COMONOIDAL PROPERTIES OF $\mathcal{I}$-CHAINS

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Abstract. Let $\mathcal{I}$ be the category of finite sets and injections. We prove that the homotopy colimit functor from the category of functors from $\mathcal{I}$ to chain complexes is an E_∞-monoidal functor. In particular, it maps a cocommutative comonoid to an E_∞-coalgebra. However, we also show that there is no model category structure on cocommutative comonoids in $\mathcal{I}$-chains whose weak equivalences and cofibrations are determined by the forgetful functor to $\mathcal{I}$-chains. The E_∞-coalgebra of the singular chains of a space therefore cannot be modelled in $\mathcal{I}$-chains in this way.

1. Introduction

Rational homotopy theory provides several algebraic models for the homotopy category of rational nilpotent spaces of finite type: there is Sullivan's differential graded commutative model of the rational cochains on a space [Sul77], there is a cocommutative coalgebra model for the rational chains on a space and there is a differential graded Lie model for the shifted rational homotopy groups of a space [Qui69, Nei78]. If one doesn't want to work over the rationals, then one option is to work with functor categories like $\mathsf{Ch}(k)^{\mathcal{D}}$ for a suitable small category $\mathcal{D}$.

In [RS20] we constructed a commutative model for the cochains of a space over an arbitrary commutative ring using the diagram category $\mathcal{I}$ of finite sets and injections. This category has two key features: its classifying space is contractible because it has an initial object and the endomorphisms of an object are the symmetric group and this gives enough Σ-freeness to replace some of the arguments for rational chain complexes with analogous arguments for functors from $\mathcal{I}$ to $\mathsf{Ch}(k)$ for an arbitrary ground ring k.

Over the integers our model of the cochains on a space is strong enough to determine the homotopy type of a nilpotent space of finite type. One could hope that one can also use the category $\mathcal{I}$ in order to construct Lie algebra and cocommutative coalgebra models. When I started this project, I actually believed that it is possible to find a cocommutative coalgebra model, but I had to learn that this is not possible, at least not in a way that is compatible with the Sullivan-like model from [RS20].

For the cochain model, Sagave and I show that the homotopy colimit of our version $A^{\mathcal{I},*}(X;k)$ of the Sullivan model of the rational singular cochains of a space X is an E_∞-algebra that is weakly equivalent as an E_∞-algebra to the cochain algebra of a space $S^*(X;k)$. It is true, that the homotopy colimit maps cocommutative comonoids in $\mathcal{I}$-chain complexes to E_∞-coalgebras:

Theorem 1.1. *The functor* $\mathsf{hocolim}_{\mathcal{I}}\colon \mathsf{Ch}(k)^{\mathcal{I}} \to \mathsf{Ch}(k)$ *is an* E_∞*-comonoidal functor. In particular, if* C_* *is a cocommutative comonoid in the category* $\mathsf{Ch}(k)^{\mathcal{I}}$*, then* $\mathsf{hocolim}_{\mathcal{I}} C_*$ *is an* E_∞*-coalgebra in* $\mathsf{Ch}(k)$*.*

So for any commutative ring k one could try to find a cocommutative comonoid $C_*^{\mathcal{I}}(X;k)$ in the category of $\mathcal{I}$-chain complexes $\mathsf{Ch}(k)^{\mathcal{I}}$ such that $\mathsf{hocolim}_{\mathcal{I}} C_*^{\mathcal{I}}(X;k)$ is weakly equivalent as an E_∞-coalgebra to the E_∞-coalgebra of the singular chains on a space X with coefficients in k, $S_*(X;k)$. However, this would not fit into a model categorical setting as one would expect from rational

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homotopy theory. Neisendorfer established a model category structure on connected cocommutative rational differential graded coalgebras in [Nei78, §5] in a way that the weak equivalences and cofibrations are determined by the forgetful functor to chain complexes.

If the homotopy colimit should compare a model in $\mathsf{Ch}(k)^{\mathcal{I}}$ to the E_∞-coalgebra of chains on a space, one would naturally consider the model structure on $\mathsf{Ch}(k)^{\mathcal{I}}$ that has those morphisms of $\mathcal{I}$-chain complexes as weak equivalences that induce quasi-isomorphisms after the application of the homotopy colimit functor. As one should work with an analogue of connected comonoids and as we want to model something that is a chain complex that is concentrated in non-negative degrees – like $S_*(X;k)$ – one would proceed as follows.

One first considers the projective model structure on $\mathsf{Ch}(k)^{\mathcal{I}}$ with levelwise fibrations and levelwise weak equivalences. We will actually consider a positive variant of this, so maps are required to be fibrations and weak equivalences in every positive level. One then uses left Bousfield localizations so that the weak equivalences are the ones that induce weak equivalences on the homotopy colimit. We call this model structure the *positive $\mathcal{I}$-model structure*. As we aim at modelling the singular chain complex of a connected space, we modify the base category to chain complexes that are concentrated in degrees $\geqslant 1$, $\mathsf{Ch}(k)_{\geqslant 1}$.

There is a cofree functor that sends an $\mathcal{I}$-chain complex to the cofree connected cocommutative coalgebra on that $\mathcal{I}$-chain complex. This functor is right adjoint to the forgetful functor, so in order to obtain a model category structure one would use the left-induced model structure: in this structure the weak equivalences and cofibrations are determined by the forgetful functor. This works for differential graded connected cocommutative coalgebras over the rationals [Nei78, §5]. However, this does *not* work in the setting of $\mathcal{I}$-chain complexes, at least if you want to avoid to work over $\mathbb{Q}$.

Theorem 1.2. *Assume that $\mathbb{Q} \not\subset k$. Then there is a fibrant acyclic $\mathcal{I}$-chain complex whose cofree cocommutative comonoid is* not *acyclic. Hence, there is no left-induced model structure on the category of connected cocommutative comonoids whose cofibrations and weak equivalences are determined by the cofibrations and weak equivalences in the positive $\mathcal{I}$-model structure on $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1}$.*

At the heart of the above result is the fact that the norm map is not an isomorphism in general in the context of $\mathcal{I}$-chain complexes, even if one requires the zeroth level to be trivial. This causes a discrepancy between the cofree cocommutative comonoid generated by an $\mathcal{I}$-chain complex and the free commutative monoid generated by the very same $\mathcal{I}$-chain complex. In the contexts of rational homotopy theory and symmetric sequences one has such norm isomorphisms and hence in these context acyclic objects create acyclic cofree cocommutative comonoids. Despite the fact that $\mathcal{I}$-chains provide a full integral analogue of Sullivan's differential graded commutative model of the rational cochains of a space, $\mathcal{I}$-chains fail to provide a coalgebraic model of the E_∞-coalgebra of the chains on a space.

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When I started this project I was very optimistic that E_∞-coalgebras actually could be modelled via $\mathcal{I}$-chain complexes and Hendrik Laß worked as a master student on some of the aspects. He realized that some things didn't quite work out as I had hoped and that slowly made me turn to search for counterexamples rather than proofs. I thank Steffen Sagave for several very helpful comments.

2. Basics on $\mathcal{I}$-chain complexes

Let $\mathcal{I}$ be the skeleton of the category of finite sets and injections whose objects are the sets $\{1,\dots,n\} =: \mathbf{n}$ for $n \geqslant 0$ with $\mathbf{0} = \varnothing$. The morphism set $\mathcal{I}(\mathbf{n},\mathbf{m})$ consists of all injective functions from $\mathbf{n}$ to $\mathbf{m}$. The category $\mathcal{I}$ is symmetric monoidal under concatenation of sets: $\mathbf{n} \sqcup \mathbf{m} := \mathbf{n} + \mathbf{m}$. The initial object $\mathbf{0}$ is the unit of this symmetric monoidal structure.

Morphisms in $\mathcal{I}$ can be expressed as follows. Let $\varphi \in \mathcal{I}(\mathbf{n},\mathbf{m})$ with $n > 0$. Then we can uniquely decompose φ as $\varphi = i \circ \sigma$ with $\sigma \in \Sigma_n$ and i and order preserving injection: Let us denote the subset $\varphi(\mathbf{n}) \subset \mathbf{m}$ as the ordered set $x_1 < \ldots < x_n$ and set $\sigma(i) := \varphi^{-1}(x_i)$. Then $\sigma \in \Sigma_n$ and $\varphi \circ \sigma^{-1}(j) = x_j$ is an order preserving injection.

We will consider several functor categories from $\mathcal{I}$ to categories $\mathcal{C}$, $\mathcal{C}^{\mathcal{I}}$. If $\mathcal{C}$ is the category of k-modules for some commutative unital ring k, then we will call the corresponding category the category of $\mathcal{I}$-modules, $\mathsf{mod}^{\mathcal{I}}$. We call functors from $\mathcal{I}$ to the category of chain complexes *$\mathcal{I}$-chain complexes* and denote the corresponding functor category by $\mathsf{Ch}(k)^{\mathcal{I}}$.

Definition 2.1. Let $\mathcal{C}$ be an abelian category. An object $M \in \mathcal{C}^{\mathcal{I}}$ is *reduced*, if $M(\mathbf{0}) = 0$.

For every $n \geqslant 0$ there is an evaluation functor $\mathrm{Ev}_n\colon \mathsf{Ch}(k)^{\mathcal{I}} \to \mathsf{Ch}(k)$ that takes an $\mathcal{I}$-chain complex X_* to the chain complex $X_*(\mathbf{n})$. These functors have left adjoints

$$F_n^{\mathcal{I}}\colon \mathsf{Ch}(k) \to \mathsf{Ch}(k)^{\mathcal{I}}$$

with

$$F_n^{\mathcal{I}}(C_*)(\mathbf{m}) = \bigoplus_{\mathcal{I}(\mathbf{n},\mathbf{m})} C_* \cong k\{\mathcal{I}(\mathbf{n},\mathbf{m})\} \otimes_k C_*.$$

Here, for a set S we denote by $k\{S\}$ the free k-module generated by S. As $\mathbf{0}$ is initial, $F_0^{\mathcal{I}}$ maps a chain complex C_* to the constant $\mathcal{I}$-diagram with value C_*.

The Day convolution product gives $\mathsf{Ch}(k)^{\mathcal{I}}$ a symmetric monoidal structure. Explicitly, for two $\mathcal{I}$-chain complexes X_*, Y_*

$$(X_* \boxtimes Y_*)(\mathbf{n}) = \mathrm{colim}_{\mathcal{I}(\mathbf{p}\sqcup\mathbf{q},\mathbf{n})} X_*(\mathbf{p}) \otimes Y_*(\mathbf{q}).$$

Abstract nonsense yields

$$F_n^{\mathcal{I}} C_* \boxtimes F_m^{\mathcal{I}} D_* \cong F_{n+m}^{\mathcal{I}}(C_* \otimes D_*).$$

In $\mathsf{Ch}(k)$ we write S^q for the chain complex with k concentrated in degree $q \in \mathbb{Z}$, and D^q for the chain complex with $(D^q)_i = k$ if $i \in \{q, q-1\}$, with $(D^q)_i = 0$ for all other i, and with $d_q = \mathrm{id}$. As S^0 is the symmetric monoidal unit in $\mathsf{Ch}(k)$, the $\mathcal{I}$-chain complex $F_0^{\mathcal{I}}(S^0)$ is the unit for the Day convolution product. We denote $F_0^{\mathcal{I}}(S^0)$ by $U^{\mathcal{I}}$.

The category of $\mathcal{I}$-chain complexes is tensored over chain complexes: if $C_* \in \mathsf{Ch}(k)$, $X_* \in \mathsf{Ch}(k)^{\mathcal{I}}$, then

$$(C_* \otimes X_*)(\mathbf{m}) := C_* \otimes (X_*(\mathbf{m}))$$

and hence $C_* \otimes X_* = F_0^{\mathcal{I}}(C_*) \boxtimes X_*$.

Connection to symmetric sequences. The symmetric group on n letters is equal to the endomorphisms of the object $\mathbf{n}$ of $\mathcal{I}$. Let Σ denote the skeleton of the category of finite sets and bijections whose objects are the same as the ones of $\mathcal{I}$ and with

$$\Sigma(\mathbf{n},\mathbf{m}) = \begin{cases} \Sigma_n, & \text{for } n = m, \\ \varnothing, & \text{for } n \neq m. \end{cases}$$

Here we use the convention that Σ_0 and Σ_1 are both the trivial group.

