REFLEXIVE HOMOLOGY AND INVOLUTIVE HOCHSCHILD HOMOLOGY AS EQUIVARIANT LODAY CONSTRUCTIONS

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Abstract. For associative rings with anti-involution several homology theories exists, for instance reflexive homology as studied by Graves and involutive Hochschild homology defined by Fernàndez-València and Giansiracusa. We prove that the corresponding homology groups can be identified with the homotopy groups of an equivariant Loday construction of the one-point compactification of the sign-representation evaluated at the trivial orbit, if we assume that 2 is invertible and if the underlying abelian group of the ring is flat. We also show a relative version where we consider an associative k-algebra with an anti-involution where k is an arbitrary ground ring.

1. Introduction

In [LRZb] we introduced equivariant Loday constructions. These generalize the non-equivariant Loday constructions, which include (topological) Hochschild homology, higher order Hochschild homology and torus homology.

In the equivariant case we fix a finite group G. The starting point for a Loday construction is a G-commutative monoid in the sense of Hill and Hopkins [HH]. In the setting of G-equivariant stable homotopy theory these are genuine G-commutative ring spectra whereas in the algebraic setting of Mackey functors G-commutative monoids are G-Tambara functors. Some equivariant homology theories such as the twisted cyclic nerve of Blumberg-Gerhardt-Hill-Lawson [BGHL19] and Hesselholt-Madsen's Real topological Hochschild homology, THR, [DMPR21] can be identified with such equivariant Loday constructions [LRZb, §7]. Here, THR is a homology theory for associative algebra spectra with anti-involution A and we identified this in the commutative case with the Loday construction over the one-point compactification of the sign-representation, $\mathsf{THR}(A) \simeq \mathcal{L}^{C_2}_{S^\sigma}(A)$. In [LRZb, Proposition 6.1], we show that for any G-simplicial set X, if we apply the functor $\underline{\pi}_0$ levelwise to the equivariant Loday construction of a connective genuine commutative G-algebra spectrum A to obtain a simplicial G-Tambara functor,

$$\underline{\pi}_0(\mathcal{L}^G_X(A)) \cong \mathcal{L}^G_X(\underline{\pi}_0(A)),$$

which relates $\mathcal{L}^{C_2}_{S^\sigma}$ of C_2-Tambara functors to THR.

There is an algebraic version of THR, called Real Hochschild homology [AKGH, Definition 6.15] that takes associative algebras with anti-involution as input. These are associative k-algebras for some commutative ring k, such that $\tau(a) := \bar{a}$ satisfies $\overline{ab} = \bar{b}\bar{a}$ and such that the C_2-action is k-linear. In her thesis Chloe Lewis developed a Bökstedt-type spectral sequence for THR [Lew23] whose E^2-term consists of Real Hochschild homology groups. Other homology theories for associative algebras with anti-involution are reflexive homology [Gra24] and involutive Hochschild homology [FVG18]. Reflexive homology, HR_*, is a homology theory associated to the crossed simplicial group that is the cyclic group of order two, $C_2 = \langle\tau\rangle$, in every simplicial degree, where we do *not* view C_2 as a constant simplicial group, but let τ interact with the category Δ by reversing the simplicial structure. Involutive Hochschild homology, iHH_*, was

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defined in [FVG18]; the corresponding cohomology theory was developed by Braun [Bra14], who developed a cohomology theory for involutive A_∞-algebras, motivated by work of Costello on open Klein topological conformal field theories [Cos07]. We slightly generalize the definition in [FVG18] and work over arbitrary commutative rings instead of fields.

We prove the identification of reflexive homology as the homotopy groups of an equivariant Loday construction in section 6 and the one for involutive Hochschild homology in section 7:

Theorem (Theorems 6.4 and 7.2) Assume that R is a commutative ring with involution and that 2 is invertible in R. If the underlying abelian group of R is flat, then

$$\mathsf{iHH}_*^{\mathbb{Z}}(R) \cong \pi_*(\mathcal{L}_{S^\sigma}^{C_2}(\underline{R}^{\mathrm{fix}})(C_2/C_2)) \cong \mathsf{HR}_*^{+,\mathbb{Z}}(R,R).$$

If we work relative to a commutative ground ring k, then we obtain a corresponding result:

Theorem (Theorems 6.5 and 7.3) Assume that R is a commutative k-algebra with a k-linear involution and that 2 is invertible in R. If the underlying module of R is flat over k, then

$$\mathsf{iHH}_*^{k}(R) \cong \pi_*(\mathcal{L}_{S^\sigma}^{C_2,\underline{k}^c}(\underline{R}^{\mathrm{fix}})(C_2/C_2)) \cong \mathsf{HR}_*^{+,k}(R,R).$$

In hindsight, this identifies the Loday construction over the C_2-Burnside Tambara functor with the Loday construction relative to $\underline{\mathbb{Z}}^c$ under the above assumptions. We consider the examples of $\mathbb{F}_2$ and $\mathbb{Z}$ with the trivial C_2-action in section 8 in order to understand what happens if we drop these assumptions. There, the homotopy groups of the Loday constructions do *not* agree with neither reflexive homology nor with involutive Hochschild homology.

The relationship to the Real Hochschild homology of [AKGH] is unsatisfactory: The latter takes all dihedral groups into account and for $D_2 = C_2$ their definition agrees with ours. But for the higher D_{2m} the relationship to equivariant Loday constructions is unclear. We plan to tackle this problem in future work.

In section 9 we extend our results to the associative case, where we consider associative rings R and associative k-algebras with anti-involution where k is an arbitrary commutative ground ring. Usually, one cannot form Loday constructions without assuming commutativitiy, but the simplicial model of the one-point compactification of the sign-representation consists of two glued copies of the simplicial 1-simplex with its intrinsic ordering, so we can extend the definition to equivariant associative monoids in this case and we get results generalizing the above theorems:

Theorem (Theorem 9.4) Assume that R is an associative ring with anti-involution and that 2 is invertible in R. If the underlying abelian group of R is flat, then

$$\mathsf{iHH}_*^{\mathbb{Z}}(R) \cong \pi_*(\mathcal{L}_{S^\sigma}^{C_2}(\underline{R}^{\mathrm{fix}})(C_2/C_2)) \cong \mathsf{HR}_*^{+,\mathbb{Z}}(R,R).$$

If we work relative to a commutative ground ring k, then we obtain a corresponding result:

Theorem (Theorem 9.6) Assume that A is an associative k-algebra with a k-linear anti-involution and that 2 is invertible in A. If the underlying module of A is flat over k, then

$$\mathsf{iHH}_*^{k}(A) \cong \pi_*(\mathcal{L}_{S^\sigma}^{C_2,\underline{k}^c}(\underline{A}^{\mathrm{fix}})(C_2/C_2)) \cong \mathsf{HR}_*^{+,k}(A,A).$$

The proofs, however, are different: In the case of an associative ring R with anti-involution for instance the fixed point Mackey functor $\underline{R}^{\mathrm{fix}}$ does not carry any multiplicative structure and there is no norm construction. But we construct a substitute of the norm-restriction, $\tilde{N}_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}}$, such that $\underline{R}^{\mathrm{fix}}$ is a bimodule over it.

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2. Equivariant Loday constructions

We recall the basic facts about equivariant Loday constructions for G-Tambara functors from [LRZb] for an arbitrary finite group G. We work with unital rings. We assume that ring maps preserve the unit, and that the unit acts as the identity on any module over the ring.

We consider simplicial G-sets X that are finite in every degree and call them finite simplicial G-sets. For every G-Tambara functor $\underline{T}$ and every such X the simplicial G-Tambara functor $\mathcal{L}^G_X(\underline{T})$ is the G-Loday construction for X and $\underline{T}$. In simplicial degree n we define:

$$\mathcal{L}^G_X(\underline{T})_n = X_n \otimes \underline{T}$$

where the formation of the tensor product with the finite G-set X_n uses the fact that G-Tambara functors are the G-commutative monoids in the setting of G-Mackey functors. This was proved by Mazur [Maz13] for cyclic p-groups for a prime p and by Hoyer [Hoy14] in the case of a general finite group G. As they show that the construction $X_n \otimes \underline{T}$ is functorial in X_n, the Loday construction is well-defined.

The above tensor can be made explicit. Every finite G-set is isomorphic to a finite disjoint union of orbits and Mazur and Hoyer show that for an orbit G/H we obtain

$$G/H \otimes \underline{T} \cong N^G_H i^*_H \underline{T}.$$

Here, i^*_H restricts a G-Tambara functor to H, so for a finite H-set Y, $i^*_H\underline{T}(Y) := \underline{T}(G \times_H Y)$. The restriction functor has the norm functor N^G_H as a left adjoint. A disjoint union of G-sets X, X', $X \sqcup X'$ is sent to

$$(X \sqcup X') \otimes \underline{T} \cong (X \otimes \underline{T}) \Box (X' \otimes \underline{T}),$$

so this determines every $X_n \otimes \underline{T}$ up to isomorphism.

3. Basic results about fixed-point Tambara functors

In this section we study C_2-Mackey and Tambara functors. If L is an abelian group with involution $a \mapsto \bar{a}$, there is a C_2-Mackey functor $\underline{L}^{\mathrm{fix}}$ given by

$$\underline{L}^{\mathrm{fix}} = \begin{cases} L^{C_2} & \text{at } C_2/C_2, \\ L & \text{at } C_2/e, \end{cases}$$

where $\mathsf{tr}(a) = a + \bar{a}$ for all $a \in L$ and $\mathsf{res}(a) = a$ for all $a \in L^{C_2}$. If R is a commutative ring whose multiplication is compatible with its involution, then we can define $\mathsf{norm}(a) = a\bar{a}$ and get a C_2-Tambara functor structure on $\underline{R}^{\mathrm{fix}}$.

For a set Y we denote by $\mathbb{Z}\{Y\}$ the free abelian group generated by Y and for $y \in Y$ the corresponding generator in $\mathbb{Z}\{Y\}$ is $\{y\}$. When R is a commutative ring with involution the norm restriction of $\underline{R}^{\mathrm{fix}}$ is given by

$$N^{C_2}_e i^*_e \underline{R}^{\mathrm{fix}} = \begin{cases} (\mathbb{Z}\{R\} \oplus (R \otimes R)/C_2)/\mathrm{TR} & \text{at } C_2/C_2 \\ R \otimes R & \text{at } C_2/e, \end{cases}$$

where C_2 acts on $R \otimes R$ via $\tau(a \otimes b) = \bar{b} \otimes \bar{a}$, $[a \otimes b]$ denotes the equivalence class of $a \otimes b$ in $(R \otimes R)/C_2$, and Tambara Reciprocity, TR, identifies $\{a + b\} \sim \{a\} + \{b\} + [a \otimes \bar{b}]$. Here $\mathsf{norm}(a \otimes b) = \{a\bar{b}\}$ and $\mathsf{tr}(a \otimes b) = [a \otimes b]$ for all $a \otimes b \in R \otimes R$, $\mathsf{res}(\{a\}) = a \otimes \bar{a}$, and $\mathsf{res}([a \otimes b]) = a \otimes b + \bar{b} \otimes \bar{a}$ (see [HM19] for properties of the norm functor, especially Fact 4.4 in loc. cit.).

