-1} = t^2$ the boundary map $\mathrm{id} - t$ in P_* turns into $\mathrm{id} - t^2$ in the twisted complex. We need maps f_n such that the diagram

$$\begin{tikzcd} \ldots \arrow[r, "\mathrm{id}-t"] & \mathbb{Z}[C_3] \arrow[r, "N"] \arrow[d, "f_2"] & \mathbb{Z}[C_3] \arrow[r, "\mathrm{id}-t"] \arrow[d, "f_1"] & \mathbb{Z}[C_3] \arrow[d, "f_0"] \\ \ldots \arrow[r, "\mathrm{id}-t^2"] & \mathbb{Z}[C_3] \arrow[r, "N"] & \mathbb{Z}[C_3] \arrow[r, "\mathrm{id}-t^2"] & \mathbb{Z}[C_3] \end{tikzcd}$$

commutes.

We claim that the definitions

$$f_{2n} = (-1)^n t^n, \quad f_{2n-1} = (-1)^n t^n$$

work. We have $f_{2n} \circ (\mathrm{id} - t) = (-1)^n t^n - (-1)^n t^{n+1}$. On the other hand

$$(\mathrm{id} - t^2) \circ f_{2n+1} = (\mathrm{id} - t^2) \circ (-1)^{n+1} t^{n+1} = (-1)^{n+1} t^{n+1} - (-1)^{n+1} t^{n+3}.$$

As $t^{n+3} = t^n$, both terms agree.

For the squares involving the norm map we get

$$f_{2n-1} \circ N = (-1)^n t^n N = (-1)^n N$$

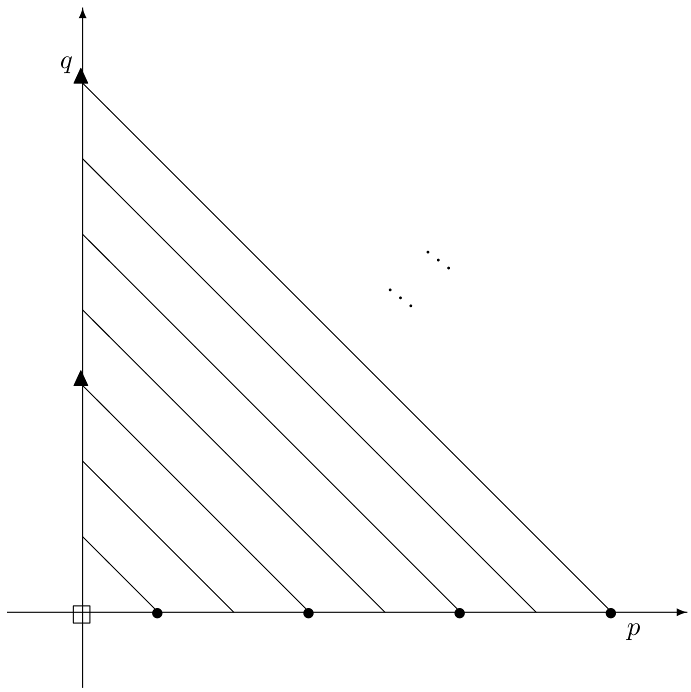
and $N \circ f_{2n} = (-1)^n N t^n = (-1)^n N$, so these agree as well. Therefore these f_n constitute a chain map.

We are only interested in the effect in odd degrees. If the degree is of the form $4n+1$, then f_{4n+1} is the multiplication by $-t^{2n+1}$, so we get an action by a minus sign. For degrees of the form $4n+3$ f_{4n+3} is multiplication by $t^{2(n+1)}$, so no sign is introduced.

Therefore

$$(H_{4n+1}(C_3;\mathbb{Z}))_{\mathbb{Z}/2\mathbb{Z}} = (\mathbb{Z}/3\mathbb{Z})/(x \sim -x) = 0, \quad (H_{4n+3}(C_3;\mathbb{Z}))_{\mathbb{Z}/2\mathbb{Z}} = (\mathbb{Z}/3\mathbb{Z})/(x \sim x) = \mathbb{Z}/3\mathbb{Z}.$$

We have the terms $H_p(\mathbb{Z}/2\mathbb{Z};\mathbb{Z})$ on the $(q=0)$-line and the above terms in the $(p=0)$-line and nothing in the middle. In the picture $\square$ stands for $\mathbb{Z}$ in $(0,0)$, $\bullet$ for $\mathbb{Z}/2\mathbb{Z}$ for odd p and $q=0$ and a $\blacktriangle$ stands for a $\mathbb{Z}/3\mathbb{Z}$ in q-degrees of the form $4n+3$ and $q=0$. The lines indicate constant total degree.



There cannot be any non-trivial differentials for degree reasons. We don't actually get extension problems because the elements in total degree of the form $4n+3$ give a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$.

So in total we obtain:

$$H_n(D_6,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n=0, \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1 \bmod 4, \\ \mathbb{Z}/6\mathbb{Z}, & n \equiv 3 \bmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

Remark V.5.8. You actually get a similar answer for all dihedral groups D_{2m} with m odd:

$$H_n(D_{2m},\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n=0, \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1 \bmod 4, \\ \mathbb{Z}/2m\mathbb{Z}, & n \equiv 3 \bmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

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