The category Σ is symmetric monoidal again by the ordered sum $\mathbf{n} \sqcup \mathbf{m} := \mathbf{n} + \mathbf{m}$. If $\mathcal{C}$ is symmetric monoidal and cocomplete, then $\mathcal{C}^\Sigma$ carries a symmetric monoidal structure via the Day convolution product and for $C_1, C_2 \in \mathcal{C}^\Sigma$ we denote their product by $C_1 \odot C_2$ in order to distinguish it from the $\boxtimes$-product in $\mathcal{C}^\mathcal{I}$. Explicitly for $C_1, C_2 \in \mathsf{Ch}(k)^\Sigma$ and $n \in \mathbb{N}_0$ we have

$$(C_1 \odot C_2)(\mathbf{n}) = \bigoplus_{p+q=n} k[\Sigma_n] \otimes_{k[\Sigma_p \times \Sigma_q]} C_1(\mathbf{p}) \otimes C_2(\mathbf{q}).$$

Here, for a group G we denote the group k-algebra on G by $k[G]$. The canonical inclusion functor $i\colon \Sigma \to \mathcal{I}$ induces a restriction functor

$$i^*\colon \mathcal{C}^\mathcal{I} \to \mathcal{C}^\Sigma.$$

If $\mathcal{C}$ is cocomplete, then left Kan extension gives a left adjoint to restriction

$$i_!\colon \mathcal{C}^\Sigma \to \mathcal{C}^\mathcal{I}.$$

Recall the explicit form of the left Kan extension. We denote by $i \downarrow \mathbf{n}$ the category whose objects are morphisms in $\mathcal{I}$ from some $i(\mathbf{m})$ to $\mathbf{n}$ and whose morphisms from $f \in \mathcal{I}(i(\mathbf{m}), \mathbf{n})$ to $g \in \mathcal{I}(i(\mathbf{m}), \mathbf{n})$ are given by $\sigma \in \Sigma_m$ such that

$$g \circ i(\sigma) = f.$$

Then for any $C_* \in \mathsf{Ch}(k)^\Sigma$

$$i_! C_*(\mathbf{n}) = \mathrm{colim}_{i \downarrow \mathbf{n}} C_* = \mathrm{colim}_{\varphi\colon i(\mathbf{m}) \to \mathbf{n}} C_*(\mathbf{m}).$$

As there are no morphisms in Σ that connect different objects, the above expression splits as

$$i_! C_*(\mathbf{n}) = \bigoplus_{m \geqslant 0} \mathrm{colim}_{\mathcal{I}(\mathbf{m},\mathbf{n})} C_*(\mathbf{m}).$$

But here the colimit just identifies elements that lie in the same Σ_m-orbit so we obtain

$$i_! C_*(\mathbf{n}) \cong \bigoplus_{m \geqslant 0} \left(\bigoplus_{\mathcal{I}(\mathbf{m},\mathbf{n})} C_*(\mathbf{m}) \right) / \Sigma_m \cong \bigoplus_{m \geqslant 0} k\{\mathcal{I}(\mathbf{m},\mathbf{n})\} \otimes_{k[\Sigma_m]} C_*(\mathbf{m}). \tag{2.1}$$

This proves the following result:

Lemma 2.2. *For every $C_* \in \mathsf{Ch}(k)^\mathcal{I}$ the left Kan extension $i_!(C_*)$ satisfies*

$$i_!(C_*)(\mathbf{n}) = k\{\mathcal{I}(i(-), \mathbf{n})\} \otimes_\Sigma C_*$$

where $k\{\mathcal{I}(i(-), \mathbf{n})\}$ is the Σ^{op}-module that sends $\mathbf{m}$ to $k\{\mathcal{I}(i(\mathbf{m}), \mathbf{n})\}$ and where the tensor product denotes the tensor product of the Σ^{op}-module $k\{\mathcal{I}(i(-), \mathbf{n})\}$ with the Σ-chain complex C_*.

Similar to the category $\mathcal{C}^\mathcal{I}$ there are evaluation functors

$$\mathrm{Ev}_n\colon \mathcal{C}^\Sigma \to \mathcal{C}, \quad M \mapsto M(\mathbf{n}).$$

For $\mathcal{C} = \mathsf{Ch}(k)$ these functors have left adjoints

$$F_n^\Sigma\colon \mathsf{Ch}(k) \to \mathsf{Ch}(k)^\Sigma, \quad X_* \mapsto F_n^\Sigma(X_*)$$

where

$$F_n^\Sigma(X_*)(\mathbf{m}) = \begin{cases} \bigoplus_{\Sigma_n} X_* \cong k\{\Sigma_n\} \otimes_k X_*, & \text{for } n = m, \\ 0, & \text{for } m \neq n. \end{cases}$$

Lemma 2.3.

(1) *For any chain complex X_* and every $p \geqslant 0$ there is an isomorphism of $\mathcal{I}$-chain complexes $i_!(F_p^\Sigma(X_*)) \cong F_p^\mathcal{I}(X_*)$.*

(2) *The left Kan extension $i_!$ is strong symmetric (co)monoidal.*

Proof.
(1) The first claim follows from abstract nonsense, because the corresponding composition of the right adjoint functors commutes.
(2) The functor $i_!$ is left adjoint, hence right-exact and it preserves colimits. Every object in $\mathsf{Ch}(k)^\Sigma$ can be written as an epimorphic image of a sum of suitable $F_n^\Sigma(X_*)$'s and similarly for $\mathsf{Ch}(k)^\mathcal{I}$. It hence suffices to check the claim on free objects. But here by (1) we get

$$\begin{aligned} i_!(F_n^\Sigma(X_*)) \boxtimes i_!(F_m^\Sigma(Y_*)) &\cong F_n^\mathcal{I}(X_*) \boxtimes F_m^\mathcal{I}(Y_*) \\ &\cong F_{n+m}^\mathcal{I}(X_* \otimes Y_*) \\ &\cong i_!(F_{n+m}^\Sigma(X_* \otimes Y_*)) \\ &\cong i_!(F_n^\Sigma(X_*) \odot F_m^\Sigma(Y_*)). \end{aligned}$$

□

3. Homotopy colimits

We need to control homotopy colimits of $\mathcal{I}$-chain complexes. In certain cases, these can be identified in a very explicit manner, for instance for $\mathcal{I}$-chain complexes of the form $i_!(C_*)$ with $C_* \in \mathsf{Ch}(k)^\Sigma$.

Recall the definition of the homotopy colimit for an $\mathcal{I}$-chain complex Y_* from [RS20, §2]: $\mathsf{hocolim}_\mathcal{I} Y_*$ is the total complex associated to the bicomplex whose bidegree (p,q)-part is

$$\bigoplus_{[f_p|\ldots|f_1]\in N_p\mathcal{I}} Y_q(sf_1)$$

where sf_1 denotes the source of f_1. The vertical differential is induced by the internal differential of Y_*. The horizontal differential is the one of the chain complex associated to the nerve. The face map involving f_1 induces an action $Y_q(f_1)\colon Y_q(sf_1) \to Y_q(tf_1) = Y_q(sf_2)$ where tf_1 denotes the target of f_1.

Rodríguez González [RG14] developes a general framework that allows to use these nice and concrete Bousfield-Kan like models of the homotopy colimits in certain contexts. Dugger [Dug01, Theorem 5.2] considers an axiomatic framework that yields that the homotopy colimit over $\mathcal{I}$ gives rise to an $\mathcal{I}$-model structure whose weak equivalences are those morphisms that induce a quasi-isomorphism on homotopy colimits. Joachimi proves this for $\mathcal{I}$-chain complexes in a direct way [Joa11].

There is a canonical projection map

$$\pi_{Y_*}\colon \mathsf{hocolim}_\mathcal{I} Y_* \to \operatorname{colim}_\mathcal{I} Y_* \tag{3.1}$$

that takes the cokernel of the horizontal differential $d\colon \bigoplus_{[f_1]\in N_1\mathcal{I}} Y_*(sf_1) \to \bigoplus_{n\geqslant 0} Y_*(\mathbf{n})$.

The following fact is well-known [Sch18, Proof of Proposition 2.54]; compare [SaS12, Proposition 6.15]: Free $\mathcal{I}$-chain complexes on a chain complex X_* have X_* as the homotopy colimit:

Lemma 3.1. *For all $m \geqslant 0$ and all chain complexes X_*:*

$$\mathsf{hocolim}_\mathcal{I} F_m^\mathcal{I}(X_*) \simeq X_*.$$

Proof. One has

$$\bigoplus_{[f_p|\ldots|f_1]\in N_p\mathcal{I}} F_m^\mathcal{I}(X_*)(sf_1) = \bigoplus_{[f_p|\ldots|f_1]\in N_p\mathcal{I}} \bigoplus_{\mathcal{I}(\mathbf{m},sf_1)} X_* \cong \bigoplus_{N_p(\mathbf{m}\downarrow\mathcal{I})} X_*,$$

where $\mathbf{m} \downarrow \mathcal{I}$ is the category of object in $\mathcal{I}$ under $\mathbf{m}$. This category has an initial object and hence the nerve is contractible. □

For general $\mathcal{I}$-chain complexes of the form $i_! C_*$ we will use an operad to express the homotopy colimit.

Lemma 3.2. *The categories $C(m) := \mathbf{m} \downarrow \mathcal{I}$ form an operad in the category of small categories.*

Proof. The right-Σ_m action on $C(m)$ is defined by precomposition.

Let $f\colon \mathbf{m} \to \mathbf{n}$ and $g_i\colon \mathbf{k}_i \to \mathbf{n}_i$ be objects of $C(m)$ and $C(k_i)$ respectively. We define the operadic composite $\gamma(f; g_1, \ldots, g_m)$ as

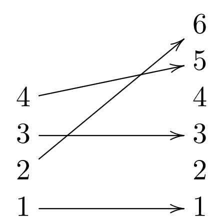
$$(\tilde{g}_{f^{-1}(1)} \sqcup \ldots \sqcup \tilde{g}_{f^{-1}(n)}) \circ f(\mathbf{k}_1, \ldots, \mathbf{k}_m). \tag{3.2}$$

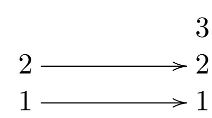
Here, $f(\mathbf{k}_1, \ldots, \mathbf{k}_m)$ maps the blocks of numbers $\mathbf{k}_1, \ldots, \mathbf{k}_m$ as f maps $1, \ldots, m$ and

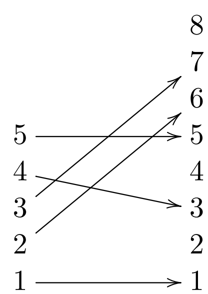
$$\tilde{g}_{f^{-1}(j)} = \begin{cases} \mathrm{id}_{\mathbf{1}}, & \text{if } f^{-1}(j) = \varnothing, \\ g_\ell, & \text{if } f(\ell) = j. \end{cases}$$

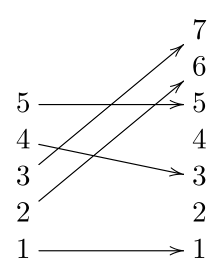
The identity $1 \in C(1)$ is then defined to be $\mathrm{id}_{\mathbf{1}}$. It is straightforward to check that the composition is equivariant. We leave the tedious proof of associativity to the brave reader. This is best done using the $\circ_i$-definition of a pseudo-operad in the sense of Markl. □

Example 3.3. Let $f \in \mathcal{I}(\mathbf{4}, \mathbf{6})$ be the map that sends 1 to 1, 2 to 6, 3 to 3 and 4 to 5



and let $g \in \mathcal{I}(\mathbf{2}, \mathbf{3})$ be the standard inclusion given by $g(i) = i$ for $i = 1, 2$:  .

Then $f \circ_2 g\colon \mathcal{I}(\mathbf{5}, \mathbf{8})$ is the injection  because $f(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$ is the map



and we compose it with $\mathrm{id}_{\mathbf{1}} \sqcup \mathrm{id}_{\mathbf{1}} \sqcup \mathrm{id}_{\mathbf{1}} \sqcup \mathrm{id}_{\mathbf{1}} \sqcup \mathrm{id}_{\mathbf{1}} \sqcup g = \mathrm{id}_{\mathbf{5}} \sqcup g$.

Note, that the operad composition defined in (3.2) agrees with the operad structure on the categorical version of the Barratt-Eccles operad if all morphisms are actually bijective.