Lemma 3.1. *Assume that M and N are two abelian groups with involution and assume that 2 is invertible in M or N. Then there is an equivalence of C_2-Mackey functors*

$$\underline{M}^{\mathrm{fix}} \Box \underline{N}^{\mathrm{fix}} \cong \underline{(M \otimes N)}^{\mathrm{fix}}$$

which is natural in M and N. Here C_2 acts on $M\otimes N$ by the diagonal action. If, in addition, M and N are both commutative rings with involution, $\underline{M}^{\mathrm{fix}}$, $\underline{N}^{\mathrm{fix}}$, and $\underline{(M\otimes N)}^{\mathrm{fix}}$ are C_2-Tambara functors and the above equivalence is an equivalence of C_2-Tambara functors.

Proof. Without loss of generality assume that 2 is invertible in M. At the free orbit C_2/e, both sides give $M\otimes N$, so we check the result at C_2/C_2, where

$$(\underline{M}^{\mathrm{fix}}\Box\underline{N}^{\mathrm{fix}})(C_2/C_2) = (M^{C_2}\otimes N^{C_2}\oplus (M\otimes N)/C_2)/\mathrm{FR},$$

with Frobenius Reciprocity, FR, identifying $(m+\bar{m})\otimes n$ with $[m\otimes n]$ for all $m\in M$, $n\in N^{C_2}$ and $m\otimes(n+\bar{n})$ with $[m\otimes n]$ for all $m\in M^{C_2}$, $n\in N$.

Any $a\in M^{C_2}$ can be written as $a=m+\bar{m}$ for $m=a/2$. Here m is also fixed by C_2 since 2 is invertible. Note that 2 is invertible in $M\otimes N$.

Frobenius Reciprocity in fact identifies the first summand $M^{C_2}\otimes N^{C_2}$ into the second summand $(M\otimes N)/C_2$. A priori, a term of the form $(m+\bar{m})\otimes(n+\bar{n})$ could be identified in two ways with $(M\otimes N)/C_2$, either as $[m\otimes(n+\bar{n})]=[m\otimes n]+[m\otimes\bar{n}]$ or as $[(m+\bar{m})\otimes n]=[m\otimes n]+[\bar{m}\otimes n]$. But since $[m\otimes\bar{n}]=[\bar{m}\otimes n]$, the two ways agree and so we are left with $(M\otimes N)/C_2$ with no new relations.

As 2 is invertible in $M\otimes N$, the C_2-fixed points and C_2-coinvariants can be identified via $x\mapsto[x/2]$, $[x]\mapsto x+\bar{x}$. Since we argued that $(\underline{M}^{\mathrm{fix}}\Box\underline{N}^{\mathrm{fix}})(C_2/C_2)\cong(M\otimes N)/C_2$, in order to show the compatibility of res and tr with our identification we only need to show it for terms of that form. For $\underline{M}^{\mathrm{fix}}\Box\underline{N}^{\mathrm{fix}}$, $\mathsf{res}([a\otimes b])=a\otimes b+\bar{a}\otimes\bar{b}$. This is exacly what we get in $\underline{(M\otimes N)}^{\mathrm{fix}}$. Similarly, $\mathsf{tr}(a\otimes b)=[a\otimes b]$ in $\underline{M}^{\mathrm{fix}}\Box\underline{N}^{\mathrm{fix}}$ which is identified with $a\otimes b+\bar{a}\otimes\bar{b}$, the trace in $\underline{(M\otimes N)}^{\mathrm{fix}}$. So our correspondence preserves the C_2-Mackey structure.

If M and N are commutative rings with involution, although the identification of fixed points and coinvariants does not seem to be a ring map, it actually is: since res preserves the ring structure, in $\underline{M}^{\mathrm{fix}}\Box\underline{N}^{\mathrm{fix}}$ we have

$$\begin{aligned}\mathsf{res}([a\otimes b]\cdot[c\otimes d])) &= \mathsf{res}([a\otimes b])\mathsf{res}([c\otimes d]) = (a\otimes b+\bar{b}\otimes\bar{a})(c\otimes d+\bar{c}\otimes\bar{d})\\ &= ac\otimes bd+a\bar{c}\otimes b\bar{d}+\bar{a}c\otimes\bar{b}d+\bar{a}\bar{c}\otimes\bar{b}\bar{d}=\mathsf{res}([ac\otimes bd]+[\bar{a}c\otimes\bar{b}d]).\end{aligned}$$

We already know that $\underline{M}^{\mathrm{fix}}\Box\underline{N}^{\mathrm{fix}}\cong\underline{(M\otimes N)}^{\mathrm{fix}}$ as C_2-Mackey functors and res is injective on fixed-point Mackey functors. Hence we must have $[a\otimes b]\cdot[c\otimes d]=[ac\otimes bd]+[\bar{a}c\otimes\bar{b}d]$ in $\underline{M}^{\mathrm{fix}}\Box\underline{N}^{\mathrm{fix}}(C_2/C_2)$. Then the above calculation exactly says that the isomorphism $[a\otimes b]\mapsto a\otimes b+\bar{a}\otimes\bar{b}$ from C_2-coinvariants to C_2-invariants (and therefore also its inverse) is multiplicative. Note that the multiplicative unit of $(M\otimes N)/C_2$ is $[\frac{1}{2}\otimes 1]$.

When M and N, and therefore also $M\otimes N$, are commutative rings with involution, then in $\underline{M}^{\mathrm{fix}}\Box\underline{N}^{\mathrm{fix}}$, $N(a\otimes b)=a\bar{a}\otimes b\bar{b}\in M^{C_2}\otimes N^{C_2}$ which gets identified with $\frac{1}{2}[a\bar{a}\otimes b\bar{b}]\in(M\otimes N)/C_2$ which in turn is identified with $a\bar{a}\otimes b\bar{b}=(a\otimes b)(\bar{a}\otimes\bar{b})$ in $\underline{(M\otimes N)}^{\mathrm{fix}}(C_2/C_2)$. So our correspondence preserves the C_2-Tambara structure. □

Lemma 3.2. *If R is a commutative ring with involution in which 2 is invertible and if M is an abelian group with an involution, then there is an equivalence of C_2-Mackey functors*

$$(N_e^{C_2}i_e^*\underline{R}^{\mathrm{fix}})\Box\underline{M}^{\mathrm{fix}}\cong\underline{(R\otimes R\otimes M)}^{\mathrm{fix}}, \tag{3.1}$$

which is natural in M and R. Here, C_2 acts on $R\otimes R\otimes M$ by $\tau(a\otimes b\otimes m)=\bar{b}\otimes\bar{a}\otimes\bar{m}$. If M is also a commutative ring with involution, then (3.1) is an equivalence of C_2-Tambara functors.

Proof. The formula for the box product (see e.g. [HM19, Definition 3.1]) gives

$$(N_e^{C_2}i_e^*\underline{R}^{\mathrm{fix}})\Box\underline{M}^{\mathrm{fix}} = \begin{cases}\big((\mathbb{Z}\{R\}\oplus(R\otimes R)/C_2)/\mathrm{TR}\otimes M^{C_2}\oplus(R\otimes R\otimes M)/C_2\big)/\mathrm{FR} & \text{at } C_2/C_2\\ R\otimes R\otimes M & \text{at } C_2/e,\end{cases}$$

so we get the correct value at C_2/e. At C_2/C_2, we can send the three summands to $(R \otimes R \otimes M)^{C_2}$. We define the map as follows:

$$\begin{aligned} \{a\} \otimes m &\mapsto a \otimes \bar{a} \otimes m \quad \text{for } a \in R, m \in M^{C_2}, \\ [a \otimes b] \otimes m &\mapsto a \otimes b \otimes m + \bar{b} \otimes \bar{a} \otimes m \quad \text{for } a, b \in R, m \in M^{C_2}, \\ [a \otimes b \otimes m] &\mapsto a \otimes b \otimes m + \bar{b} \otimes \bar{a} \otimes \bar{m} \quad \text{for } a, b \in R, m \in M. \end{aligned}$$

Then Tambara Reciprocity is respected since

$$(\{a+b\} - \{a\} - \{b\} - [a \otimes \bar{b}]) \otimes m \mapsto ((a+b) \otimes (\bar{a} + \bar{b}) - a \otimes \bar{a} - b \otimes \bar{b} - a \otimes \bar{b} - b \otimes \bar{a}) \otimes m = 0.$$

Frobenius Reciprocity is also respected:

It identifies $\mathsf{tr}(a \otimes b) \otimes m$ with $[a \otimes b \otimes \mathsf{res}(m)]$ for all $a, b \in R$, $m \in M^{C_2}$, and by either name this element is mapped to $a \otimes b \otimes m + \bar{b} \otimes \bar{a} \otimes m$.

The terms $\{a\} \otimes \mathsf{tr}(m)$ are identified with $[\mathsf{res}(\{a\}) \otimes m]$ for all $a \in R$, $m \in M$, and this is mapped to

$$a \otimes \bar{a} \otimes (m + \bar{m}) = a \otimes \bar{a} \otimes m + a \otimes \bar{a} \otimes \bar{m}.$$

Finally, FR gives $[a \otimes b] \otimes \mathsf{tr}(m) \sim [\mathsf{res}([a \otimes b]) \otimes m]$ for all $a, b \in R$, $m \in M$, and this is mapped to

$$(a \otimes b + \bar{b} \otimes \bar{a}) \otimes (m + \bar{m}) = (a \otimes b + \bar{b} \otimes \bar{a}) \otimes m + (\bar{b} \otimes \bar{a} + a \otimes b) \otimes \bar{m}.$$

The relations described above explain how Frobenius Reciprocity actually identifies the first two summands $(\mathbb{Z}\{R\} \oplus (R \otimes R)/C_2)/\mathrm{TR} \otimes M^{C_2}$ in $(N_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}} \Box \underline{M}^{\mathrm{fix}})(C_2/C_2)$ into the last summand $(R \otimes R \otimes M)/C_2$. We know that on this last summand, the map we used, $[a \otimes b \otimes m] \mapsto a \otimes b \otimes m + \bar{b} \otimes \bar{a} \otimes \bar{m}$ is in fact an isomorphism between $(R \otimes R \otimes M)/C_2$ and $(R \otimes R \otimes M)^{C_2}$ because 2 is invertible in R and hence in $R \otimes R \otimes M$. The fact that it can be extended to a map $(N_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}} \Box \underline{M}^{\mathrm{fix}})(C_2/C_2) \to (R \otimes R \otimes M)^{C_2}$ means that the Frobenius Reciprocity relations did not impose additional relations within $(R \otimes R \otimes M)/C_2$, and so we get an isomorphism $(N_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}} \Box \underline{M}^{\mathrm{fix}})(C_2/C_2) \cong (R \otimes R \otimes M)^{C_2}$. The proof that this identification respects the C_2-Mackey structure is completely analogous to that in the previous lemma.

If M is also a commutative ring with involution and so $\underline{M}^{\mathrm{fix}}$ and $N_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}} \Box \underline{M}^{\mathrm{fix}}$ as well as $\underline{R}^{\mathrm{fix}}$ and $N_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}}$ are C_2-Tambara functors, this isomorphism is multiplicative and respects the norm: The product in $(R \otimes R \otimes M)/C_2$ is, for similar reasons to those in the previous lemma,

$$[a \otimes b \otimes m] \cdot [c \otimes d \otimes n] = [ac \otimes bd \otimes mn] + [a\bar{d} \otimes b\bar{c} \otimes m\bar{n}]$$

and the identification of the coinvariants with the fixed points respects this. For $a, b \in R$ and $m \in M$,

$$N(a \otimes b \otimes m) = \{a\bar{b}\} \otimes m\bar{m}$$

in $(N_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}}) \Box \underline{M}^{\mathrm{fix}}$ maps to

$$a\bar{b} \otimes \bar{a}b \otimes m\bar{m} = (a \otimes b \otimes m)(\bar{b} \otimes \bar{a} \otimes m) = N(a \otimes b \otimes m)$$

in $(R \otimes R \otimes M)^{\mathrm{fix}}$. □

Note also that for a map of commutative C_2-rings $f\colon R \to C$ where 2 is invertible in both rings, the sequence of maps

$$N_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}} \Box \underline{C}^{\mathrm{fix}} \to \underline{R}^{\mathrm{fix}} \Box \underline{C}^{\mathrm{fix}} \cong \underline{(R \otimes C)}^{\mathrm{fix}} \to \underline{(C \otimes C)}^{\mathrm{fix}} \to \underline{C}^{\mathrm{fix}}$$

that is given by the counit of the $(N_e^{C_2}, i_e^*)$-adjunction, the identification of Lemma 3.1, the map f, and multiplication, the corresponding map $\underline{(R \otimes R \otimes C)}^{\mathrm{fix}} \to \underline{C}^{\mathrm{fix}}$ is given by the multiplication in R and the R-module structure on C induced by the map f.