Corollary 3.4. *The sequence of nerves of the categories $\mathbf{m} \downarrow \mathcal{I}$, $(N(\mathbf{m} \downarrow \mathcal{I}))_{m \geqslant 0}$ forms an operad in the category of simplicial sets. Applying the free k-module functor yields that $(k\{N(\mathbf{m} \downarrow \mathcal{I})\})_{m \geqslant 0}$ forms an operad in the category of simplicial k-modules.*

The associated chain complexes $(O(m) := C_*(\{N(\mathbf{m} \downarrow \mathcal{I})\})$ form an E_∞-operad in the category of chain complexes.

Proof. As the functors involved are all at least lax symmetric monoidal, it is clear that we actually obtain operads. It remains to show that

$$(O(m) = C_*(\{N(\mathbf{m} \downarrow \mathcal{I})\}))_{m \geqslant 0}$$

is an E_∞-operad.

Using the contractibility of the nerve of $\mathbf{m} \downarrow \mathcal{I}$ as in the proof of Lemma 3.1, we get that each $C_*(\{N(\mathbf{m} \downarrow \mathcal{I})\}))_{m \geqslant 0}$ is acyclic. The set $N_p(\mathbf{m} \downarrow \mathcal{I})$ is free as a Σ_m set and we can write it as the orbit of elements $(S \subset \mathbf{n}_0, |S| = m, [f_p \mid \ldots \mid f_1])$, identifying an injective map $f_0 \colon \mathbf{m} \to \mathbf{n}_0$ with its image.

□

We know that $F_m^{\mathcal{I}}(X_*)$ is an $\mathcal{I}$-chain complex of the form $i_! F_m^{\Sigma}(X_*)$, so one can ask whether one can extend the identification from Lemma 3.1 to all $\mathcal{I}$-chain complexes of the form $i_! Y_*$ with $Y_* \in \mathsf{Ch}(k)^{\Sigma}$. We start with the simple case where $Y_* = M$ is a Σ-module viewed as a Σ-chain complex concentrated in degree zero.

Lemma 3.5. *For all $M \in \mathsf{mod}^{\Sigma}$:*

$$H_* \mathsf{hocolim}_{\mathcal{I}}\, i_!(M) \cong \bigoplus_{m \in \mathbb{N}_0} H_*(\Sigma_m; M(\mathbf{m})).$$

Proof. By the very definition of $i_!(M)$ we get

$$\begin{aligned} \mathsf{hocolim}_{\mathcal{I}} i_!(M)_p &= \bigoplus_{[f_p|\ldots|f_1] \in N\mathcal{I}_p} i_!(M)(sf_1) \\ &\cong \bigoplus_{[f_p|\ldots|f_1] \in N\mathcal{I}_p} k\{\mathcal{I}(i(-), sf_1)\} \otimes_\Sigma M(-) \\ &\cong \bigoplus_{[f_p|\ldots|f_1] \in N\mathcal{I}_p} \bigoplus_{m \geqslant 0} k\{\mathcal{I}(i(\mathbf{m}), sf_1)\} \otimes_{\Sigma_m} M(\mathbf{m}). \end{aligned}$$

But the latter is isomorphic to

$$\bigoplus_{m \geqslant 0} k\{N(i(\mathbf{m}) \downarrow \mathcal{I})_p\} \otimes_{\Sigma_m} M(\mathbf{m}).$$

The differential is induced by the face maps in the nerve and we showed above that this complex is contractible and Σ_m-free in every degree. Hence the homotopy colimit calculates the Tor-groups $\bigoplus_{m \geqslant 0} \mathsf{Tor}_*^{k[\Sigma_m]}(k, M(\mathbf{m}))$. □

Remark 3.6.

(1) Note that we can also consider the homotopy colimit with respect to the category Σ, see [Ric20, 11.4.7, §16.3] or where the only difference is that we work with the nerve of Σ instead of the nerve of $\mathcal{I}$. The homotopy colimit over Σ splits as

$$\mathsf{hocolim}_{\Sigma} M = \bigoplus_{m \geqslant 0} \mathsf{hocolim}_{\Sigma_m} M$$

where Σ_m is the category with one object $\mathbf{m}$ and Σ_m as morphisms. The homology of $\mathsf{hocolim}_{\Sigma_m} M$ is $H_*(\Sigma_m; M(\mathbf{m}))$ because we can identify the chain complex $\mathsf{hocolim}_{\Sigma_m} M(\mathbf{m})$ with the chain complex $C_*(k\{N\mathcal{E}_{\Sigma_m}\}) \otimes_{\Sigma_m} M(\mathbf{m})$. Here, $\mathcal{E}_{\Sigma_m}$ denotes the translation category of Σ_m. Thus $H_* \mathsf{hocolim}_{\Sigma} M$ agrees with $H_* \mathsf{hocolim}_{\mathcal{I}}\, i_!(M)$.

(2) The proof of Lemma 3.5 actually tells us that we can express the homotopy colimit of an $\mathcal{I}$-module $i_! M$ as

$$\mathsf{hocolim}_{\mathcal{I}} i_! M \cong \bigoplus_{m \geqslant 0} O(m) \otimes_{\Sigma_m} M(\mathbf{m}).$$

This is true in broader generality.

Proposition 3.7. *Let $C_* \in \mathsf{Ch}(k)^{\Sigma}$. Then*

$$\mathsf{hocolim}_{\mathcal{I}} i_! C_* \cong \bigoplus_{m \geqslant 0} O(m) \otimes_{\Sigma_m} C_*(\mathbf{m}).$$

As C_* is a chain complex, the homotopy colimit is the total complex of a bicomplex and the total grading corresponds to the usual total grading of the tensor product of chain complexes, *i.e.*,

$$(\mathsf{hocolim}_{\mathcal{I}} i_! C_*)_n \cong \bigoplus_{r+s=n} \bigoplus_{m \geqslant 0} O(m)_r \otimes_{\Sigma_m} C_s(\mathbf{m}).$$

Proof. By Lemma 3.5 we get the claimed correspondence for every fixed chain degree. The internal differential d_C on C_* induces the summand $\mathrm{id} \otimes d_C$ of the differential on the tensor product of chain complexes. $\square$

As the free functors $F_n^{\mathcal{I}}(C_*)$ are of the form $i_! F_n^{\Sigma}(C_*)$ we obtain the following variant of Lemma 3.1:

Corollary 3.8. *For all $n \geqslant 0$ and all chain complexes C_**

$$\mathsf{hocolim}_{\mathcal{I}} F_n^{\mathcal{I}}(C_*) \cong \bigoplus_{m \geqslant 0} O(m) \otimes_{\Sigma_m} F_n^{\Sigma}(C_*)(\mathbf{m}) \cong O(n) \otimes C_*.$$

As $O(n)$ is acyclic, this gives of course the same result as Lemma 3.1.

Definition 3.9. Let V_* be an $\mathcal{I}$-chain complex. The *tensor algebra on V_** is

$$\mathsf{T}^{\mathcal{I}}(V_*) := \bigoplus_{k \geqslant 0} V_*^{\boxtimes k} \tag{3.3}$$

with the usual convention that $X_*^{\boxtimes 0} = U^{\mathcal{I}}$.

The functor $\mathsf{T}^{\mathcal{I}}$ is left adjoint to the forgetful functor from associative monoids in $\mathsf{Ch}(k)^{\mathcal{I}}$, $\mathsf{As}^{\mathcal{I}}$, to $\mathsf{Ch}(k)^{\mathcal{I}}$ where we view $\mathsf{T}^{\mathcal{I}}(X_*)$ as an associative monoid via the concatenation product

$$\mu^T \colon \mathsf{T}^{\mathcal{I}}(X_*) \boxtimes \mathsf{T}^{\mathcal{I}}(X_*) \to \mathsf{T}^{\mathcal{I}}(X_*)$$

that is induced by the canonical isomorphisms

$$c_{rx} \colon X^{\boxtimes r} \boxtimes X^{\boxtimes s} \cong X_*^{\boxtimes r+s}. \tag{3.4}$$

As the category of $\mathcal{I}$-modules is symmetric monoidal we get an action of Σ_n on $M^{\boxtimes n}$. Here we consider the *right* Σ_n-action. The free commutative monoid generated by an $\mathcal{I}$-chain complex V_* is

$$\mathsf{S}^{\mathcal{I}}(V_*) = \bigoplus_{n \geqslant 0} V_*^{\boxtimes n}/\Sigma_n.$$

The functor $\mathsf{S}^{\mathcal{I}}$ is left adjoint to the forgetful functor from commutative $\mathcal{I}$-chain complexes to $\mathcal{I}$-chain complexes. If we consider $\mathsf{S}^{\mathcal{I}}(F_1^{\mathcal{I}}(C_*))$ for a chain complex C_*, then we can actually identify the homotopy colimit of the free commutative monoid $\mathsf{S}^{\mathcal{I}}(F_1^{\mathcal{I}}(C_*))$ with the free O-algebra generated by C_*:

Theorem 3.10. *For all chain complexes C_*:*

$$\mathsf{hocolim}_{\mathcal{I}} \mathsf{T}^{\mathcal{I}}(F_1^{\mathcal{I}}(C_*)) \cong \bigoplus_{m \geqslant 0} O(m) \otimes C_*^{\otimes m}$$

and

$$\mathsf{hocolim}_{\mathcal{I}} \mathsf{S}^{\mathcal{I}}(F_1^{\mathcal{I}}(C_*)) \cong \bigoplus_{m \geqslant 0} O(m) \otimes_{\Sigma_m} C_*^{\otimes m}.$$

Proof. As $i_!$ is strong symmetric monoidal and as it is a left adjoint, we know by Proposition 3.7 that

$$\begin{aligned}
\mathsf{hocolim}_{\mathcal{I}} \mathsf{T}^{\mathcal{I}}(F_1^{\mathcal{I}}(C_*)) &\cong \mathsf{hocolim}_{\mathcal{I}} \mathsf{T}^{\mathcal{I}}(i_! F_1^{\Sigma}(C_*)) \\
&\cong \mathsf{hocolim}_{\mathcal{I}} i_! \bigoplus_{\ell \geqslant 0} (F_1^{\Sigma}(C_*))^{\odot \ell} \\
&\cong \mathsf{hocolim}_{\mathcal{I}} i_! \bigoplus_{\ell \geqslant 0} F_\ell^{\Sigma}(C_*^{\otimes \ell}) \\
&\cong \bigoplus_{m \geqslant 0} \bigoplus_{\ell \geqslant 0} O(m) \otimes_{\Sigma_m} (F_\ell^{\Sigma}(C_*^{\otimes \ell}))(\mathbf{m}).
\end{aligned}$$

As $(F_\ell^{\Sigma}(C_*^{\otimes \ell}))(\mathbf{m})$ is trivial for all $\ell \neq m$, this sum reduces to

$$\bigoplus_{m \geqslant 0} O(m) \otimes_{\Sigma_m} (F_m^{\Sigma}(C_*^{\otimes m}))(\mathbf{m}) = \bigoplus_{m \geqslant 0} O(m) \otimes_{\Sigma_m} (k\{\Sigma_m\} \otimes C_*^{\otimes m}) \cong \bigoplus_{m \geqslant 0} O(m) \otimes C_*^{\otimes m}.$$

For the free commutative monoid we obtain as above

$$\mathsf{S}^{\mathcal{I}}(F_1^{\mathcal{I}}(C_*)) \cong \mathsf{S}^{\mathcal{I}}(i_! F_1^{\Sigma}(C_*)) \cong i_! \mathsf{S}^{\Sigma}(F_1^{\Sigma}(C_*)).$$

With the help of Proposition 3.7 we can identify the homotopy colimit of the latter as

$$\mathsf{hocolim}_{\mathcal{I}} i_! \mathsf{S}^{\Sigma}(F_1^{\Sigma}(C_*)) \cong \bigoplus_{m \geqslant 0} O(m) \otimes_{\Sigma_m} \mathsf{S}^{\Sigma}(F_1^{\Sigma}(C_*))(\mathbf{m}).$$

We get

$$(F_1^{\Sigma}(C_*))^{\odot m}(\mathbf{m})/\Sigma_m \cong F_m^{\Sigma}(C_*^{\otimes m})/\Sigma_m \cong (k\{\Sigma_m\} \otimes C_*^{\otimes m})/\Sigma_m \cong C_*^{\otimes m}$$

and therefore in total we obtain

$$\bigoplus_{m \geqslant 0} O(m) \otimes_{\Sigma_m} \mathsf{S}^{\Sigma}(F_1^{\Sigma}(C_*))(\mathbf{m}) \cong \bigoplus_{m \geqslant 0} O(m) \otimes_{\Sigma_m} C_*^{\otimes m}.$$

$\square$

The result above actually generalizes to all free algebras in $\mathsf{Ch}(k)^{\mathcal{I}}$ over an operad in the category of modules: If $(P(m))_{m\geqslant 0}$ is an operad in the category of modules and if C_* is a chain complex, then the free P-algebra generated by $F_1^\Sigma(C_*)$ is

$$P(F_1^\Sigma(C_*)) = \bigoplus_{n\geqslant 0} P(n)\otimes_{\Sigma_n} F_1^\Sigma(C_*)^{\odot n} \cong \bigoplus_{n\geqslant 0} P(n)\otimes_{\Sigma_n} F_n^\Sigma(C_*^{\otimes n}).$$

The argument in the proof of Theorem 3.10 then identifies the homotopy colimit of $i_! P(F_1^\Sigma(C_*))$ as

$$\begin{aligned}\mathsf{hocolim}_{\mathcal{I}} i_! P(F_1^\Sigma(C_*)) &= \bigoplus_{m\geqslant 0} O(m)\otimes_{\Sigma_m}\left(\bigoplus_{n\geqslant 0} P(n)\otimes_{\Sigma_n} F_n^\Sigma(C_*^{\otimes n})\right)(\mathbf{m})\\ &\cong \bigoplus_{m\geqslant 0} O(m)\otimes_{\Sigma_m}\left(P(m)\otimes_{\Sigma_m}(k[\Sigma_m]\otimes C_*^{\otimes m})\right).\end{aligned}$$

Here, the left copy of Σ_m acts on the right on $O(m)$ and on the left on $k[\Sigma_m]\otimes C_*^{\otimes m}$ whereas the right copy of Σ_m acts from the right on $P(m)$ and from the left on $k[\Sigma_m]\otimes C_*^{\otimes m}$. We can cancel the left hand copy of Σ_m with the $k[\Sigma_m]$-tensor factor in $k[\Sigma_m]\otimes C_*^{\otimes m}$ and can therefore identify the homotopy colimit with

$$\bigoplus_{m\geqslant 0}(O(m)\otimes P(m))\otimes_{\Sigma_m} C_*^{\otimes m}$$

where Σ_m acts diagonally on $O(m)\otimes P(m)$ and on the left on $C_*^{\otimes m}$ by the usual permutation of tensor factors. This proves the following result:

Theorem 3.11. *If P is an operad in the category of modules, then* $\mathsf{hocolim}_{\mathcal{I}} i_!(P(F_1^\Sigma(C_*))$ *is the free $O\otimes P$-algebra generated by C_*.*

For example, if we consider the free Lie algebra on $F_1^\Sigma(C_*)$ we obtain in the homotopy colimit the free $O\otimes \mathsf{Lie}^{\mathcal{I}}$-algebra generated by C_*. This is the tensor product of an E_∞-operad with the Lie operad, hence it parametrizes Lie algebras up to homotopy.

4. Comonoids

In this section we study homotopy colimits of (cocommutative) comonoids and construct a cofree (cocommutative) coalgebra functor.

Definition 4.1. An *$\mathcal{I}$-chain coalgebra* C_* is a comonoid in $(\mathsf{Ch}(k)^{\mathcal{I}}, \boxtimes, U^{\mathcal{I}})$, so there is a diagonal map $\Delta\colon C_*\to C_*\boxtimes C_*$ and a counit map $\varepsilon\colon C_*\to U^{\mathcal{I}}$ such that Δ is coassociative and

$$(\varepsilon\boxtimes \mathrm{id}_{C_*})\circ\Delta = \mathrm{id}_{C_*} = (\mathrm{id}_{C_*}\boxtimes\varepsilon)\circ\Delta.$$

We call such a coalgebra *cocommutative*, if $\tau\circ\Delta=\Delta$ where τ is the symmetry in the symmetric monoidal category $(\mathsf{Ch}(k)^{\mathcal{I}}, \boxtimes, U^{\mathcal{I}})$.

A *morphism of $\mathcal{I}$-chain coalgebras* is defined as a morphism of $\mathcal{I}$-chain complexes commuting with the diagonal and the counit.

4.1. **Homotopy colimits of comonoids.** We first show that the homotopy colimit for $\mathcal{I}$-chains is an E_∞-comonoidal functor. We then use the operad O in order to produce an explicit E_∞-coalgebra structure on homotopy colimits of $\mathcal{I}$-chain complexes of the form $i_!(X_*)$ whenever X_* is a cocommutative comonoid in $\mathsf{Ch}(k)^\Sigma$.

We state a more detailed version of Theorem 1.1:

Theorem 4.2. *The homotopy colimit* $\mathsf{hocolim}_{\mathcal{I}}\colon \mathsf{Ch}(k)^{\mathcal{I}} \to \mathsf{Ch}(k)$ *is an* E_∞*-comonoidal functor,* i.e.*, there is an* E_∞*-operad* $\mathcal{E}$ *such that for all* $n \geqslant 0$ *and for all* $C_*^1, \dots, C_*^n \in \mathsf{Ch}(k)^{\mathcal{I}}$ *there are morphisms*

$$\theta_n\colon \mathcal{E}(n) \otimes \mathsf{hocolim}_{\mathcal{I}}(C_*^1 \boxtimes \dots \boxtimes C_*^n) \to \mathsf{hocolim}_{\mathcal{I}}(C_*^1) \otimes \dots \otimes \mathsf{hocolim}_{\mathcal{I}}(C_*^n)$$

that are natural in C_*^1 *up to* C_*^n *and that satisfy coherence relations dual to those in* [Ric00, Definition 3.4].

In particular, this implies that for any cocommutative comonoid C_* in $\mathsf{Ch}(k)^{\mathcal{I}}$, the homotopy colimit $\mathsf{hocolim}_{\mathcal{I}} C_*$ is cocommutative up to homotopy and all higher homotopies, so it is an differential graded E_∞-coalgebra. Here, the structure maps are

$$\begin{tikzcd} \mathcal{E}(n) \otimes \mathsf{hocolim}_{\mathcal{I}} C_* \arrow[r, "\mathrm{id}\otimes \mathsf{hocolim}_{\mathcal{I}}\Delta^{(n-1)}"] & \mathcal{E}(n) \otimes \mathsf{hocolim}_{\mathcal{I}}(C_*^{\boxtimes n}) \arrow[d, "\theta_n"] \\ & \mathsf{hocolim}_{\mathcal{I}}(C_*)^{\otimes n}, \end{tikzcd}$$

where the first map is induced by the iterated coproduct on C_* and the second one is given by the coaction of the operad.

Proof. The homotopy colimit is the composite of three functors

$$\mathsf{hocolim}_{\mathcal{I}} = \mathsf{Tot} \circ C_* \circ \mathsf{srep}.$$

- By [RS20] the functor Tot is strong symmetric monoidal, hence it is also strong symmetric comonoidal.
- In [Ric00, §7] we show that the Moore chain functor C_* is an E_∞-comonoidal functor from simplicial modules to chain complexes. This proof adapts to the setting of viewing C_* as a functor from simplicial chain complexes to bicomplexes:

  We take the E_∞ operad from [Ric00, §7] and call it $\mathcal{E}(n)$. Consider n simplicial chain complexes $A^1_{\bullet,*}, \dots, A^n_{\bullet,*}$. Here, $\bullet$ denotes the simplicial degree and $*$ the chain degree. In bidegree (p,q) the bicomplex $C_*(A^1_{\bullet,*} \hat{\otimes} \dots \hat{\otimes} A^n_{\bullet,*})$ is

$$\bigoplus_{\ell_1+\dots+\ell_n=q} A^1_{p,\ell_1} \otimes \dots \otimes A^n_{p,\ell_n} \tag{4.1}$$

  whereas in the same bidegree we get for $C_*(A^1_{\bullet,*}) \otimes \dots \otimes C_*(A^n_{\bullet,*})$:

$$(C_*(A^1_{\bullet,*}) \otimes \dots \otimes C_*(A^n_{\bullet,*}))_{p,q} = \bigoplus_{a_1+\dots+a_n=p} \bigoplus_{\ell_1+\dots+\ell_n=q} A^1_{a_1,\ell_1} \otimes \dots \otimes A^n_{a_n,\ell_n}, \tag{4.2}$$

  thus we get an action map

$$\theta_n\colon \mathcal{E}(n) \otimes C_*(A^1_{\bullet,*} \hat{\otimes} \dots \hat{\otimes} A^n_{\bullet,*}) \to C_*(A^1_{\bullet,*}) \otimes \dots \otimes C_*(A^n_{\bullet,*})$$

  by applying θ_n to every direct summand in (4.1). By construction, the action map commutes with the horizontal differential. The vertical differential on (4.1) is

$$d_v = \sum_{i=1}^n (-1)^{\ell_1+\dots+\ell_{i-1}} \mathrm{id}_{A^1_{p,\ell_1}} \otimes \dots \otimes \mathrm{id}_{A^{i-1}_{p,\ell_{i-1}}} \otimes d \otimes \mathrm{id}_{A^{i+1}_{p,\ell_{i+1}}} \otimes \dots \otimes \mathrm{id}_{A^n_{p,\ell_n}}$$

and the one on (4.2) is

$$d_v = \sum_{i=1}^{n} (-1)^{\ell_1+\ldots+\ell_{i-1}} \mathrm{id}_{A^1_{a_1,\ell_1}} \otimes \ldots \otimes \mathrm{id}_{A^{i-1}_{a_{i-1},\ell_{i-1}}} \otimes d \otimes \mathrm{id}_{A^{i+1}_{a_{i+1},\ell_{i+1}}} \otimes \ldots \otimes \mathrm{id}_{A^n_{a_n,\ell_n}};$$

see [RS20, §3] for the sign-conventions on simplicial chain complexes and bicomplexes. Naturality of the $\mathcal{E}$-action then implies that the action maps θ_n also commute with the vertical differential.

- We show that srep is lax symmetric comonoidal. So let's assume that X and Y are $\mathcal{I}$-chain complexes. In simplicial degree p we obtain

$$\mathsf{srep}(X \boxtimes Y)_p = \bigoplus_{\mathbf{n}_p \xrightarrow{\alpha_p} \ldots \xrightarrow{\alpha_1} \mathbf{n}_0} \mathrm{colim}_{\mathbf{a} \sqcup \mathbf{b} \to \mathbf{n}_p} X(\mathbf{a}) \otimes Y(\mathbf{b}),$$

whereas $\mathsf{srep}(X)_p \otimes \mathsf{srep}(Y)_p$ is of the form

$$\left(\bigoplus_{\mathbf{m}_p \xrightarrow{\beta_p} \ldots \xrightarrow{\beta_1} \mathbf{m}_0} X(\mathbf{m}_p) \right) \otimes \left(\bigoplus_{\mathbf{q}_p \xrightarrow{\gamma_p} \ldots \xrightarrow{\gamma_1} \mathbf{q}_0} Y(\mathbf{q}_p) \right).$$

We define $\theta_{X,Y} \colon \mathsf{srep}(X \boxtimes Y)_p \to \mathsf{srep}(X)_p \otimes \mathsf{srep}(Y)_p$ on a generator

$$[\alpha_p \mid \ldots \mid \alpha_1] \otimes [(\varphi \colon \mathbf{a} \sqcup \mathbf{b} \to \mathbf{n}_p) \otimes x \otimes y]$$

as

$$[\alpha_p \mid \ldots \mid \alpha_1] \otimes X(\varphi \circ i_a)(x) \otimes [\alpha_p \mid \ldots \mid \alpha_1] \otimes Y(\varphi \circ j_b)(y)$$

where $i_a \colon \mathbf{a} \to \mathbf{a} \sqcup \mathbf{b}$ sends $\ell \in \mathbf{a}$ to ℓ and $j_b \colon \mathbf{b} \to \mathbf{a} \sqcup \mathbf{b}$ maps $k \in \mathbf{b}$ to $k + a$.