4. Working relative to a commutative ground ring

In [LRZb, §8] we defined a G-equivariant Loday construction relative to a map of G-Tambara functors $\underline{R} \to \underline{T}$. In general, this construction is rather involved because its building blocks are relative norm-restriction terms: For an orbit G/H we set

$$G/H \otimes_{\underline{R}} \underline{T} := (G/H \otimes \underline{T}) \Box_{(G/H \otimes \underline{R})} \underline{R} = N_H^G i_H^*(\underline{T}) \Box_{N_H^G i_H^*(\underline{R})} \underline{R} =: N_H^{G,\underline{R}} i_H^*(\underline{T}). \tag{4.1}$$

This uses the naturality of $N_H^G i_H^*(-)$ and the counit of the augmentation $N_H^G i_H^*(\underline{R}) \to \underline{R}$.

In [LRZb] we define the relative equivariant Loday construction for any finite simplicial G-set X:

$$\mathcal{L}_X^{G,\underline{R}}(\underline{T}) := \mathcal{L}_X^G(\underline{T}) \Box_{\mathcal{L}_X^G(\underline{R})} \underline{R}.$$

If we consider fixed-point C_2-Tambara functors and if we assume that 2 is invertible, then these terms simplify drastically. If the C_2-action on R is trivial, then we emphasize this by writing $\underline{R}^{\text{fix}} = \underline{R}^c$ and we call $\underline{R}^c$ the constant Tambara functor on R.

The purpose of this section is to relate the relative Loday construction of C_2-Tambara functors as follows to the fixed-point Tambara functor of the non-equivariant relative Loday construction:

Theorem 4.1. *If $k \to R$ is a map of commutative C_2-rings where C_2 acts trivially on k and 2 is invertible in R,*

$$\mathcal{L}_X^{C_2,\underline{k}^c}(\underline{R}^{\text{fix}}) \cong \underline{\mathcal{L}_X^k(R)}^{\text{fix}},$$

where C_2 acts on each level $\mathcal{L}_{X_n}^k(R)$ by simultaneously using the action induced from the C_2-action on X_n (exchanging copies of R as needed) by naturality and acting on all copies of R.

Proof. Theorem 4.1 follows directly from the following two results: Proposition 4.2 says that for free orbits C_2/e,

$$\mathcal{L}_{C_2/e}^{C_2,\underline{k}^c}(\underline{R}^{\text{fix}}) \cong \underline{\mathcal{L}_{C_2/e}^k(R)}^{\text{fix}}.$$

Clearly for one-point orbits,

$$\mathcal{L}_{C_2/C_2}^{C_2,\underline{k}^c}(\underline{R}^{\text{fix}}) = \underline{R}^{\text{fix}} = \underline{\mathcal{L}_{C_2/C_2}^k(R)}^{\text{fix}}.$$

If X and Y are disjoint C_2-sets

$$\mathcal{L}_{X \sqcup Y}^{C_2,\underline{k}^c}(\underline{R}^{\text{fix}}) \cong \mathcal{L}_X^{C_2,\underline{k}^c}(\underline{R}^{\text{fix}}) \Box_{\underline{k}^c} \mathcal{L}_Y^{C_2,\underline{k}^c}(\underline{R}^{\text{fix}}).$$

Then Lemma 4.4 implies that the identification for the free and trivial orbits can be assembled into a statement about disjoint unions of orbits. □

It is important to remember that the equivariant Loday construction is *not* the Loday construction relative to $\underline{\mathbb{Z}}^c$, but rather the Loday construction relative to the C_2-Burnside Tambara functor, and these are different. For example, taking the relative norm-restriction term from (4.1) $N_e^{C_2,\underline{\mathbb{Z}}^c} i_e^* \underline{\mathbb{Z}}^c$ gives $\underline{\mathbb{Z}}^c$, whereas taking $N_e^{C_2} i_e^* \underline{\mathbb{Z}}^c$ gives the C_2-Burnside Tambara functor as explained for instance in [LRZb, (5.1)].

Proposition 4.2. *Let $k \to R$ be a map of commutative C_2-rings where C_2 acts trivially on k, and assume that 2 is invertible in R. Then*

$$N_e^{C_2,\underline{k}^c} i_e^*(\underline{R}^{\text{fix}}) \cong \underline{(R \otimes_k R)}^{\text{fix}},$$

where C_2 acts on $R \otimes_k R$ by $\tau(a \otimes b) = \bar{b} \otimes \bar{a}$.

Proof. The relative box-product $N_e^{C_2,\underline{k}^c} i_e^*(\underline{R}^{\text{fix}}) = N_e^{C_2} i_e^*(\underline{R}^{\text{fix}}) \Box_{N_e^{C_2} i_e^*(\underline{k}^c)} \underline{k}^c$ is the coequalizer of the diagram

$$N_e^{C_2} i_e^*(\underline{R}^{\text{fix}}) \Box N_e^{C_2} i_e^* \underline{k}^c \Box \underline{k}^c \underset{\text{id}\Box\nu'}{\overset{\nu\Box\text{id}}{\rightrightarrows}} N_e^{C_2} i_e^*(\underline{R}^{\text{fix}}) \Box \underline{k}^c$$

where ν is the composite of the map $N_e^{C_2} i_e^* \underline{k}^c \to N_e^{C_2} i_e^*(\underline{R}^{\text{fix}})$ and the multiplication map of $N_e^{C_2} i_e^*(\underline{R}^{\text{fix}})$ and ν' uses the counit of the adjunction $\varepsilon\colon N_e^{C_2} i_e^* \underline{k}^c \to \underline{k}^c$ and the multiplication in $\underline{k}^c$. As $\underline{k}^c$ is the fixed-point Tambara functor for the trivial action we can use the fact that i_e^* and $N_e^{C_2}$ are strong symmetric monoidal and Lemma 3.1 to get that

$$N_e^{C_2} i_e^*(\underline{R}^{\text{fix}}) \Box N_e^{C_2} i_e^*(\underline{k}^c) = N_e^{C_2} i_e^*(\underline{R}^{\text{fix}} \Box \underline{k}^c) = N_e^{C_2} i_e^*(\underline{R}^{\text{fix}} \Box \underline{k}^{\text{fix}}) \cong N_e^{C_2} i_e^*(\underline{(R \otimes k)}^{\text{fix}}).$$

Then we can use Lemma 3.2 to rewrite the diagram as

$$\underline{((R \otimes k) \otimes (R \otimes k) \otimes k)}^{\text{fix}} \underset{\text{id}\Box\nu'}{\overset{\nu\Box\text{id}}{\rightrightarrows}} \underline{(R \otimes R \otimes k)}^{\text{fix}}$$

where now ν uses the map $k \to R$ and the induced k-module structure on R and ν' uses the multiplication in k.

As $R \otimes_k R \cong (R \otimes R) \otimes_{k \otimes k} k$, we will show that when 2 is invertible, taking the fixed-point Tambara functor commutes with forming coequalizers. To this end we show that the functor that assigns to a commutative C_2-ring in which 2 is invertible its fixed-point C_2-Tambara functor is left adjoint to the functor from fixed-point Tambara functors where 2 is invertible to commutative C_2-rings where 2 is invertible that evaluates such a Tambara functor at the free level C_2/e.

Of course, if we have a map of C_2-Tambara functors $\underline{S}^{\text{fix}} \to \underline{T}^{\text{fix}}$ and if 2 is invertible in S and T, then the map at the free level is a map of commutative C_2-rings $S \to T$ and 2 is invertible in S and T.

For the converse, assume that $f\colon S \to T$ is a map of commutative C_2-rings and that 2 is invertible in S and T. We claim that the map

$$g(x) := \frac{1}{2}\mathsf{tr} f(\mathsf{res}(x))$$

is a map at the trivial orbit C_2/C_2 such that the pair (g, f) is a map of fixed-point Tambara functors $(g, f)\colon \underline{S}^{\text{fix}} \to \underline{T}^{\text{fix}}$. Note that the value $g(x)$ can actually be identified with $f(x)$ because the restriction map is an inclusion and tr is just the multiplication by 2. We have to show that g is a ring map and that it is compatible with res, tr and norm.

- As $\mathsf{tr}(1) = 2$ in any fixed C_2-Tambara functor and as f and res are ring maps, we obtain
$$g(1) = \frac{1}{2}\mathsf{tr} f(\mathsf{res}(1)) = \frac{1}{2}\mathsf{tr}(1) = 1.$$
- Our maps commute with the restriction maps because
$$\mathsf{res} g(x) = \mathsf{res}\frac{1}{2}\mathsf{tr} f(\mathsf{res}(x)) = \frac{1}{2}(f(\mathsf{res} x) + \overline{f(\mathsf{res} x)}),$$
and as $x \in S^{C_2}$ and as res just incluces S^{C_2} into S, this is $f(\mathsf{res}(x))$.
- The map g preserves products because f and res are ring maps, hence
$$g(xy) = \frac{1}{2}\mathsf{tr} f(\mathsf{res}(xy)) = \frac{1}{2}\mathsf{tr}(f(\mathsf{res}(x)) f(\mathsf{res}(y))) = \frac{1}{2} \cdot 2 \cdot f(\mathsf{res}(x)) f(\mathsf{res}(y)).$$
On the other hand
$$g(x)g(y) = \frac{1}{2}\mathsf{tr} f(\mathsf{res}(x))\frac{1}{2}\mathsf{tr} f(\mathsf{res}(y)) = \frac{1}{2} \cdot 2 \cdot f(\mathsf{res}(x)) \cdot \frac{1}{2} \cdot 2 \cdot f(\mathsf{res}(y)).$$
- That g preserves addition follows because res and f are ring maps.
- We calculate
$$g(\mathsf{tr}(y)) = g(y + \bar{y}) = \frac{1}{2}\mathsf{tr} f(\mathsf{res}(y + \bar{y})) = f(y + \bar{y}) = f(\mathsf{tr}(y))$$
and hence our maps commute with tr.

- For the compatibility with the norm map we observe that
$$g(\mathsf{norm}(y)) = \frac{1}{2}\mathsf{tr}f(\mathsf{res}(\mathsf{norm}(y))) = \frac{1}{2}\mathsf{tr}f(y \cdot \bar{y}) = f(y \cdot \bar{y}) = f(\mathsf{norm}(y)).$$

□

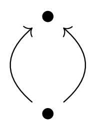
Remark 4.3. Beware that not all C_2-Tambara functors satisfy $\mathsf{tr}(1) = 2$, even if 2 is invertible. For instance, let $\underline{A}[\frac{1}{2}]$ be the C_2-Burnside Tambara functor with 2 inverted. Then $\underline{A}[\frac{1}{2}](C_2/C_2) = \mathbb{Z}[\frac{1}{2}][t]/t^2 - 2t$ and with $s := \frac{1}{2}t$ this can be re-written as $\mathbb{Z}[\frac{1}{2}][s]/s^2 - s$. As $\mathsf{tr}(1) = t$, we get $\mathsf{tr}(1) = 2s \neq 2$.