The map $\theta_{X,Y}$ is well-defined: for $f \colon \mathbf{c} \to \mathbf{a}$, $g \colon \mathbf{d} \to \mathbf{b}$, $z \in X(\mathbf{c})$ and $w \in Y(\mathbf{d})$ we obtain that $[\alpha_p \mid \ldots \mid \alpha_1] \otimes [\varphi \circ (f \sqcup g) \otimes z \otimes w]$ maps to

$$[\alpha_p \mid \ldots \mid \alpha_1] \otimes X(\varphi \circ (f \sqcup g) \circ i_c)(z) \otimes [\alpha_p \mid \ldots \mid \alpha_1] \otimes Y(\varphi \circ (f \sqcup g) \circ j_d)(w)$$

and $[\alpha_p \mid \ldots \mid \alpha_1] \otimes [\varphi \otimes X(f)(z) \otimes Y(g)(w)]$ is sent to

$$[\alpha_p \mid \ldots \mid \alpha_1] \otimes X(\varphi \circ i_a)(X(f)(z)) \otimes [\alpha_p \mid \ldots \mid \alpha_1] \otimes Y(\varphi \circ j_b)(Y(g)(w)).$$

As $\varphi \circ (f \sqcup g) \circ i_c = \varphi \circ i_a \circ f$ and $\varphi \circ (f \sqcup g) \circ j_d = \varphi \circ j_b \circ g$, these terms agree.

If we apply the symmetry of $\boxtimes$ first to

$$[\alpha_p \mid \ldots \mid \alpha_1] \otimes [(\varphi \colon \mathbf{a} \sqcup \mathbf{b} \to \mathbf{n}_p) \otimes x \otimes y]$$

we get

$$(-1)^{|x||y|}[\alpha_p \mid \ldots \mid \alpha_1] \otimes [(\varphi \circ \chi_{b,a} \colon \mathbf{b} \sqcup \mathbf{a} \to \mathbf{n}_p) \otimes y \otimes x]$$

and $\theta_{Y,X}$ maps this to

$$(-1)^{|x||y|}[\alpha_p \mid \ldots \mid \alpha_1] \otimes Y(\varphi \circ \chi_{b,a} \circ i_b)(y) \otimes [\alpha_p \mid \ldots \mid \alpha_1] \otimes X(\varphi \circ \chi_{b,a} \circ j_a)(x).$$

Applying the twist to $\theta_{X,Y}(\alpha_p \mid \ldots \mid \alpha_1] \otimes [\varphi \otimes x \otimes y])$ yields

$$(-1)^{|x||y|}[\alpha_p \mid \ldots \mid \alpha_1] \otimes Y(\varphi \circ j_b)(y) \otimes [\alpha_p \mid \ldots \mid \alpha_1] \otimes X(\varphi \circ i_a)(x).$$

But as $\chi_{b,a} \circ i_b = j_b$ and $\chi_{b,a} \circ j_a = i_a$, both values coincide so θ is compatible with the symmetry.

It is straightforward to see that θ is associative. There is a canonical map from the constant simplicial chain complex on S^0 to $\mathsf{srep}(F_0^{\mathcal{I}}(S^0))$ that sends $1 \in S^0$ in simplicial degree p to the element

$$[\mathrm{id}_{\mathbf{0}} \mid \ldots \mid \mathrm{id}_{\mathbf{0}}] \otimes 1 \in \bigoplus_{\mathbf{q}_p \xrightarrow{\gamma_p} \ldots \xrightarrow{\gamma_1} \mathbf{q}_0} (F_0^{\mathcal{I}}(S^0)(\mathbf{q}_p))_0 = \bigoplus_{\mathbf{q}_p \xrightarrow{\gamma_p} \ldots \xrightarrow{\gamma_1} \mathbf{q}_0} (S^0)_0.$$

□

Remark 4.3. The functor $\mathsf{hocolim}_{\mathcal{I}}$ is lax comonoidal. The functor C_* is E_∞-comonoidal because the structure map $C_*(A_\bullet \hat{\otimes} B_\bullet) \to C_*(A_\bullet) \otimes C_*(B_*)$ for simplicial modules $A_\bullet$ and $B_\bullet$ is the Alexander-Whitney map, so the functor C_* is lax comonoidal.

We will also use the E_∞-operad $\mathcal{E}$ from above for the specific E_∞-comonoidal structure on Kan extensions of cocommutative comonoids.

Lemma 4.4. *For every $m \geqslant 0$ and every n-tuple of numbers $(p_1, \ldots, p_n)$ with $p_1 + \ldots + p_n = m$ and $p_i \in \mathbb{N}_0$ there is a $\Sigma_{p_1} \times \ldots \times \Sigma_{p_n}$-equivariant map*

$$\psi_{p_1,\ldots,p_n} \colon \mathcal{E}(n) \otimes O(m) \to O(p_1) \otimes \ldots \otimes O(p_1) \tag{4.3}$$

that satisfies the axioms of a coaction of the operad $\mathcal{E}$ on the $\mathbb{N}_0$-graded chain complex O.

Proof. As we constructed the operad O starting from the category of small categories we do the same for the maps $\psi_{p_1,\ldots,p_n}$: For every $m \geqslant 0$ and all $p_1, \ldots, p_n$ with $p_1 + \ldots + p_n = m$ there is a functor

$$P_{p_1,\ldots,p_n} \colon \mathbf{m} \downarrow \mathcal{I} \to \mathbf{p}_1 \downarrow \mathcal{I} \times \ldots \times \mathbf{p}_n \downarrow \mathcal{I}.$$

On objects, $P_{p_1,\ldots,p_n}$ sends a $\varphi \colon \mathbf{m} \to \mathbf{q}$ to the n-tuple $(\varphi \circ i_{1,\ldots,p_1}, \ldots, \varphi \circ i_{p_1+\ldots+p_{n-1}+1,\ldots,m})$, where $i_{p_1+\ldots+p_{j-1}+1,\ldots,p_1+\ldots+p_j} \in \mathcal{I}(\mathbf{p}_j, \mathbf{m})$ is given by $i_{p_1+\ldots+p_{j-1}+1,\ldots,p_1+\ldots+p_j}(\ell) = p_1 + \ldots + p_{j-1} + \ell$. A morphism g in $\mathbf{m} \downarrow \mathcal{I}$ is sent to the n-tuple of morphisms given by postcomposition by g.

If $\tau_j \in \Sigma_{p_j}$, then for $\tau_1 \oplus \ldots \oplus \tau_n \in \Sigma_{p_1} \times \ldots \times \Sigma_{p_n}$ we get

$$(\tau_1 \oplus \ldots \oplus \tau_n) \circ i_{p_1+\ldots+p_{j-1}+1,\ldots,p_1+\ldots+p_j} = i_{p_1+\ldots+p_{j-1}+1,\ldots,p_1+\ldots+p_j} \circ \tau_j.$$

Hence $P_{p_1,\ldots,p_n}$ is $\Sigma_{p_1} \times \ldots \times \Sigma_{p_n}$-equivariant.

As the nerve functor is strong symmetric monoidal, we obtain a $\Sigma_{p_1} \times \ldots \times \Sigma_{p_n}$-equivariant morphism of simplicial sets

$$N(P_{p_1,\ldots,p_n}) \colon N(\mathbf{m} \downarrow \mathcal{I}) \to N(\mathbf{p}_1 \downarrow \mathcal{I}) \times \ldots \times N(\mathbf{p}_n \downarrow \mathcal{I})$$

and also an equivariant morphism of simplicial k-modules

$$k\{N(P_{p_1,\ldots,p_n})\} \colon k\{N(\mathbf{m} \downarrow \mathcal{I})\} \to k\{N(\mathbf{p}_1 \downarrow \mathcal{I})\} \hat{\otimes} \ldots \hat{\otimes} k\{N(\mathbf{p}_n \downarrow \mathcal{I})\}.$$

The induced map on the corresponding chain complexes is

$$C_*(k\{N(P_{p_1,\ldots,p_n})\}) \colon O(m) = C_*(k\{N(\mathbf{m} \downarrow \mathcal{I})\}) \to C_*(k\{N(\mathbf{p}_1 \downarrow \mathcal{I})\} \hat{\otimes} \ldots \hat{\otimes} k\{N(\mathbf{p}_n \downarrow \mathcal{I})\}).$$

We use the action of the operad $\mathcal{E}$ to get a map

$$\theta_n \colon \mathcal{E}(n) \otimes C_*(k\{N(\mathbf{p}_1 \downarrow \mathcal{I})\} \hat{\otimes} \ldots \hat{\otimes} k\{N(\mathbf{p}_n \downarrow \mathcal{I})\}) \to O(p_1) \otimes \ldots \otimes O(p_n)$$

and we define the map $\psi_{p_1,\ldots,p_n}$ as the composite

$$\begin{tikzcd}
\mathcal{E}(n)\otimes O(m) \arrow[r, "\mathrm{id}_{\mathcal{E}(n)\otimes C_*(k\{N(P_{p_1,\ldots,p_n})\})}"] & \mathcal{E}(n)\otimes C_*(k\{N(\mathbf{p}_1\downarrow\mathcal{I})\}\hat{\otimes}\ldots\hat{\otimes}k\{N(\mathbf{p}_n\downarrow\mathcal{I})\}) \arrow[d, "\theta_n"] \\
& O(p_1)\otimes\ldots\otimes O(p_n).
\end{tikzcd}$$

As we only applied functors that are at least lax symmetric monoidal and as the θ_ns assemble into a coaction of $\mathcal{E}$, the coherence claim follows. □

Theorem 4.5. *If X_* is a cocommutative comonoid in $\mathsf{Ch}(k)^\Sigma$, then $\mathsf{hocolim}_{\mathcal{I}} i_!(X_*)$ is an E_∞ differential graded coalgebra employing the equivariant maps $\psi_{p_1,\ldots,p_n}$ from Lemma 4.4.*

Proof. As the functor $i_!$ is strong symmetric monoidal, $i_!(X_*)$ is a cocommutative comonoid in $\mathcal{I}$-chain complexes. The counit $X_*\to F_0^\Sigma S^0$ gives rise to a counit $i_!(X_*)\to i_!F_0^\Sigma S^0\cong F_0^{\mathcal{I}}S^0$ and by Corollary 3.8 we obtain for the corresponding homotopy colimits a morphism

$$\mathsf{hocolim}_{\mathcal{I}} i_!(X_*)\to O(0)\otimes S^0\cong S^0.$$

An iteration of the coproduct Δ_{X_*} on X_* induces a morphism

$$\delta^h\colon \mathsf{hocolim}_{\mathcal{I}} i_!(X_*)\xrightarrow{\mathsf{hocolim}_{\mathcal{I}} i_!(\Delta_{X_*}^{(n-1)})}\mathsf{hocolim}_{\mathcal{I}} i_!(X_*^{\odot n}).$$

As $\mathsf{hocolim}_{\mathcal{I}} i_!(X_*^{\odot n})$ is the chain complex

$$\begin{aligned}
&\bigoplus_{m\geqslant 0} O(m)\otimes_{\Sigma_m}(X_*^{\odot n})(\mathbf{m})\\
=&\bigoplus_{m\geqslant 0} O(m)\otimes_{\Sigma_m}\Big(\bigoplus_{p_1+\ldots+p_n=m} k[\Sigma_m]\otimes_{k[\Sigma_{p_1}\times\ldots\times\Sigma_{p_n}]} X_*(\mathbf{p}_1)\otimes\ldots\otimes X_*(\mathbf{p}_n)\Big)\\
\cong&\bigoplus_{m\geqslant 0}\bigoplus_{p_1+\ldots+p_n=m} O(m)\otimes_{k[\Sigma_{p_1}\times\ldots\times\Sigma_{p_n}]} X_*(\mathbf{p}_1)\otimes\ldots\otimes X_*(\mathbf{p}_n),
\end{aligned}$$

we can use the maps $\psi_{p_1,\ldots,p_n}$ from the previous lemma in order to obtain a morphism to

$$\begin{aligned}
&\bigoplus_{m\geqslant 0}\bigoplus_{p_1+\ldots+p_n=m} O(p_1)\otimes\ldots\otimes O(p_n)\otimes_{k[\Sigma_{p_1}\times\ldots\times\Sigma_{p_n}]} X_*(\mathbf{p}_1)\otimes\ldots\otimes X_*(\mathbf{p}_n)\\
\cong&\bigoplus_{m\geqslant 0}\bigoplus_{p_1+\ldots+p_n=m} O(p_1)\otimes_{k[\Sigma_{p_1}]} X_*(\mathbf{p}_1)\otimes\ldots\otimes O(p_n)\otimes_{k[\Sigma_{p_n}]} X_*(\mathbf{p}_n)
\end{aligned}$$

and this is nothing but $\mathsf{hocolim}_{\mathcal{I}} i_!(X_*)^{\otimes n}$. □

4.2. **Connected cofree objects.** As for ordinary chain complexes, the tensor algebra from (3.3) has a deconcatenation coproduct

$$\Delta_{\mathrm{dec}}\colon \mathsf{T}^{\mathcal{I}}(C_*)\to\mathsf{T}^{\mathcal{I}}(C_*)\boxtimes\mathsf{T}^{\mathcal{I}}(C_*).$$

Here, Δ_{dec} on the direct summand $C_*^{\boxtimes n}$ is defined as the sum of the canonical isomorphisms $c_{i,n-i}^{-1}\colon C_*^{\boxtimes n}\to C_*^{\boxtimes i}\boxtimes C_*^{\boxtimes n-i}$ (see 3.4)

$$C_*^{\boxtimes n}\cong U^{\mathcal{I}}\boxtimes C_*^{\boxtimes n}\cong C_*\boxtimes C_*^{\boxtimes n-1}\cong\ldots\cong C_*^{\boxtimes n-1}\boxtimes C_*\cong C_*^{\boxtimes n}.$$

If we work in unbounded chain complexes, then $\mathsf{T}^{\mathcal{I}}(C_*)$ will not be a cofree object. But if we use 1-truncated chain complexes, then this works out. We denote by $\mathsf{Ch}(k)_{\geqslant 1}$ the full subcategory of $\mathsf{Ch}(k)$ consisting of chain complexes that are concentrated in degrees $\geqslant 1$. Let $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$ be the category of reduced objects in $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1}$.

Definition 4.6. The category of *connective comonoids in* $\mathsf{Ch}(k)^{\mathcal{I}}$, $\mathsf{cCoalg}^{\mathcal{I}}$, is the full subcategory of the category of comonoids in $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 0}$ consisting of $\mathcal{I}$-chain coalgebras C_* with a morphism of coalgebras $U^{\mathcal{I}} \to C_*$ such that for $\bar{C}_*$ with $C_* \cong \bar{C}_* \oplus U^{\mathcal{I}}$ we have that $\bar{C}_*(\mathbf{0}) = 0$, $\bar{C}_i(\mathbf{n}) = 0$ for all $n \in \mathcal{I}_+$ and $i \leqslant 0$.

Note that the assumptions ensure that $C_*(\mathbf{0}) = U^{\mathcal{I}}(\mathbf{0}) = S^0$ and that the only elements in chain degree zero come from $U^{\mathcal{I}}$.

Lemma 4.7. *The functor $\mathsf{T}^{\mathcal{I}}$ is right adjoint to the functor $U\colon \mathsf{cCoalg}^{\mathcal{I}} \to \mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$ that sends a connected comonoid C_* to $\bar{C}_*$.*

Proof. The proof of the lemma is adapted from the one in [Qui69, Appendix B].

If $\psi\colon D_* \to \mathsf{T}^{\mathcal{I}}(C_*)$ is a morphisms in $\mathsf{cCoalg}^{\mathcal{I}}$ for $C_* \in \mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$, then $U(\varphi)\colon U(D_*) \to U(\mathsf{T}^{\mathcal{I}}(C_*))$ and we define the morphism $f_\psi\colon U(D_*) \to C_*$ as the composite of $U(\varphi)$ with the morphism that is induced by the projection $p_1\colon \mathsf{T}^{\mathcal{I}}(C_*) \to C_*$ onto the tensor strings of length one.

Now assume, that $f\colon U(D_*) \to C_*$ is a morphism in $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$. Let $\Delta^{(m)}$ denote the m-fold iteration of the diagonal on D_*. As we have coassociativity, one incarnation of this map is $(\Delta \boxtimes \mathrm{id}_D^{\boxtimes m-1}) \circ \ldots \circ (\Delta \boxtimes \mathrm{id}_{D_*}) \circ \Delta$.

We extend $f\colon U(D_*) \to C_*$ in $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$ to $\bar{f}\colon D_* \to C_*$ by setting $\bar{f}|_{U^{\mathcal{I}}} = 0$ and $\bar{f}|_{\bar{D}_*} = f$.

We then define $\psi_f\colon D_* \cong \bar{D}_* \oplus U^{\mathcal{I}} \to \mathsf{T}^{\mathcal{I}}(C_*)$ by setting $\psi_f|_{U^{\mathcal{I}}} = \mathrm{id}_{U^{\mathcal{I}}}$ and

$$\psi_f|_{\bar{D}_*} = \sum_{n \geqslant 1} \bar{f}^{\boxtimes n} \circ \Delta^{(n-1)}\colon \bar{D}_* \to \mathsf{T}^{\mathcal{I}}(C_*) \tag{4.4}$$

We have to show that the map ψ_f is well-defined.

As $\bar{D}(\mathbf{n})_0 = 0$, as $\bar{D}(\mathbf{0}) = 0$ and as the $\boxtimes$-product is still built out of the ordinary tensor product of complexes, any $d \in \bar{D}(\mathbf{n})_\ell$ for positive ℓ only has a finite number of non-trivial values of $\bar{f}^{\boxtimes m+1} \circ \Delta^{(m)}(d)$ because $\bar{f}$ is trivial on $U^{\mathcal{I}}$ and $U(D_*) \in \mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$. Therefore the sum in (4.4) is finite.

We claim that ψ_f is a morphism of comonoids, so we have to show that the diagram

$$\begin{tikzcd} D_* \arrow[r, "\psi_f"] \arrow[d, "\Delta"'] & \mathsf{T}^{\mathcal{I}}(C_*) \arrow[d, "\Delta_{\mathrm{dec}}"] \\ D_* \boxtimes D_* \arrow[r, "\psi_f \boxtimes \psi_f"] & \mathsf{T}^{\mathcal{I}}(C_*) \boxtimes \mathsf{T}^{\mathcal{I}}(C_*) \end{tikzcd} \tag{4.5}$$

commutes and it suffices to check the claim on $\bar{D}_*$.

Coassociativity gives as in [Qui69, Appendix B, (4.2)] that for all natural numbers p and q

$$(\Delta^{p-1} \boxtimes \Delta^{q-1}) \circ \Delta = \Delta^{p+q-1}.$$

Applying Δ_{dec} to an n-fold $\boxtimes$-product gives deconcatenations at the ith spot with i from 0 to n and we denote such a deconcatenation by dec_i. This yields

$$\begin{aligned}\Delta_{\mathrm{dec}} \circ \psi_f &= \Delta_{\mathrm{dec}} \circ \sum_{n\geqslant 1} \bar{f}^{\boxtimes n} \circ \Delta^{(n-1)} \\ &= \sum_{n\geqslant 1}\sum_{i=0}^{n} \mathrm{dec}_i \circ \bar{f}^{\boxtimes n} \circ (\Delta^{i-1} \boxtimes \Delta^{n-i-1}) \circ \Delta \\ &= \sum_{n\geqslant 1}\sum_{i=0}^{n} (\bar{f}^{\boxtimes i} \boxtimes \bar{f}^{\boxtimes n-i}) \circ (\Delta^{i-1} \boxtimes \Delta^{n-i-1}) \circ \Delta \\ &= \Big(\big(\sum_{i\geqslant 1} \bar{f}^{\boxtimes i} \circ \Delta^{i-1}\big) \boxtimes \big(\sum_{j\geqslant 1} \bar{f}^{\boxtimes j} \circ \Delta^{j-1}\big)\Big) \circ \Delta \\ &= (\psi_f \boxtimes \psi_f) \circ \Delta.\end{aligned}$$

It is clear that $f_{\psi_f} = f$ for any $f \in \mathsf{Ch}(k)_{\geqslant 1}(U(D_*), C_*)$. So it remains to show that every map ψ in $\mathsf{cCoalg}^{\mathcal{I}}$ from D_* to $\mathsf{T}^{\mathcal{I}}(C_*)$ with $f_\psi = f$ is of the form ψ_f. But it follows from diagram (4.5) with an induction on the chain degree of the element d in positive level that every such map ψ satisfies that the component of $\psi(d)$ in $C_*^{\boxtimes n}$ is determined by $\boxtimes$-powers of f and $\Delta^{(n-1)}$. □

If D_* is cocommutative, then its diagonal takes values in the Σ-invariants $\bigoplus_{n\geqslant 0} \big(D_*^{\boxtimes n}\big)^{\Sigma_n}$. We denote by $\mathsf{ccCoalg}^{\mathcal{I}}$ the full subcategory of $\mathsf{cCoalg}^{\mathcal{I}}$ consting of cocommutative connected comonoids in $\mathcal{I}$-chain complexes.

For any $\mathcal{I}$-chain complex C_* we denote by $\Gamma^{\mathcal{I}}(C_*)$ the $\mathcal{I}$-chain complex

$$\Gamma^{\mathcal{I}}(C_*) := \bigoplus_{\ell\geqslant 0} \Big(C_*^{\boxtimes \ell}\Big)^{\Sigma_\ell}. \tag{4.6}$$

We use $\Gamma^{\mathcal{I}}$ because the analoguous construction in the category of modules or chain complexes would result in the free (dg) divided power algebra.

The argument from [Qui69, Appendix B] then translates to the setting of $\mathcal{I}$-chain complexes:

Proposition 4.8. *The functor $\Gamma^{\mathcal{I}}$ is right adjoint to the functor $U\colon \mathsf{ccCoalg}^{\mathcal{I}} \to \mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$ that sends a connected cocommutative comonoid C_* to $\bar{C}$.* □

Thus $\Gamma^{\mathcal{I}}(C_*)$ is the cofree connected cocommutative comonoid generated by C_*.

5. The functors I_n

In the category of Σ-chain complexes or Σ-modules there is the free functor F_0^Σ that embeds chain complexes and modules into the diagram category by placing them in level zero and by assigning zero to any other $\mathbf{n}$. Reducing objects then corresponds to discarding $F_0^\Sigma(S^0)$ or $F_0^\Sigma(k)$ respectively. One can do the same thing in the context of $\mathcal{I}$-diagrams, but here the functor won't behave nicely and discarding it doesn't solve problems. However, it plays an important role: Tensoring with this object gives rise to the H_0-construction of [CEF15, Definition 2.3.7].

Definition 5.1. Let X_* be a chain complex. Then $I_0(X_*)$ is the $\mathcal{I}$-chain complex with

$$I_0(X_*)(\mathbf{n}) = \begin{cases} X_*, & n = 0, \\ 0, & \text{otherwise.}\end{cases}$$

All morphisms that are not the identity on $\mathbf{0}$ induce the zero map.

This actually defines a functor $I_0\colon \mathsf{Ch}(k) \to \mathsf{Ch}(k)^{\mathcal{I}}$.

Lemma 5.2. *The functor $I_0\colon \mathsf{Ch}(k) \to \mathsf{Ch}(k)^{\mathcal{I}}$ is lax symmetric monoidal.*

Proof. There is a canonical projection $\pi\colon U^{\mathcal{I}} \to I_0(S^0)$ that is the identity in level zero and the trivial map in positive levels.

For two chain complexes C_*, D_* we have by definition

$$I_0(C_*) \boxtimes I_0(D_*)(\mathbf{n}) = \mathrm{colim}_{\mathbf{p}\sqcup\mathbf{q}\to\mathbf{n}}\, I_0(C_*)(\mathbf{p}) \otimes I_0(D_*)(\mathbf{q}).$$

If n is zero, then we just obtain

$$I_0(C_*) \boxtimes I_0(D_*)(\mathbf{0}) = I_0(C_*)(\mathbf{0}) \otimes I_0(D_*)(\mathbf{0}) = C_* \otimes D_* = I_0(C_* \otimes D_*)(\mathbf{0}).$$

For positive n we can write

$$I_0(C_*) \boxtimes I_0(D_*)(\mathbf{n}) = \left(\bigoplus_{p+q\leqslant n} I_0(C_*)(\mathbf{p}) \otimes I_0(D_*)(\mathbf{q})\right) / \sim$$

where the equivalence relation is generated by $x \otimes y \sim \varphi_*(x) \otimes \psi_*(y)$ for $xI_0(C_*)(\mathbf{p})$, $y \in I_0(D_*)(\mathbf{q})$ and $\varphi \in \mathcal{I}(\mathbf{p},\mathbf{r})$, $\psi \in \mathcal{I}(\mathbf{q},\mathbf{r})$. But these $x \otimes y$ are only possibly non-trivial for $p = q = 0$ and the equivalence relation identifies those elements with something trivial.