Lemma 4.4. *If k is a commutative ring with trivial C_2 action and M and N are two k-modules with a k-linear involution and 2 is invertible in M or in N, then there is an equivalence of C_2-Mackey functors*
$$\underline{M}^{\text{fix}} \Box_{\underline{k}^c} \underline{N}^{\text{fix}} \cong \underline{(M \otimes_k N)}^{\text{fix}}$$
which is natural in M and N. Here C_2 acts on $M \otimes N$ by the diagonal action. If M and N are both also commutative k-algebras, this is an equivalence of C_2-Tambara functors.

Proof. Using Lemma 3.1 we know that
$$\underline{M}^{\text{fix}} \Box \underline{k}^c \Box \underline{N}^{\text{fix}} = \underline{M}^{\text{fix}} \Box \underline{k}^{\text{fix}} \Box \underline{N}^{\text{fix}} \cong \underline{(M \otimes k \otimes N)}^{\text{fix}}$$
and
$$\underline{M}^{\text{fix}} \Box \underline{N}^{\text{fix}} \cong \underline{(M \otimes N)}^{\text{fix}}.$$
The result in the k-module case then followed from the fact that taking the fixed-point Mackey functor commutes with forming coequalizers, which is completely analogous to the fact that the fixed-point Tambara functor commutes with forming coequalizers which was shown in the proof of Proposition 4.2 above. And in the case of k-algebras, it follows directly by the argument in the proof there. □

5. Identifying $\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\text{fix}})$

In this section we will continue to work with the cyclic group of order 2, $C_2 = \langle \tau \mid \tau^2 = e \rangle$, and we will consider the C_2-simplicial set S^σ which is the one-point compactification of the real sign-representation,

$S^\sigma =$ 

where the C_2-action flips the two arcs.

By [LRZb, (7.4)], for any C_2-Tambara functor $\underline{T}$ we can express the C_2-Loday construction of $\underline{T}$ with respect to S^σ as a two-sided bar construction

(5.1) $$\mathcal{L}^{C_2}_{S^\sigma}(\underline{T}) \cong B(\underline{T}, N_e^{C_2} i_e^* \underline{T}, \underline{T}).$$

We will simplify this for the C_2-Tambara functor $\underline{R}^{\text{fix}}$ associated to a commutative ring R with involution $a \mapsto \bar{a}$. We will repeatedly use the commutative C_2-algebra $R \otimes R$, with

(5.2) $$\tau(a \otimes b) = \bar{b} \otimes \bar{a}.$$

For a ring spectrum A with an anti-involution, Dotto, Moi, Patchkoria and Reeh observed [DMPR21, p. 84], that
$$B(A, N_e^{C_2} i_e^* A, A) \simeq B(A, A \wedge A, A),$$
where they use the flip-C_2-action on $A \wedge A$ (switching coordinates and acting on them, as in (5.2)). They identify $\mathsf{THR}(A)$ with $B(A, N_e^{C_2} i_e^* A, A)$ in [DMPR21, Theorem 2.23] under a flatness assumption on A.

The following result is an algebraic version of this result where we use the C_2-action on $R \otimes R$ that exchanges the coordinates and acts on both.

Theorem 5.1. *If R is a commutative ring with involution and 2 is invertible in R, then there is a natural equivalence of simplicial C_2-Tambara functors*

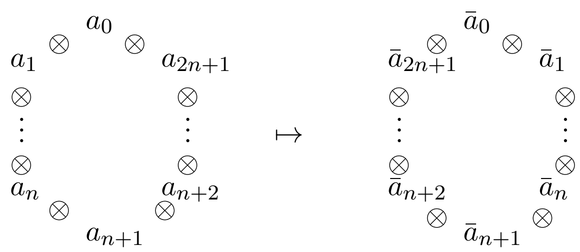
$$\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\text{fix}}) \cong B(\underline{R}^{\text{fix}}, N_e^{C_2} i_e^* \underline{R}^{\text{fix}}, \underline{R}^{\text{fix}}) \cong \underline{B(R, R \otimes R, R)}^{\text{fix}}$$

where C_2 acts on $R \otimes R$ as in (5.2).

So in every simplicial degree n, $\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\text{fix}})_n = \underline{R}^{\text{fix}} \Box (N_e^{C_2} i_e^* \underline{R}^{\text{fix}})^{\Box n} \Box \underline{R}^{\text{fix}}$ is the fixed-point Tambara functor of the C_2-ring $R \otimes (R \otimes R)^{\otimes n} \otimes R$ with C_2-action given by

$$\begin{aligned} &\tau(a_0 \otimes (a_1 \otimes a_{2n+1}) \otimes (a_2 \otimes a_{2n}) \otimes \cdots \otimes (a_n \otimes a_{n+2}) \otimes a_{n+1}) \\ &\quad = \bar{a}_0 \otimes (\bar{a}_{2n+1} \otimes \bar{a}_1) \otimes (\bar{a}_{2n} \otimes \bar{a}_2) \otimes \cdots \otimes (\bar{a}_{n+2} \otimes \bar{a}_n) \otimes \bar{a}_{n+1}. \end{aligned}$$

One can visualize this C_2-action as



Remark 5.2. Note that $N_e^{C_2} i_e^* \underline{R}^{\text{fix}}$ is *not* equal to $\underline{(R \otimes R)}^{\text{fix}}$, even in very simple cases! For example, for $R = \mathbb{Z}$ with the trivial C_2-action, $\underline{(R \otimes R)}^{\text{fix}}$ is just $\underline{\mathbb{Z}}^c$ with respect to the constant action, while $N_e^{C_2} i_e^* \underline{\mathbb{Z}}^c$ is the C_2-Burnside Tambara functor (see for instance [LRZb, (5.1)]). We need an outer copy of $\underline{R}^{\text{fix}}$ in Theorem 5.1 as a catalyst in order to achieve the desired simplification.

Proof. The proof follows by induction on n. The base case $n = 0$ is Lemma 3.1 applied to $M = N = R$, and the inductive step can be done with the help of Lemma 3.2 for $M = R \otimes (R \otimes R)^{\otimes (n-1)} \otimes R$. Note that both lemmas proceed by identifying all the terms to the C_2-coinvariant (second) part of the box product on C_2/C_2, so these identifications of the term $\underline{R}^{\text{fix}} \Box (N_e^{C_2} i_e^* \underline{R}^{\text{fix}})^{\Box n} \Box \underline{R}^{\text{fix}}$ with $\underline{(R \otimes (R \otimes R)^{\otimes n} \otimes R)}^{\text{fix}}$ behave as one would expect for internal multiplications. See also the comment below Lemma 3.2. Therefore, these identifications are compatible with the simplicial structure maps. $\Box$

Remark 5.3. If R is a commutative ring with involution and if M is an R-module with involution compatible with the involution on R in the sense that $\overline{rm} = \bar{r}\bar{m}$ for all $r \in R$, $m \in M$, then the C_2-Mackey functor $\underline{M}^{\text{fix}}$ is a symmetric bimodule over the C_2-Tambara functor $\underline{R}^{\text{fix}}$.

Loday constructions on based G-simplicial sets of a G-Tambara functor with coefficients in a G-Mackey functor which is a bimodule over the G-Tambara functor are defined analogously to those in the non-equivariant case. We place the coefficients at the basepoint in each simplicial degree.

The proof of Theorem 5.1 shows that if 2 is invertible in R, we obtain that

$$\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\text{fix}}; \underline{M}^{\text{fix}}) \cong B(\underline{R}^{\text{fix}}, N_e^{C_2} i_e^* \underline{R}^{\text{fix}}, \underline{M}^{\text{fix}}) \cong \underline{B(R, R \otimes R, M)}^{\text{fix}}$$

where C_2 acts on $R \otimes R$ as in (5.2).

6. Relating $\mathcal{L}^{C_2}_{S^\sigma}(\underline{R})$ to reflexive homology

Let us for now consider a more general context: Let k be a commutative ring and let A be an associative k-algebra. We assume that A carries an anti-involution that we denote by $a \mapsto \bar{a}$ and which we assume to be k-linear. Let M be an A-bimodule with an involution $m \mapsto \bar{m}$ that is compatible with the bimodule structure over A in the sense that $\overline{amb} = \bar{b}\bar{m}\bar{a}$ for all $a, b \in A$, $m \in M$. All tensor products will be over k in this section, unless otherwise indicated.

Graves [Gra24, Definition 1.8] defines an involution on every level of the Hochschild complex $\mathsf{CH}^k_n(A; M) = M \otimes A^{\otimes n}$ by

$$r_n(m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \bar{m} \otimes \bar{a}_n \otimes \cdots \otimes \bar{a}_2 \otimes \bar{a}_1. \tag{6.1}$$

For the face maps of the Hochschild complex we get that $r_{n-1} \circ d_i = d_{n-i} \circ r_n$, so these levelwise maps do not preserve the simplicial structure but they reverse it. Since this relation implies that $d \circ r_n = (-1)^n r_{n-1} \circ d$, applying r_n at each level n does *not* induce a map on the associated chain complexes, unless we adjust the signs.

The C_2-actions given by the r_n-maps together with the simplicial structure maps on $\mathsf{CH}^k_\cdot(A; M)$ turn $\mathsf{CH}^k_\cdot(A; M)$ into a functor from the crossed simplicial group ΔR^{op} in the sense of Fiedorowicz-Loday [FL91] to the category of k-modules. In [Gra24, Definition 1.9], Graves defines reflexive homology as functor homology as follows:

$$\mathsf{HR}^{+,k}_*(A; M) = \mathsf{Tor}^{\Delta R^{\mathrm{op}}}_*(k^*, \mathsf{CH}^k_\cdot(A; M)).$$

Here k^* is the constant right ΔR^{op}-module with value k at all objects. In [Gra24, Definition 2.1], he defines a bicomplex $C_{*,*}$ which is a bi-resolution of k^*. With its help he shows in [Gra24, Proposition 2.4] that $\mathsf{HR}^{+,k}_*(A; M)$ is the homology of the complex $\mathsf{CH}^k_*(A; M)/(1 - r)$, where r is obtained from the maps r_n of (6.1) by

$$r(m \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^{\frac{n(n+1)}{2}} r_n(m \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^{\frac{n(n+1)}{2}} \bar{m} \otimes \bar{a}_n \otimes \cdots \otimes \bar{a}_1. \tag{6.2}$$

With this choice of sign, the map r is a chain map, so the quotient by $1 - r$ is still a chain complex.

Theorem 6.1. *Assume that k is a commutative ring, that A is an associative k-algebra with an anti-involution as above whose underlying k-module is flat. Let M be an A-bimodule with a compatible involution as above, and assume that 2 is invertible in A. Then there is a C_2-equivariant quasi-isomorphism of chain complexes*

$$B^k_*(A, A \otimes A^{\mathrm{op}}, M) \to \mathsf{CH}^k_*(A; M). \tag{6.3}$$

Here the generator τ of C_2 acts diagonally on $B^k_*(A, A \otimes A^{\mathrm{op}}, M)$, where the action on $A \otimes A^{\mathrm{op}}$ is given by $\tau(a \otimes b) = \bar{b} \otimes \bar{a}$. On the Hochschild chain complex C_2 acts via r.