Hence,

$$I_0(C_*) \boxtimes I_0(D_*) \cong I_0(C_* \otimes D_*).$$

The identification is compatible with the twist map and satisfies associativity. □

Proposition 5.3. *For any chain complex X_*, the homotopy colimit of $I_0(X_*)$ is acyclic:*

$$H_\ell(\mathsf{hocolim}_{\mathcal{I}} I_0(X_*)) = 0 \textit{ for all } \ell \geqslant 0.$$

Proof. In the proof of [SaS12, Corollary 5.9] it is shown that the inclusion $\mathcal{I}^+ \subset \mathcal{I}$ is homotopy cofinal. This allows us to replace $\mathsf{hocolim}_{\mathcal{I}} I_0(X_*)$ by $\mathsf{hocolim}_{\mathcal{I}_+} I_0(X_*)$, but the latter is trivial on the nose.

□

For any $\mathcal{I}$-chain complex Y_* Church-Ellenberg-Farb [CEF15] define a Σ-chain complex $H_0(Y_*)$ by setting $H_0(Y_*)(\mathbf{n}) := Y_*(\mathbf{n})/(Y_*)(\leqslant n-1)$ where we denote by $(Y_*)(\leqslant n-1)$ everything in $Y_*(\mathbf{n})$ that is in the image of $Y_*(\varphi)$ for some $\varphi \in \mathcal{I}(\mathbf{m},\mathbf{n})$ with $m < n$.

We view H_0 as an endofunctor on the category of $\mathcal{I}$-chain complexes by embedding Σ-chain complexes into $\mathcal{I}$-chain complexes.

Proposition 5.4. *For all $\mathcal{I}$-chain complexes Y_*:*

$$I_0(S^0) \boxtimes Y_* \cong H_0(Y_*) \cong Y_* \boxtimes I_0(S^0).$$

Proof. Again, we use the explicit form of the $\boxtimes$-product and get

$$I_0(S^0) \boxtimes Y_*(\mathbf{n}) = \left(\bigoplus_{p+q\leqslant n} I_0(S^0)(\mathbf{p}) \otimes Y_*(\mathbf{q})\right) / \sim .$$

If $q \neq n$, then there are terms in the colimit with $p > 0$, so that possible non-trivial elements coming from $I_0(S^0)(\mathbf{0}) \otimes Y_*(\mathbf{q})$ are identified with zero. Elements in the summand for $p = 0$ and $q = n$ have the potential to survive coming from elements in

$$I_0(S^0)(\mathbf{0}) \otimes Y_*(\mathbf{n}) = S^0 \otimes Y_*(\mathbf{n}) \cong Y_*(\mathbf{n}).$$

However, elements in $Y_*(\mathbf{n})$ of the form $\psi_*(y)$ for any $\psi \in \mathcal{I}(\mathbf{m},\mathbf{n})$ for $m < n$ are again identified with zero. $\square$

The category $F\mathcal{I}^{\#}$ is the category whose objects are the objects of $\mathcal{I}$, but morphisms in $F\mathcal{I}^{\#}$ from $\mathbf{m}$ to $\mathbf{n}$ are tripels (S, σ, T) with $S \subset \mathbf{m}$, $T \subset \mathbf{n}$ and where $\sigma\colon S \cong T$ is a bijection. The extension of the functor H_0 to a functor from the category $F\mathcal{I}^{\#}$-modules to Σ-modules is actually an equivalence of categories [CEF15, Theorem 4.1.5]. We will see in Lemma 7.4 why the functor I_0 causes trouble for the norm map.

Remark 5.5. If P is a projective module, then $F_m^{\mathcal{I}}(P)$ is a projective object in the category of $\mathcal{I}$-modules. However, $I_0(P)$ will in general *not* be projective: Assume that $q\colon M \to Q$ is an arbitrary epimorphism of $\mathcal{I}$-modules and that $f\colon I_0(P) \to Q$ is a morphism of $\mathcal{I}$-modules. We always get a lift $\xi(\mathbf{0})\colon P \to M(\mathbf{0})$ of $f(\mathbf{0})$ because P is projective. If this lift were natural in $\mathcal{I}$, then the unique $i_m \in \mathcal{I}(\mathbf{0},\mathbf{m})$ would have to satisfy $M(i_m) \circ \xi(\mathbf{0}) = 0$. But for general M the morphism $M(i_m)$ doesn't have to be trivial.

We generalize I_0 to functors $I_n\colon \mathsf{Ch}(k)^{\Sigma_n} \to \mathsf{Ch}(k)^{\mathcal{I}}$:

Definition 5.6. Let C_* be a chain complex with Σ_n-action for $n \geqslant 0$. We define $I_n(C_*) \in \mathsf{Ch}(k)^{\mathcal{I}}$ as the $\mathcal{I}$-chain complex with

$$I_n(C_*)(\mathbf{m}) := \begin{cases} C_*, & \text{for } n = m, \\ 0, & \text{otherwise.} \end{cases}$$

Morphisms in Σ_n act via the Σ_n-action on C_* and all other morphisms in $\mathcal{I}$ induce the zero map.

Similarly, we can consider $\Sigma_n(C_*)$ as the symmetric sequence with C_* in level n and 0 in all other levels.

Lemma 5.7. *For all $n, m \geqslant 0$ there are isomorphisms $I_n(C_*) \boxtimes I_m(D_*) \cong I_{n+m}(\Sigma_n(C_*) \odot \Sigma_m(D_*))$.*

Proof. As before it is not hard to see that $I_n(C_*) \boxtimes I_m(D_*)(\mathbf{r})$ is trivial if $r \neq n+m$: For $r < n+m$ and any $\varphi\colon \mathbf{p} \to \mathbf{q} \to \mathbf{r}$, p or q has to be smaller than n, respectively m, so we get trivial factors in the tensor product. Any $\varphi\colon \mathbf{p} \sqcup \mathbf{q} \to \mathbf{r}$ with $r > n+m$ factors as $\psi \circ (f_1 \sqcup f_2)$ with $f_1\colon \mathbf{p} \to \mathbf{s}$ and $f_2\colon \mathbf{q} \to \mathbf{t}$ with s or t greater than n respectively m.

For $\mathbf{r} = \mathbf{n} + \mathbf{m}$ we obtain

$$I_n(C_*) \boxtimes I_m(D_*)(\mathbf{n}+\mathbf{m}) = \mathrm{colim}_{\mathbf{p}\sqcup\mathbf{q}\to\mathbf{n}+\mathbf{m}}\, I_n(C_*)(\mathbf{p}) \otimes I_m(D_*)(\mathbf{q}) \cong \mathrm{colim}_{\mathbf{n}\sqcup\mathbf{m}\to\mathbf{n}+\mathbf{m}}\, C_* \otimes D_*$$

and this term is isomorphic to

$$\mathbb{Z}[\Sigma_{n+m}] \otimes_{\mathbb{Z}[\Sigma_n \times \Sigma_m]} C_* \otimes D_* = (\Sigma_n(C_*) \odot \Sigma_m(D_*))(\mathbf{n}+\mathbf{m}).$$

$\square$

6. No Künneth isomorphisms in general

Another feature of $\mathcal{I}$-chain complexes is the absense of a Künneth isomorphism, even if one works with chain complexes over a field. For any $X_* \in \mathsf{Ch}(k)^{\mathcal{I}}$ let $H_*(X_*)$ be the graded $\mathcal{I}$-module with $H_*(X_*)(\mathbf{n}) := H_*(X_*(\mathbf{n}))$:

Proposition 6.1. *In general, the canonical map $H_*X_* \boxtimes H_*Y_* \to H_*(X_* \boxtimes Y_*)$ is not an isomorphism of graded $\mathcal{I}$-modules, even if we work over a field k.*

The reason for this failure is that the colimit that is used to build the Day convolution product does not commute with homology. As a proof of Proposition 6.1 we offer the following basic counterexample.

Example 6.2. Let $\mathsf{Sym}(D^1)$ denote the $\mathcal{I}$-chain complex with $\mathsf{Sym}(D^1)(\mathbf{n}) = (D^1)^{\otimes n}$ with $S^0 = (D^1)^{\otimes 0} = \mathsf{Sym}(D^1)(\mathbf{0})$ and such that the unique map from $\mathbf{0}$ to any $\mathbf{n}$ sends S^0 to

$$S^0 \cong (S^0)^{\otimes n} \subset (D^1)^{\otimes n}.$$

Then $(H_*(\mathsf{Sym}(D^1))) \boxtimes H_*(\mathsf{Sym}(D^1)))(\mathbf{1})$ is trivial. By definition

$$(H_*(\mathsf{Sym}(D^1) \boxtimes H_*(\mathsf{Sym}(D^1))_\ell(\mathbf{n}) = \mathrm{colim}_{\mathbf{p}\sqcup\mathbf{q}\to\mathbf{n}} \bigoplus_{r+s=\ell} H_r((D^1)^{\otimes p}) \otimes H_s((D^1)^{\otimes q}).$$

The right-hand side is isomorphic to $p+q$-tensor powers of the homology of D^1. If $n = 1$, then we obtain the trivial colimit because the colimit is a quotient of the trivial module.

However, $H_1(\mathsf{Sym}(D^1) \boxtimes \mathsf{Sym}(D^1))(\mathbf{1}) \neq 0$: For $H_1(\mathsf{Sym}(D^1) \boxtimes \mathsf{Sym}(D^1))(\mathbf{1})$ we obtain the first homology of the colimit

$$\begin{tikzcd} S^0 \otimes S^0 \arrow[r] \arrow[d] & S^0 \otimes D^1 \\ D^1 \otimes S^0 & \end{tikzcd}$$

and this is isomorphic to the homology of the pushout $D^1 \oplus_{S^0} D^1$ where the degree zero parts of the two 1-disks are identified. This pushout has a nontrivial 1-cycle

$$(1, -1) \in D^1 \oplus_{S^0} D^1$$

which is not a boundary.

Similar to [Sch07, Lemma 3.1] one can show that for all $S^0 \to X_*$ and $S^0 \to Y_*$ in $\mathsf{Ch}(k)$ there is an isomorphism of $\mathcal{I}$-chain complexes

$$\mathsf{Sym}(X_*) \boxtimes \mathsf{Sym}(Y_*) \cong \mathsf{Sym}(X_* \oplus_{S^0} Y_*).$$

We will see later in Lemma 8.2 that $\mathsf{hocolim}_{\mathcal{I}} C_* \otimes \mathsf{hocolim}_{\mathcal{I}} D_*$ and $\mathsf{hocolim}_{\mathcal{I}}(C_* \boxtimes D_*)$ are quasi-isomorphic in good cases.

Remark 6.3. Note that for symmetric sequences of chain complexes over a field there *is* a Künneth isomorphism: For $X_*, Y_* \in \mathsf{Ch}(k)^\Sigma$ the ℓth homology of $X_* \odot Y_*(\mathbf{n})$ is the ℓth homology of

$$\bigoplus_{p+q=n} k[\Sigma_n] \otimes_{k[\Sigma_p\times\Sigma_q]} X_*(\mathbf{p}) \otimes Y_*(\mathbf{q}).$$

But as the $\Sigma_p \times \Sigma_q$-action on Σ_n is free, we can identify this as the ℓth homology of

$$\bigoplus_{p+q=n} \bigoplus_{\Sigma_n/\Sigma_p\times\Sigma_q} X_*(\mathbf{p}) \otimes Y_*(\mathbf{q})$$

and as we work over a field this is isomorphic to

$$\bigoplus_{p+q=n} k[\Sigma_n] \otimes_{k[\Sigma_p\times\Sigma_q]} (H_*X_*(\mathbf{p}) \otimes H_*Y_*(\mathbf{q}))_\ell.$$

7. The norm map

Stover showed in [Sto93, §9] that the norm map $N = \sum_{\sigma \in \Sigma_n}$ induces an isomorphism

$$N\colon M^{\odot n}/\Sigma_n \to (M^{\odot n})^{\Sigma_n}$$

for every reduced symmetric sequence M and every $n \geqslant 0$. His approach is to find a suitable combinatorial description of $M^{\odot n}$ in terms of unshuffles and to use this to choose an explicit set of representatives for the Σ_n-action on $M^{\odot n}$.

We call an object $\mathbf{m}$ of $\mathcal{I}$ *positive* if $|\mathbf{m}| \geqslant 1$ and write $\mathcal{I}_+$ for the full subcategory of positive objects in $\mathcal{I}$.