Corollary 6.2. *Under the assumptions of Theorem 6.1, we get homology isomorphisms*

$$H_*(B^k_*(A, A \otimes A^{\mathrm{op}}, M)) \cong \mathsf{HH}^k_*(A; M) \tag{6.4}$$

$$H_*(B^k_*(A, A \otimes A^{\mathrm{op}}, M)^{C_2}) \cong \mathsf{HR}^{+,k}_*(A; M). \tag{6.5}$$

Proof. We get the first isomorphism because the map of Theorem 6.1 is a quasi-isomorphism. It also follows from the fact that both complexes calculate $\mathsf{Tor}^{A \otimes A^{\mathrm{op}}}_*(A, M)$ because of the assumption that A is flat over k. Note that in the case $M = A$ the first isomorphism also follows from the fact that the bar construction on the left is isomorphic to the Segal-Quillen subdivision of the Hochschild complex [Seg73].

The second isomorphism follows from the fact that 2 is invertible in both complexes: As 2 is invertible in A, the unit of A, $k \to A$ factors through $k[\frac{1}{2}]$. We can express every level of each of the complexes as the direct sum of the $+1$-eigenspace and the -1-eigenspace of the action of the generator of C_2 on them. Since the actions commute with d, in fact each of the complexes breaks up as the direct sum of a positive subcomplex and a negative subcomplex.

Since the quasi-isomorphism is a C_2-map, it preserves this decomposition, and as it is a quasi-isomorphism, it must be a quasi-isomorphism on the positive and negative subcomplexes, respectively. That means that we get a quasi-isomorphism

$$H_*(B^k_*(A, A\otimes A^{\mathrm{op}}, M)^{C_2}) \to H_*(\mathsf{CH}^k_*(A; M)^{C_2}),$$

but since 2 is invertible, we have a chain isomorphism

$$\mathsf{CH}^k_*(A; M)^{C_2} \to \mathsf{CH}^k_*(A; M)_{C_2} = \mathsf{CH}^k_*(A; M)/(1-r),$$

and hence the claim follows with [Gra24, Proposition 2.4]. □

Proof of Theorem 6.1. We consider two $A\otimes A^{\mathrm{op}}$-flat resolutions of A: We use $B^k_*(A, A\otimes A^{\mathrm{op}}, A\otimes A^{\mathrm{op}})$ with $A\otimes A^{\mathrm{op}}$ acting on the rightmost coordinate and $B^k_*(A, A, A)$ where A^{op} acts on the left and A on the right, as in the Tor-identification of Hochschild homology. We let C_2 act on $B_*(A, A\otimes A^{\mathrm{op}}, A\otimes A^{\mathrm{op}})$ by acting diagonally on all the coordinates, and denote the action of the generator on it by τ. This action is simplicial, and therefore commutes with d. We let C_2 act on $B^k_*(A, A, A)$ by setting

$$r(a_0\otimes a_1\otimes\cdots\otimes a_n\otimes a_{n+1}) = (-1)^{\frac{n(n+1)}{2}}\bar a_{n+1}\otimes\bar a_n\otimes\cdots\otimes\bar a_1\otimes\bar a_0.$$

Because of the sign adjustment, r is a chain map. We only know that the two resolutions are flat, not that they are projective. But any chain map between them that covers the identity on A induces an isomorphism on H_0, which is the only nontrivial homology group for both complexes, and therefore is a quasi-isomorphism.

We define $f_n\colon B^k_n(A, A\otimes A^{\mathrm{op}}, A\otimes A^{\mathrm{op}}) \to B^k_n(A, A, A)$ as

$$\begin{aligned} f_n(a_0 &\otimes (a_1\otimes a_{2n+2})\otimes(a_2\otimes a_{2n+1})\otimes\cdots\otimes(a_{n+1}\otimes a_{n+2})) \\ &= a_{n+2}a_{n+3}\cdots a_{2n+1}a_{2n+2}a_0\otimes a_1\otimes a_2\otimes\cdots\otimes a_{n+1}. \end{aligned}$$

This is a simplicial $A\otimes A^{\mathrm{op}}$-map, and covers the identity on the A being resolved since in level 0 it sends $a_0\otimes(a_1\otimes a_2)$ to $a_2a_0\otimes a_1$ and both of these map down to $a_2a_0a_1\in A$. This map is not C_2-equivariant, but if we define $g := r\circ f\circ\tau$, we get $g_n\colon B^k_n(A, A\otimes A^{\mathrm{op}}, A\otimes A^{\mathrm{op}}) \to B^k_n(A, A, A)$ with

$$\begin{aligned} g_n(a_0 &\otimes (a_1\otimes a_{2n+2})\otimes(a_2\otimes a_{2n+1})\otimes\cdots\otimes(a_{n+1}\otimes a_{n+2})) \\ &= (-1)^{\frac{n(n+1)}{2}} a_{n+2}\otimes a_{n+3}\otimes\cdots\otimes a_{2n+1}\otimes a_{2n+2}\otimes a_0a_1a_2\cdots a_{n+1}. \end{aligned}$$

This is not a simplicial map but it is an $A\otimes A^{\mathrm{op}}$-map and it is a chain map since r, f, and τ are chain maps. Again, it covers the identity on A since on level 0, $a_0\otimes(a_1\otimes a_2)\mapsto a_2\otimes a_0a_1$ and both of these map down to $a_2a_0a_1\in A$.

We now use the fact that 2 is invertible in A and consider the map

$$\frac{f+g}{2}\colon B^k_*(A, A\otimes A^{\mathrm{op}}, A\otimes A^{\mathrm{op}}) \to B^k_*(A, A, A),$$

which is an $A\otimes A^{\mathrm{op}}$-map and covers the identity on A since f and g are such maps. This map is also equivariant because

$$r\circ\frac{f+g}{2} = r\circ\frac{f+r\circ f\circ\tau}{2} = \frac{r\circ f+f\circ\tau}{2} = \frac{r\circ f\circ\tau+f}{2}\circ\tau = \frac{f+g}{2}\circ\tau.$$

So $\frac{f+g}{2}$ is a quasi-isomorphism of flat $A\otimes A^{\mathrm{op}}$-complexes. By Lemma 6.3 below, if we tensor it over $A\otimes A^{\mathrm{op}}$ with the $A\otimes A^{\mathrm{op}}$-module M, we get a quasi-isomorphism

$$\frac{f+g}{2}\otimes\mathrm{id}_M\colon B^k_*(A, A\otimes A^{\mathrm{op}}, M) \to \mathsf{CH}^k_*(A; M).$$

This map is equivariant because it is the tensor product of two equivariant maps. □

Lemma 6.3. *Let R be an associative ring and let $\phi\colon C_* \to D_*$ be a quasi-isomorphism between two bounded below chain complexes of flat right R-modules. Let M be a left R-module. Then $\phi \otimes \mathrm{id}_M\colon C_* \otimes_R M \to D_* \otimes_R M$ is a quasi-isomorphism as well.*

Proof. Since ϕ is a quasi-isomorphism, its mapping cone, cone(ϕ), is acyclic. The mapping cone is also a bounded-below chain complex of flat right R-modules, so it can be viewed as a flat resolution of the 0-module, possibly with a shift. We suspend it, so that $\Sigma^a\mathrm{cone}(\phi)$ is a non-negative chain complex whose bottom chain group is in degree zero. Since flat resolutions can be used to calculate Tor,

$$H_*(\Sigma^a\mathrm{cone}(\phi) \otimes_R M) \cong \mathsf{Tor}_*^R(0, M) = 0$$

for all $*$. So $\Sigma^a\mathrm{cone}(\phi) \otimes_R M$ and hence $\mathrm{cone}(\phi) \otimes_R M = \mathrm{cone}(\phi \otimes_R \mathrm{id}_M)$ is acyclic. But that forces $\phi \otimes_R \mathrm{id}_M$ to be a quasi-isomorphism. □

Taking our identification of $\mathcal{L}^{C_2}_{S^\sigma}(\underline{R})$ with $\underline{B(R, R \otimes R, R)}^{\mathrm{fix}}$ from Theorem 5.1 together with Corollary 6.2 we obtain the following comparison result between the homology groups of the C_2-Loday construction for the circle S^σ and $\underline{R}^{\mathrm{fix}}$ on the one hand and the reflexive homology groups on the other hand:

Theorem 6.4. *Assume that R is a commutative ring with involution and that 2 is invertible in R. If the underlying abelian group of R is flat over $\mathbb{Z}$, then*

$$\pi_*(\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\mathrm{fix}})(C_2/C_2)) \cong \mathsf{HR}^{+,\mathbb{Z}}_*(R, R).$$

The relative version follows directly from Corollary 6.2, Theorem 4.1, and the identification in (5.1):

Theorem 6.5. *Assume that R is a commutative k-algebra with a k-linear involution and that 2 is invertible in R. If the underlying module of R is flat over k, then*

$$\pi_*(\mathcal{L}^{C_2,\underline{k}^c}_{S^\sigma}(\underline{R}^{\mathrm{fix}})(C_2/C_2)) \cong \mathsf{HR}^{+,k}_*(R, R).$$

7. Involutive Hochschild homology as a Loday construction

Involutive Hochschild cohomology was defined in [Bra14]. Fernàndez-València and Giansiracusa extended the definition to involutive homology. The input is an associative algebra with anti-involution and in [FVG18] the authors work relative to a field k.

A straightforward generalization of their definition [FVG18, Definition 3.3.1] to arbitrary commutative ground rings is as follows:

Definition 7.1. Let k be a commutative ring, let A be an associative algebra with anti-involution and let M be an involutive A-bimodule. The involutive Hochschild homology groups of A with coefficients in M are

$$\mathsf{iHH}^k_*(A; M) = \mathsf{Tor}^{A^{ie}}_*(A; M).$$

Here A^{ie} is the involutive enveloping algebra. As in the classical case its role is to describe (involutive) A-bimodules: There is an equivalence of categories between the category of involutive A-bimodules and the category of modules over A^{ie} [FVG18, Proposition 2.2.1]. As a k-module

$$A^{ie} = A \otimes_k A \otimes_k k[C_2]$$

and the multiplication on A^{ie} is determined by

$$(a \otimes b \otimes \tau^i) \cdot (c \otimes d \otimes \tau^j) = (a \otimes b) \cdot \tau^i(c \otimes d) \otimes \tau^{i+j}.$$

Here, $\tau(c \otimes d)$ is again $\bar{d} \otimes \bar{c}$, so

$$(a \otimes b \otimes \tau) \cdot (c \otimes d \otimes \tau^j) = (a\bar{d} \otimes \bar{c}b) \otimes \tau^{1+j}.$$

Hence we can view A^{ie} as a twisted group algebra $(A \otimes A^{\mathrm{op}})[C_2]$. As before, every involutive algebra A is an involutive A-bimodule.

Of course we know from the classical setting of Hochschild homology that the above definition does not yield what you want if A is not flat as a k-module.

We obtain a comparison theorem between involutive Hochschild homology and the homology of the C_2-Loday construction of the circle S^σ for $\underline{R}^{\mathrm{fix}}$.

Theorem 7.2. *Let R be a commutative ring with a C_2-action. Assume that 2 is invertible in R and that the underlying abelian group of R is flat. Then*

$$\pi_*(\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\mathrm{fix}})(C_2/C_2)) \cong \mathsf{iHH}^{\mathbb{Z}}_*(R).$$

And again over a general commutative k, there is a relative version:

Theorem 7.3. *Let k be a commutative ring and let R be a commutative k-algebra with a k-linear C_2-action. Assume that 2 is invertible in R and that the underlying k-module of R is flat. Then*

$$\pi_*(\mathcal{L}^{C_2,\underline{k}^c}_{S^\sigma}(\underline{R}^{\mathrm{fix}})(C_2/C_2)) \cong \mathsf{iHH}^{k}_*(R).$$

We prove Theorem 7.2 by comparing $\mathsf{iHH}_*(R;M)$ for an involutive R-bimodule M to the C_2/C_2-level of the Mackey functor $\underline{B(R, R \otimes R^{\mathrm{op}}, M)}^{\mathrm{fix}}$ where C_2 acts on $R \otimes R$ by $\tau(a \otimes b) = \bar{b} \otimes \bar{a}$. The lemmata below should be used for $k = \mathbb{Z}$. The proof of Theorem 7.3 is similar, just over a general commutative ground ring k.

In the following we will always assume that R is a commutative k-algebra with a k-linear C_2-action, that 2 is invertible in R and that the underlying k-module of R is flat over k.

Lemma 7.4.

$$\pi_0(\underline{B^k_*(R, R \otimes_k R, M)}^{\mathrm{fix}}(C_2/C_2)) \cong R \otimes_{R^{ie}} M.$$

Proof. As 2 is invertible, taking C_2-fixed points is isomorphic to taking C_2-coinvariants and both functors are exact. Thus we have to identify the quotient of $(R \otimes M)_{C_2}$ by the bimodule action and this yields $(R \otimes_{R \otimes_k R} M)_{C_2}$ which is isomorphic to $(M/\{am - ma, a \in R, m \in M\})_{C_2}$. By [FVG18, Proposition 2.4.1], $R \otimes_{R^{ie}} M$ is isomorphic to the pushout of

$$\begin{array}{ccc} M & \longrightarrow & M_{C_2} \\ \downarrow & & \\ M/\{am - ma, a \in R, m \in M\} & & \end{array}$$

and this proves the claim. ∎

Lemma 7.5. *Assume that $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of R^{ie}-modules and abbreviate $\underline{B^k(R, R \otimes_k R, M_i)}^{\mathrm{fix}}(C_2/C_2)$ by BM_i. Then we get an induced long exact sequence*

$$\ldots \longrightarrow \pi_n BM_1 \longrightarrow \pi_n BM_2 \longrightarrow \pi_n BM_3 \longrightarrow \pi_{n-1} BM_1 \longrightarrow \ldots$$

Proof. As we assume that R is flat over k, tensoring with R is exact, and as 2 is invertible, taking fixed points is exact. Therefore, in every simplicial degree n, the sequence

$$0 \to (BM_1)_n \to (BM_2)_n \to (BM_3)_n \to 0$$

is short exact and hence we obtain a short exact sequence of the associated chain complexes

$$0 \to C_*(BM_1) \to C_*(BM_2) \to C_*(BM_3) \to 0$$

and an induced long exact sequence on homology. As $H_* C_*(BM_i) \cong \pi_*(BM_i)$, the claim follows. ∎

Lemma 7.6. *Assume that P is a projective R^{ie}-module. Then $\pi_n \underline{B^k(R, R \otimes_k R, P)}^{\mathrm{fix}}(C_2/C_2) \cong 0$ for all positive n.*

Proof. In the category of R^{ie}-modules, R^{ie} is a projective generator and every module can be written as a quotient of a direct sum of copies of R^{ie}. Our construction sends a direct sum of modules to a direct sum of simplicial objects, yielding a direct sum of associated chain complexes. Retracts of modules give retracts of the associated chain complexes. It therefore suffices to check the claim for $P = R^{ie}$.

If D is any k-module with a C_2-action such that 2 acts invertibly on D, then there is an isomorphism

$$(D \otimes_k k[C_2])^{C_2} \cong D$$

where on the left hand side we consider the diagonal C_2-action: First note that $D \otimes_k k[C_2]$ with the diagonal action is isomorphic to $D \otimes_k k[C_2]$ where the C_2-action is only on the right-hand factor. The isomorphism $\psi\colon D \otimes_k k[C_2] \to D \otimes_k k[C_2]$ sends a generator $d \otimes \tau^i$ to $\tau^{-i} d \otimes \tau^i$. Then, as 2 acts invertibly, we have

$$(D \otimes_k k[C_2])^{C_2} \cong (D \otimes_k k[C_2])_{C_2} = (D \otimes_k k[C_2]) \otimes_{k[C_2]} k.$$

So in total, $(D \otimes_k k[C_2])^{C_2} \cong D$.

Therefore, in every simplicial degree n we can identify

$$\underline{B_n^k(R, R \otimes_k R, R^{ie})}^{\mathrm{fix}}(C_2/C_2) = (R \otimes_k (R \otimes_k R)^{\otimes_k n} \otimes_k (R \otimes_k R \otimes_k k[C_2]))^{C_2}$$

with $R \otimes_k (R \otimes_k R)^{\otimes_k n} \otimes_k (R \otimes_k R)$. But then we are left with the bar construction $B^k(R, R \otimes_k R, R \otimes_k R)$ and this has trivial homotopy groups in positive degrees. □

Proposition 7.7. *Assume that R is a commutative k-algebra with a k-linear involution such that 2 is invertible in R and assume that M is an involutive R-bimodule. Then*

$$\pi_* \underline{B^k(R, R \otimes_k R, M)}^{\mathrm{fix}}(C_2/C_2) \cong \mathsf{iHH}_*^k(R; M).$$

Proof. Lemmata 7.4, 7.5 and 7.6 imply that $\pi_* \underline{B^k(R, R \otimes_k R, -)}^{\mathrm{fix}}(C_2/C_2)$ has the same axiomatic description as $\mathsf{Tor}_*^{R^{ie}}(R; -)$. □

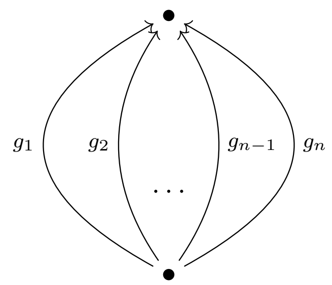
Proof of Theorems 7.2 and 7.3. Theorem 7.2 is a special case of Proposition 7.7 working with $k = \mathbb{Z}$ (although we are working over the C_2-Burnside Tambara functor, not over $\mathbb{Z}^c$) and with $M = R$. Theorem 7.3 is the relative version. □

Remark 7.8. Graves states a comparison result in [Gra24, Theorem 9.1] between reflexive homology, $\mathsf{HR}_*^{+,k}(A; M)$, and involutive Hochschild homology, $\mathsf{iHH}_*^k(A; M)$. The assumptions are slightly too restrictive there: Fernàndez-València and Giansiracusa prove in [FVG18, Proposition 3.3.3] that $\mathsf{iHH}_*^k(A; M) \cong \mathsf{HH}_*^k(A; M)_{C_2}$ if the characteristic of the ground field is different from 2 and Graves shows in [Gra24, Proposition 2.4], that $\mathsf{HH}_*^k(A; M)_{C_2} \cong \mathsf{HR}_*^{+,k}(A; M)$ if 2 is invertible in the ground ring. The assumption on A being projective as a $k[C_2]$-module comes for free if we work over a field of characteristic different from 2 thanks to Maschke's theorem. For an arbitrary ring R, we also get that an arbitrary $R[G]$-module M is projective if M is projective as an R-module and if $|G|$ is invertible in R [Mer17, Proposition 4.4].

Remark 7.9. If a finite group G carries a homomorphism $\varepsilon\colon G \to C_2$, then one can consider an associated crossed simplicial group and the corresponding (co)homology theory, see for instance [KP18, AKMP].

For an arbitrary finite group $C_2 \neq G \neq \{e\}$ without an interesting homomorphism to C_2, there is only the version of an associated crossed simplicial group by viewing G as a constant simplicial group because there is no meaningful way in which G can act on the simplicial category. Then the group elements commute with the simplicial structure maps.

On the other hand, if G is a group with n elements $\{g_1, \dots, g_n\}$, we can consider the *unreduced suspension of* G, SG. This is the graph



and the group G acts by sending an element $g \in G$ and an edge labelled by g_i to the edge gg_i. We will view this graph as a finite simplicial G-set.

Thus if R is a commutative algebra with a G-action, then

$$\pi_* \mathcal{L}^G_{SG}(\underline{R}^{\text{fix}})(G/G) \tag{7.1}$$

is a perfectly fine homology theory. We propose (7.1) as a generalization of reflexive homology to arbitrary finite groups, at least if $|G|$ is invertible, and will investigate its properties in future work.

8. The cases $\underline{\mathbb{F}_2}^c$ and $\underline{\mathbb{Z}}^c$

For our results we had to assume that 2 is invertible in our commutative ring and that the underlying abelian group is flat. So it is a natural question to ask what happens if we drop these assumptions. We first study the simplest and most extreme case.

8.1. **Comparison for $\underline{\mathbb{F}_2}^c$.** We consider $\mathbb{F}_2$ with the trivial C_2-action, so the fixed point Tambara functor is the constant Tambara functor: $\underline{\mathbb{F}_2}^c = \underline{\mathbb{F}_2}^{\text{fix}}$. Graves calculated reflexive homology of the ground ring in [Gra24, Proposition 5.1] and in the case of $\mathbb{F}_2$ we obtain

$$\mathsf{HR}^{+,\mathbb{F}_2}_*(\mathbb{F}_2) \cong H_*(BC_2, \mathbb{F}_2)$$

and this is $\mathbb{F}_2$ in all non-negative degrees. Note that here it doesn't matter whether we view $\mathbb{F}_2$ as a commutative $\mathbb{F}_2$-algebra or as a commutative ring (a commutative $\mathbb{Z}$-algebra). Similarly, we can calculate the involutive Hochschild homology of $\mathbb{F}_2$ as an involutive $\mathbb{F}_2$-algebra (or as a commutative $\mathbb{Z}$-algebra) and obtain

$$\mathsf{iHH}^{\mathbb{F}_2}_*(\mathbb{F}_2; \mathbb{F}_2) = \mathsf{Tor}^{\mathbb{F}_2[C_2]}_n(\mathbb{F}_2, \mathbb{F}_2) \cong H_*(BC_2; \mathbb{F}_2).$$

Hence, involutive Hochschild homology and reflexive homology agree in this case.

If we compare this to the 2-sided bar construction $B(\mathbb{F}_2, \mathbb{F}_2 \otimes \mathbb{F}_2, \mathbb{F}_2) \cong B^{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2 \otimes \mathbb{F}_2, \mathbb{F}_2)$, then this bar construction is just the constant simplicial object with value $\mathbb{F}_2$ and therefore here we obtain

$$\pi_n \underline{B(\mathbb{F}_2, \mathbb{F}_2 \otimes \mathbb{F}_2, \mathbb{F}_2)}^{\text{fix}}(C_2/C_2) = \pi_n B(\mathbb{F}_2, \mathbb{F}_2 \otimes \mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence in this case $\pi_* \underline{B(\mathbb{F}_2, \mathbb{F}_2 \otimes \mathbb{F}_2, \mathbb{F}_2)}^{\text{fix}}(C_2/C_2)$ agrees neither with reflexive homology nor with involutive Hochschild homology.

What about $\pi_* B(\underline{\mathbb{F}_2}^c, N_e^{C_2}(\mathbb{F}_2)(C_2/C_2), \underline{\mathbb{F}_2}^c)(C_2/C_2)$? Note that

$$N_e^{C_2}(\mathbb{F}_2)(C_2/C_2) \cong \mathbb{Z}/4\mathbb{Z} \quad \text{and } N_e^{C_2}(\mathbb{F}_2)(C_2/e) \cong \mathbb{F}_2.$$

In $\underline{\mathbb{F}_2}^c \Box N_e^{C_2}(\mathbb{F}_2)$ we obtain:

$$\begin{aligned} C_2/C_2: &\quad (\mathbb{F}_2 \otimes \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{F}_2 \otimes \mathbb{F}_2 \otimes \mathbb{F}_2)/C_2)\,/\mathrm{FR} \\ C_2/e: &\quad \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes \mathbb{F}_2 \cong \mathbb{F}_2 \end{aligned}$$

The C_2-Weyl action is trivial on $\mathbb{F}_2 \otimes \mathbb{F}_2 \otimes \mathbb{F}_2 \cong \mathbb{F}_2$. Frobenius reciprocity yields

$$[1 \otimes 1 \otimes 1] = [\mathsf{res}(1) \otimes 1 \otimes 1] \sim 1 \otimes \mathsf{tr}(1 \otimes 1) = 2 \cdot 1 \otimes 1 \otimes 1 = 0$$

so at the C_2/C_2-level we are left with one copy of $\mathbb{F}_2$ and we obtain $\underline{\mathbb{F}_2}^c \Box N_e^{C_2}(\mathbb{F}_2) \cong \underline{\mathbb{F}_2}^c$.

This identifies $B(\underline{\mathbb{F}_2}^{\mathrm{fix}}, N_e^{C_2}(\mathbb{F}_2)(C_2/C_2), \underline{\mathbb{F}_2}^{\mathrm{fix}})(C_2/C_2)$ with the constant simplicial object with value $\mathbb{F}_2$, and therefore

$$\pi_n \mathcal{L}_{S^\sigma}^{C_2}(\underline{\mathbb{F}_2}^{\mathrm{fix}})(C_2/C_2) \cong \begin{cases} \mathbb{F}_2, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

At the free orbit, we also get the constant simplicial object with value $\mathbb{F}_2$, an in total we get an isomorphism of simplicial Tambara functors between $\mathcal{L}_{S^\sigma}^{C_2}(\underline{\mathbb{F}_2}^c)$ and $\underline{B(\mathbb{F}_2, \mathbb{F}_2 \otimes \mathbb{F}_2, \mathbb{F}_2)}^c$.

8.2. **Comparison for $\underline{\mathbb{Z}}^c$.** We consider the ring of integers and this only carries a trivial C_2-action. We know that norm restriction of $\underline{\mathbb{Z}}^c$ gives the C_2-Burnside Tambara functor, $N_e^{C_2} i_e^* \underline{\mathbb{Z}}^c \cong \underline{A}$. This is the monoidal unit for the $\Box$-product. We showed in [LRZa, Lemma 5.1] that for two arbitrary commutative rings A and B, $\underline{A}^c \Box \underline{B}^c \cong \underline{(A \otimes B)}^c$ and hence

$$\underline{\mathbb{Z}}^c \Box \underline{\mathbb{Z}}^c \cong \underline{(\mathbb{Z} \otimes \mathbb{Z})}^c \cong \underline{\mathbb{Z}}^c.$$

Proposition 8.1. *There is an isomorphism of simplicial C_2-Tambara functors*

$$\mathcal{L}_{S^\sigma}^{C_2}(\underline{\mathbb{Z}}^c) \cong \underline{\mathbb{Z}}^c$$

where the right-hand side denotes the constant simplicial C_2-Tambara functor with value $\underline{\mathbb{Z}}^c$.

Proof. By the above arguments we get for an arbitrary simplicial degree n:

$$\begin{aligned} \mathcal{L}_{S^\sigma}^{C_2}(\underline{\mathbb{Z}}^c)_n &= \underline{\mathbb{Z}}^c \Box (N_e^{C_2} i_e^* \underline{\mathbb{Z}}^c)^{\Box n} \Box \underline{\mathbb{Z}}^c \\ &\cong \underline{\mathbb{Z}}^c \Box \underline{\mathbb{Z}}^c \\ &\cong \underline{\mathbb{Z}}^c. \end{aligned}$$

The simplicial structure maps induce the identity maps under these isomorphisms. □

Corollary 8.2. *The homotopy groups of $\mathcal{L}_{S^\sigma}^{C_2}(\underline{\mathbb{Z}}^c)$ are*

$$\pi_*(\mathcal{L}_{S^\sigma}^{C_2}(\underline{\mathbb{Z}}^c)) \cong \begin{cases} \underline{\mathbb{Z}}^c, & * = 0, \\ 0, & * > 0. \end{cases}$$

Corollary 8.3. *For the C_2-Tambara functor $\underline{\mathbb{Z}}^c$ the homotopy groups*

$$\pi_*(\mathcal{L}_{S^\sigma}^{C_2}(\underline{\mathbb{Z}}^c))(C_2/C_2)$$

are neither isomorphic to $\mathsf{HR}_*^{+,\mathbb{Z}}(\mathbb{Z})$ nor to $\mathsf{iHH}_*^{\mathbb{Z}}(\mathbb{Z})$.

Proof. We saw above that $\pi_*(\mathcal{L}_{S^\sigma}^{C_2}(\underline{\mathbb{Z}}^c))(C_2/C_2)$ is concentrated in degree $* = 0$ with value $\mathbb{Z}$ whereas $\mathsf{HR}_*^{+,\mathbb{Z}}(\mathbb{Z})$ and $\mathsf{iHH}_*^{\mathbb{Z}}(\mathbb{Z})$ both give $H_*(C_2; \mathbb{Z})$. □

9. The case of associative C_2-Green functors

We assume now that R is an associative ring with anti-involution. The fixed point Mackey functor $\underline{R}^{\text{fix}}$ then has no multiplicative structure: For $a, b \in R^{C_2}$ we get that $\tau(ab) = \bar{b}\bar{a} = ba$ and as a and b do not necessarily commute, ab is not, in general, a fixed point. But we can still define a replacement of the norm-restriction object that we call $\tilde{N}_e^{C_2} i_e^*(\underline{R}^{\text{fix}})$ in order to avoid confusion with the commutative case. We claim that this can be done in the setting of associative C_2-Green functors.

Definition 9.1. We define $\tilde{N}_e^{C_2} i_e^*(\underline{R}^{\text{fix}})$ at the free level as

$$\tilde{N}_e^{C_2} i_e^*(\underline{R}^{\text{fix}})(C_2/e) := R \otimes R^{\text{op}}$$

and at the trivial orbit C_2/C_2 we define

$$\tilde{N}_e^{C_2} i_e^*(\underline{R}^{\text{fix}})(C_2/C_2) := (\mathbb{Z}\{R\} \oplus (R \otimes R^{\text{op}})/C_2)/\text{TR},$$

where the Tambara reciprocity relation, TR, identifies $\{a + b\} \sim \{a\} + \{b\} + [a \otimes \bar{b}]$ for all $a, b \in R$, just as in the norm-restriction construction in the commutative case.

Lemma 9.2. *We can endow $\tilde{N}_e^{C_2} i_e^*(\underline{R}^{\text{fix}})$ with the structure of an associative C_2-Green functor.*

Proof. We first establish the structure of a C_2-Mackey functor.

The restriction map is

$$\mathsf{res}\{a\} := a \otimes \bar{a}, \qquad \mathsf{res}[a \otimes b] := a \otimes b + \bar{b} \otimes \bar{a}$$

and the transfer sends $a \otimes b$ to

$$\mathsf{tr}(a \otimes b) := [a \otimes b].$$

Then this is completely analogous to the commutative case, so indeed, this *does* define a C_2-Mackey functor.

For the structure of a Green functor we have to specify a multiplication and its compatibility with respect to the restriction map.

The multiplication is set to the usual multiplication on $R \otimes R^{\text{op}}$ at the free level and

$$\begin{aligned} \{a\}\{b\} &:= \{ab\}, \\ \{a\}[b \otimes c] &:= [ab \otimes c\bar{a}], \\ [a \otimes b]\{c\} &:= [ac \otimes \bar{c}b], \\ [a \otimes b][c \otimes d] &:= [ac \otimes db] + [a\bar{d} \otimes \bar{c}b]. \end{aligned}$$

Straighforward calculations yield

$$\{a\}[b \otimes c] = \{a\}[\bar{c} \otimes \bar{b}] \text{ and } [a \otimes b]\{c\} = [\bar{b} \otimes \bar{a}]\{c\}$$

and also

$$[a \otimes b][c \otimes d] = [\bar{b} \otimes \bar{a}][c \otimes d] = [a \otimes b][\bar{d} \otimes \bar{c}]$$

so that the multiplication is well-defined on the Weyl-classes. Another direct calculation shows that the multiplication is associative.

For the unit map $\underline{A} \to \tilde{N}_e^{C_2} i_e^* \underline{R}^{\text{fix}}$ we don't have any choice at the free level: There we have to send $\mathbb{Z} = \underline{A}(C_2/e)$ to $R \otimes R^{\text{op}}$ by sending 1 to $1 \otimes 1$. At the trivial level we have to send $1 \in \underline{A}(C_2/C_2) = \mathbb{Z}[t]/t^2 - 2t$ to $\{1\}$ and the compatibility with tr forces us to send t to $\mathsf{tr}(1 \otimes 1) = [1 \otimes 1]$. It is straightforward to check that this actually defines units of the rings at each level.

Restriction is multiplicative on norms because

$$\mathsf{res}\{ab\} = ab \otimes \overline{ab} = ab \otimes \bar{b}\bar{a} = a \otimes \bar{a} \cdot b \otimes \bar{b} = \mathsf{res}\{a\} \cdot \mathsf{res}\{b\}.$$

It is also multiplicative on transfers:

$$\mathsf{res}[a \otimes b] \cdot \mathsf{res}[c \otimes d] = ac \otimes db + a\bar{d} \otimes \bar{c}b + \bar{b}c \otimes d\bar{a} + \bar{b}\bar{d} \otimes \bar{c}\bar{a}$$

and this agrees with

$$\begin{aligned}\mathsf{res}([a\otimes b][c\otimes d]) &= \mathsf{res}([ac\otimes db]+[a\bar{d}\otimes\bar{c}b])\\ &= (ac\otimes db+\overline{db}\otimes\overline{ac})+(a\bar{d}\otimes\bar{c}b+\overline{\bar{c}b}\otimes\overline{a\bar{d}}).\end{aligned}$$

As the C_2-action is an anti-involution, these terms agree. Similarly,

$$\mathsf{res}(\{a\}[b\otimes c]) = \mathsf{res}[ab\otimes c\bar{a}] = ab\otimes c\bar{a}+a\bar{c}\otimes\bar{b}\bar{a} = \mathsf{res}\{a\}\cdot\mathsf{res}[b\otimes c]$$

and $\mathsf{res}([a\otimes b]\{c\}) = \mathsf{res}[a\otimes b]\cdot\mathsf{res}\{c\}$. Therefore res is a ring map.

Tambara reciprocity is compatible with the multiplication: $\{a+b\}\cdot\{c\}$ agrees with $(\{a\}+\{b\}+[a\otimes\bar{b}])\{c\}$ and this also holds for the reversed product. On mixed terms we obtain

$$\begin{aligned}\{a+b\}[c\otimes d] &= [(a+b)c\otimes d(\bar{a}+\bar{b})]\\ &= [ac\otimes d\bar{a}]+[ac\otimes d\bar{b}]+[bc\otimes d\bar{a}]+[bc\otimes d\bar{b}]\\ &= [ac\otimes d\bar{a}]+[bc\otimes d\bar{b}]+[ac\otimes d\bar{b}]+[a\bar{d}\otimes\bar{c}\bar{b}]\\ &= \{a\}[c\otimes d]+\{b\}[c\otimes d]+[a\otimes\bar{b}][c\otimes d].\end{aligned}$$

Here, we have used the Weyl equivalence

$$[a\bar{d}\otimes\bar{c}\bar{b}] = [\overline{\bar{c}\bar{b}}\otimes\overline{a\bar{d}}] = [bc\otimes d\bar{a}].$$

□

Proposition 9.3. *The C_2-Mackey functor $\underline{R}^{\mathrm{fix}}$ is an $\tilde{N}_e^{C_2}i_e^*(\underline{R}^{\mathrm{fix}})$-bimodule.*

Proof. We know that

$$(\tilde{N}_e^{C_2}i_e^*\underline{R}^{\mathrm{fix}})\Box\underline{R}^{\mathrm{fix}} = \begin{cases}\big((\mathbb{Z}\{R\}\oplus(R\otimes R^{\mathrm{op}})/C_2)/\mathrm{TR}\otimes R^{C_2}\oplus(R\otimes R^{\mathrm{op}}\otimes R)/C_2\big)/\mathrm{FR} & \text{at } C_2/C_2\\ R\otimes R^{\mathrm{op}}\otimes R & \text{at } C_2/e.\end{cases}$$

We define the left $\tilde{N}_e^{C_2}i_e^*\underline{R}^{\mathrm{fix}}$-module structure of $\underline{R}^{\mathrm{fix}}$ by

$$(a\otimes b)\otimes c\mapsto acb$$

at the free level. At the trivial level we have three types of terms:

(1) For $a\in R$ and $x\in R^{C_2}$ we send $\{a\}\otimes x$ to $ax\bar{a}$.
(2) Whereas for $a,b\in R$ and $x\in R^{C_2}$ we define

$$[a\otimes b]\otimes x\mapsto axb+\bar{b}x\bar{a}.$$

The resulting elements are fixed points under the anti-involution.
(3) The C_2-action on $R\otimes R^{\mathrm{op}}\otimes R$ sends a generator $a\otimes b\otimes y$ to $\bar{b}\otimes\bar{a}\otimes\bar{y}$. We send a C_2-equivalence class $[a\otimes b\otimes y]$ to $ayb+\bar{b}\bar{y}\bar{a}$.

We have to check that this action is well-defined and satisfies associativity and a unit condition.

A direct inspection shows that $[a\otimes b]\otimes x$ and $[\bar{b}\otimes\bar{a}]\otimes x$ map to the same element. Similarly, the value on $[a\otimes b\otimes y]$ and $[\bar{b}\otimes\bar{a}\otimes\bar{y}]$ agrees. It is also straightforward to see that the module structure respects Tambara reciprocity. For Frobenius reciprocity we have to compare three expressions:

- $[a\otimes b]\otimes x$ is $\mathsf{tr}(a\otimes b)\otimes x$ and this is identified with $[a\otimes b\otimes\mathsf{res}(x)] = [a\otimes b\otimes x]$. Both terms are mapped to $axb+\bar{b}x\bar{a}$ because x is a fixed point.
- A term $\{a\}\otimes\mathsf{tr}(y)$ is sent to $a(y+\bar{y})\bar{a}$. It is identified with $[\mathsf{res}\{a\}\otimes y] = [a\otimes\bar{a}\otimes y]$ and this goes to $ay\bar{a}+a\bar{y}\bar{a}$.
- We have

$$[a\otimes b]\otimes(y+\bar{y}) = [a\otimes b]\otimes\mathsf{tr}(y) = \mathsf{res}([a\otimes b])\otimes y.$$

All these terms are mapped to $ayb+\bar{b}y\bar{a}+a\bar{y}b+\bar{b}\bar{y}\bar{a}$.

As $\{1\}$ acts neutrally at the trivial level and as $1 \otimes 1$ acts neutrally at the free level, the unit condition is satisfied. Associativity can be proven with a very tedious calculation.

This shows that the map $\tilde{N}_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}} \Box \underline{R}^{\mathrm{fix}} \to \underline{R}^{\mathrm{fix}}$ defined above yields a left $\tilde{N}_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}}$-module structure on $\underline{R}^{\mathrm{fix}}$.

The following is a sketch of the construction of the right $\tilde{N}_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}}$-module structure on $\underline{R}^{\mathrm{fix}}$: We have to define $\underline{R}^{\mathrm{fix}} \Box \tilde{N}_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}} \to \underline{R}^{\mathrm{fix}}$, that is: a map from

$$\underline{R}^{\mathrm{fix}} \Box (\tilde{N}_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}}) = \begin{cases} R^{C_2} \otimes \big((\mathbb{Z}\{R\} \oplus (R \otimes R^{\mathrm{op}})/C_2)/\mathrm{TR} \oplus (R \otimes R \otimes R^{\mathrm{op}})/C_2\big)/\mathrm{FR} & \text{at } C_2/C_2 \\ R \otimes R \otimes R^{\mathrm{op}} & \text{at } C_2/e \end{cases}$$

to $\underline{R}^{\mathrm{fix}}$. At the free level we send $a \otimes (b \otimes c)$ to cab and this propagates to the trivial level where we map $[y \otimes a \otimes b]$ to $bya + \bar{a}\bar{y}\bar{b}$ and $x \otimes [a \otimes b]$ to $bxa + \bar{a}x\bar{b}$. A term $x \otimes \{a\}$ goes to $\bar{a}xa$. Then a proof dual to the above shows that this indeed gives a well-defined right module structure and that this right-module structure is compatible with the left-module structure so that we actually obtain a bimodule structure. □

We use this pseudo norm-restriction term and the bimodule structure of $\underline{R}^{\mathrm{fix}}$ over $\tilde{N}_e^{C_2} i_e^*(\underline{R}^{\mathrm{fix}})$ for the definition of $\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\mathrm{fix}})$ by declaring $C_2/e \otimes \underline{R}^{\mathrm{fix}}$ to be $\tilde{N}_e^{C_2} i_e^*(\underline{R}^{\mathrm{fix}})$ and of course $C_2/C_2 \otimes \underline{R}^{\mathrm{fix}}$ is just $\underline{R}^{\mathrm{fix}}$. As the simplices in S^σ are lined up on two copies of $\Delta(-,[1])$, that are just glued at the endpoints, the associativity of R suffices to obtain well-defined face maps and therefore a well-defined Loday construction $\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\mathrm{fix}})$. As a simplicial C_2-Mackey functor, $\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\mathrm{fix}})$ is isomorphic to $B(\underline{R}^{\mathrm{fix}}, \tilde{N}_e^{C_2} i_e^*(\underline{R}^{\mathrm{fix}}), \underline{R}^{\mathrm{fix}})$.

We now state and prove the analogue of Theorems 6.4 and 7.2:

Theorem 9.4. *Assume that R is an associative ring with anti-involution and that 2 is invertible in R. If the underlying abelian group of R is flat, then*

$$\mathsf{iHH}^{\mathbb{Z}}_*(R) \cong \pi_*(\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\mathrm{fix}})(C_2/C_2)) \cong \mathsf{HR}^{+,\mathbb{Z}}_*(R,R).$$

Proof. We only point out where the differences to the proof in the commutative case are. As in Lemma 3.2, we can show (by literally using the same proof) that there is an isomorphism of C_2-Mackey functors

$$\tilde{N}_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}} \Box \underline{M}^{\mathrm{fix}} \cong \underline{(R \otimes R^{op} \otimes M)}^{\mathrm{fix}}$$

as C_2-Mackey functors, if 2 is invertible in R and if R is an associative ring with anti-involution.

The arguments in §5 go through with the difference that we have to replace $R \otimes R$ by $R \otimes R^{op}$ and Theorem 5.1 gives an isomorphism of C_2-Mackey functors

$$\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^{\mathrm{fix}}) \cong B(\underline{R}^{\mathrm{fix}}, \tilde{N}_e^{C_2} i_e^* \underline{R}^{\mathrm{fix}}, \underline{R}^{\mathrm{fix}}) \cong \underline{B(R, R \otimes R^{op}, R)}^{\mathrm{fix}}.$$

Section 6 is already formulated for associative algebras and also the homological algebra arguments in section 7 go through but we have to replace $R \otimes R$ by the enveloping algebra $R \otimes R^{op}$. □

In the setting where we choose a commutative ring k and A is an associative k-algebra with an anti-involution that fixes k, we first have to define a relative analogue of the norm.

Note that the unit map $k \to A$ induces a map of C_2-Green functors $N_e^{C_2} i_e^* \underline{k}^c \to \tilde{N}_e^{C_2} i_e^* \underline{A}^{\mathrm{fix}}$.

Definition 9.5. We define $\tilde{N}_e^{C_2,\underline{k}^c}(\underline{A}^{\mathrm{fix}})$ as

$$\tilde{N}_e^{C_2,\underline{k}^c}(\underline{A}^{\mathrm{fix}}) := \tilde{N}_e^{C_2} i_e^* \underline{A}^{\mathrm{fix}} \Box_{N_e^{C_2} i_e^* \underline{k}^c} \underline{k}^c.$$

With this we can define $\mathcal{L}^{C_2,\underline{k}^c}_{S^\sigma}(\underline{A}^{\mathrm{fix}})$ for an associative k-algebra A with anti-involution and obtain an isomorphism of simplicial C_2-Mackey functors

$$\mathcal{L}^{C_2,\underline{k}^c}_{S^\sigma}(\underline{A}^{\mathrm{fix}}) \cong B(\underline{A}^{\mathrm{fix}}, \tilde{N}_e^{C_2,\underline{k}^c}(\underline{A}^{\mathrm{fix}}), \underline{A}^{\mathrm{fix}}).$$

We get an analogue of Theorems 6.5 and 7.3.

Theorem 9.6. *Assume that A is an associative k-algebra with a k-linear anti-involution and that 2 is invertible in A. If the underlying module of A is flat over k, then*

$$\mathsf{iHH}^k_*(A) \cong \pi_*(\mathcal{L}^{C_2,\underline{k}^c}_{S^\sigma}(\underline{A}^{\mathrm{fix}})(C_2/C_2)) \cong \mathsf{HR}^{+,k}_*(A,A).$$

Proof. We have to adapt the statement and the proof of Proposition 4.2 and claim that for an associative k-algebra A with anti-involution we obtain that

$$\tilde{N}_e^{C_2,\underline{k}^c} i_e^*(\underline{A}^{\mathrm{fix}}) \cong \underline{(A \otimes_k A^{op})}^{\mathrm{fix}},$$

where C_2 acts on $A \otimes_k A^{op}$ by $\tau(a \otimes b) = \bar{b} \otimes \bar{a}$, if 2 is invertible in A.

The proof almost goes through, but we have to change the adjunction to run between the category of C_2-fixed point Mackey functors of associative rings with anti-involution with 2 invertible and the category of rings with anti-involution and with 2 invertible. Then the proof of the adjunction can be copied and we can use the same map g that worked in the commutative case. This yields an analogue of Theorem 4.1 in the associative setting.

$$\mathcal{L}^{C_2,\underline{k}^c}_{S^\sigma}(\underline{A}^{\mathrm{fix}}) \cong \underline{\mathcal{L}^k_{S^\sigma}(A)}^{\mathrm{fix}}$$

The other changes are similar to the absolute case of an associative ring with anti-involution but of course now we have to replace $A \otimes_k A$ by the enveloping algebra $A \otimes_k A^{op}$. □

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