In a k-fold $\boxtimes$-product of an $\mathcal{I}$-chain complex V_* we take a colimit over the category $\mathcal{I} \sqcup \ldots \sqcup \mathcal{I} \downarrow \mathbf{n}$ for $X_*^{\boxtimes k}(\mathbf{n})$. If we assume that V_* is reduced, then the indexing category can be simplified to

$$\mathcal{I}_+^{\sqcup k} \downarrow \mathbf{n}.$$

The following well-known fact ensures that the category of $\mathcal{P}$-algebras in $\mathcal{I}$-chain complexes has a model structure for every operad $\mathcal{P}$ in the category $\mathsf{Ch}(k)$ [PS18]:

Lemma 7.1. *For any chain complex C_*, for every $p \geqslant 1$ and for every $m, n \geqslant 0$ the norm $N_n = \sum_{\sigma \in \Sigma_n} \sigma \in \mathbb{Z}[\Sigma_n]$ induces an isomorphism of chain complexes*

$$N_n\colon (F_p^{\mathcal{I}}(C_*)^{\boxtimes n}/\Sigma_n)(\mathbf{m}) \to ((F_p^{\mathcal{I}}(C_*)^{\boxtimes n})^{\Sigma_n}(\mathbf{m}).$$

Proof. As $F_p^{\mathcal{I}}(C_*)^{\boxtimes n} \cong F_{np}^{\mathcal{I}}(C_*^{\otimes n})$, with $\mathbf{np} = \mathbf{p}^{\sqcup n}$,

$$F_{np}^{\mathcal{I}}(C_*^{\otimes n})(\mathbf{m}) = \bigoplus_{\mathcal{I}(\mathbf{p}^{\sqcup n}, \mathbf{m})} C_*^{\otimes n}.$$

If we identify an injective map $f\colon \mathbf{p}^{\sqcup n} \to \mathbf{m}$ with its image, we can express f as an n-tuple of p-tuples $((f(1), \ldots, f(p)), \ldots, (f((n-1)p+1), \ldots, f(np))$ with pairwise distinct $f(j) \in \mathbf{m}$. We can therefore pick representatives for the Σ_n-action for instance by choosing $((f(1), \ldots, f(p)), \ldots, (f((n-1)p+1), \ldots, f(np))$ with

$$\min(f(1), \ldots, f(p)) < \ldots < \min(f((n-1)p+1), \ldots, f(np)).$$

Thus every $F_{\mathbf{p}^{\sqcup n}}^{\mathcal{I}}(C_*^{\otimes n})(\mathbf{m})$ is an extended $k[\Sigma_n]$-module and the norm map is an isomorphism. □

Remark 7.2. The requirement that p is at least one is crucial, because

$$F_0^{\mathcal{I}}(C_*)^{\boxtimes n} \cong F_0^{\mathcal{I}}(C_*^{\otimes n})$$

and the Σ_n-action permutes the tensor powers of C_*. In the category of chain complexes over an arbitrary commutative ring, the norm map is not an isomorphism in general, so the functor $F_0^{\mathcal{I}}$ imports this trouble into the world of $\mathcal{I}$-chain complexes.

Beware that the norm map $N_n = \sum_{\sigma \in \Sigma_n} \sigma \in \mathbb{Z}[\Sigma_n]$ does *not* induce an isomorphism $N_n\colon M^{\boxtimes n}/\Sigma_n \to (M^{\boxtimes n})_n^{\Sigma}$ in general, even if M is reduced. We will give explicit examples of this phenomenon later. For now we consider the cases where everything works fine:

Definition 7.3. An $\mathcal{I}$-chain complex X_* is *Tate trivial*, if the norm map

$$N_n\colon X_*^{\boxtimes n}/\Sigma_n \to (X_*^{\boxtimes n})^{\Sigma_n}$$

is an isomorphism for all n.

We saw above, that for any chain complex C_*, for every $p \geqslant 1$ and for every m the norm $N_n = \sum_{\sigma\in\Sigma_n} \sigma \in \mathbb{Z}[\Sigma_n]$ induces an isomorphism of chain complexes

$$N_n \colon (F_p^{\mathcal{I}}(C_*)^{\boxtimes n}/\Sigma_n)(\mathbf{m}) \to ((F_p^{\mathcal{I}}(C_*)^{\boxtimes n})^{\Sigma_n}(\mathbf{m}).$$

So $F_p^{\mathcal{I}}(C_*)$ is Tate-trivial for all $p \geqslant 1$. Note that this implies that the free commutative monoid on $F_p^{\mathcal{I}}(C_*)$ is isomorphic to the free divided power algebra generated on $F_p^{\mathcal{I}}(C_*)$.

We will now provide two families of examples of reduced $\mathcal{I}$-chain complexes that are *not* Tate trivial. Recall that k denotes the ground ring, so chain complexes are chain complexes of k-modules.

Lemma 7.4. *Consider the chain complexes S^{2n} and D^{2n+1} and the kernels of the projection maps*

$$\pi(S^{2n}) \colon F_0^{\mathcal{I}}(S^{2n}) \to I_0(S^{2n}) \text{ and } \pi(D^{2n+1}) \colon F_0^{\mathcal{I}}(D^{2n+1}) \to I_0(D^{2n+1}).$$

We denote the kernel of $\pi(S^{2n})$ by $\overline{F}_0^{\mathcal{I}}(S^{2n})$ and the kernel of $\pi(D^{2n+1})$ by $\overline{F}_0^{\mathcal{I}}(D^{2n+1})$. If $\mathbb{Q} \not\subset k$, then these $\mathcal{I}$-chain complexes are not Tate-trivial.

Note that all structure maps in positive degrees induce the identity map in both cases as long as we consider elements in even degrees.

For the proof we have to study the equivalence relations that are involved in forming the $\boxtimes$-product in detail. To this end we use the following notation: If $\varphi \colon \mathbf{p} \sqcup \mathbf{q} \to \mathbf{n}$ is a morphism in $\mathcal{I}$, then we denote φ by $(\varphi(1)\ldots\varphi(p) \mid \varphi(p+1)\ldots\varphi(p+q))$.

Proof. We consider the case $\overline{F}_0^{\mathcal{I}}(S^{2n})$; the case of the disk-complex is similar.

In

$$(\overline{F}_0^{\mathcal{I}}(S^{2n}) \boxtimes \overline{F}_0^{\mathcal{I}}(S^{2n}))(\mathbf{4}) = \mathrm{colim}_{\varphi\colon p\sqcup q\to 4}\, \overline{F}_0^{\mathcal{I}}(S^{2n})(\mathbf{p}) \otimes \overline{F}_0^{\mathcal{I}}(S^{2n})(\mathbf{q})$$

for positive r we denote by x_{2n}^r the generator of $\overline{F}_0^{\mathcal{I}}(S^{2n})(\mathbf{r}) = S^{2n}$ in degree $2n$. We will first show that the equivalence class of $S_2 := (12 \mid 34) \otimes x_{2n}^2 \otimes x_{2n}^2$ is equal to the class of $(34 \mid 12) \otimes x_{2n}^2 \otimes x_{2n}^2$. We write down the chain of equivalences first and then explain why they hold:

$$S_2 = (12 \mid 34) \otimes x_{2n}^2 \otimes x_{2n}^2 \sim (2 \mid 3) \otimes x_{2n}^1 \otimes x_{2n}^1 \tag{7.1}$$
$$\sim (2 \mid 13) \otimes x_{2n}^1 \otimes x_{2n}^2 \tag{7.2}$$
$$\sim (2 \mid 1) \otimes x_{2n}^1 \otimes x_{2n}^1 \tag{7.3}$$
$$\sim (32 \mid 1) \otimes x_{2n}^2 \otimes x_{2n}^1 \tag{7.4}$$
$$\sim (3 \mid 1) \otimes x_{2n}^1 \otimes x_{2n}^1 \tag{7.5}$$
$$\sim (34 \mid 12) \otimes x_{2n}^2 \otimes x_{2n}^2. \tag{7.6}$$

As we consider generators of even degree, all maps in $\mathcal{I}_+$ induce the identity map. Then the above relations hold because

$$(12 \mid 34) \circ (2 \mid 3) = (2 \mid 3) \tag{7.7}$$
$$(2 \mid 13) \circ (1 \mid 3) = (2 \mid 3) \tag{7.8}$$
$$(2 \mid 13) \circ (1 \mid 2) = (2 \mid 1) \tag{7.9}$$
$$(32 \mid 1) \circ (2 \mid 1) = (2 \mid 1) \tag{7.10}$$
$$(32 \mid 1) \circ (1 \mid 3) = (3 \mid 1) \tag{7.11}$$
$$(34 \mid 12) \circ (1 \mid 2) = (3 \mid 1). \tag{7.12}$$

If we denote the generator of Σ_2 by t, then

$$t.[S_2] = t.[(12 \mid 34) \otimes x_{2n}^2 \otimes x_{2n}^2] = [(34 \mid 12) \otimes x_{2n}^2 \otimes x_{2n}^2]$$

so $[S_2] = [(12 \mid 34) \otimes x_{2n}^2 \otimes x_{2n}^2]$ is an element of the Σ_2-invariants

$$\big((\overline{F}_0^{\mathcal{I}}(S^{2n}) \boxtimes \overline{F}_0^{\mathcal{I}}(S^{2n}))(\mathbf{4})\big)^{\Sigma_2}$$

but unless 2 is invertible in the ground ring, it is not in the image of the norm map.

We now consider the Σ_m-invariants

$$\big((\overline{F}_0^{\mathcal{I}}(S^{2n})^{\boxtimes m})(\mathbf{2m})\big)^{\Sigma_m}$$

for arbitrary $m \geqslant 3$. The above chain of equivalences can be recycled to show that the transpositions $(1,2)$, $(2,3)$,…,$(m-1,m)$ leave the equivalence class of the element

$$S_m := (12 \mid 34 \mid \ldots \mid 2m{-}1\, 2m) \otimes (x_{2n}^2)^{\otimes m}$$

invariant and hence every $\sigma \in \Sigma_m$ does the same. So

$$[S_m] \in \big((\overline{F}_0^{\mathcal{I}}(S^{2n})^{\boxtimes m})(\mathbf{2m})\big)^{\Sigma_m}$$

but if $m!$ is not invertible in k this element is not in the image of the norm map. □

We show that the elements that we used in the proof of Lemma 7.4 give rise to non-trivial elements in homology.

Lemma 7.5. *The $\mathcal{I}$-chain complex $\overline{F}_0^{\mathcal{I}}(D^{2n+1})$ is acyclic, but if 2 is not invertible in k, then*

$$H_{4n}\Gamma^{\mathcal{I}}(\overline{F}_0^{\mathcal{I}}(D^{2n+1}))(\mathbf{4}) \neq 0$$

and

$$H_{4n}\mathsf{hocolim}_{\mathcal{I}}\Gamma^{\mathcal{I}}(\overline{F}_0^{\mathcal{I}}(D^{2n+1})) \neq 0.$$

Proof. The element $[(12 \mid 34) \otimes x_{2n}^2 \otimes x_{2n}^2]$ is a cycle in $\Gamma^{\mathcal{I}}(\overline{F}_0^{\mathcal{I}}(D^{2n+1}))(\mathbf{4})$ because

$$d[(12 \mid 34) \otimes x_{2n}^2 \otimes x_{2n}^2] = [(12 \mid 34) \otimes d(x_{2n}^2) \otimes x_{2n}^2 + [(12 \mid 34) \otimes x_{2n}^2 \otimes d(x_{2n}^2)]] = 0 + 0 = 0.$$

However, as the only element that can hit x_{2n}^r via the boundary map is the generator x_{2n+1}^r and as neither classes of the form $[\varphi \otimes x_{2n+1}^2 \otimes x_{2n}^2]$ nor classes such as $[\varphi \otimes x_{2n}^2 \otimes x_{2n+1}^2]$ are Σ_2-invariant (because morphisms in $\mathcal{I}$ act degree-perserving), the $4n$-cycle $[(12 \mid 34) \otimes x_{2n}^2 \otimes x_{2n}^2]$ survives.

In the bicomplex for the homotopy colimit the element $[(12 \mid 34) \otimes x_{2n}^2 \otimes x_{2n}^2]$ sits in bidegree $(0, 4n)$. We saw above that it cannot be hit by a vertical differential because this is induced by the internal differential of $\big(\overline{F}_0^{\mathcal{I}}(D^{2n+1}) \boxtimes \overline{F}_0^{\mathcal{I}}(D^{2n+1})\big)^{\Sigma_2}$. The horizontal differential of an element in bidegree $(1, 4n)$ with $f\colon \mathbf{n} \to \mathbf{m} \in \mathcal{I}$, $\varphi\colon \mathbf{p} \sqcup \mathbf{q} \to \mathbf{n}$, $x \in \big(\overline{F}_0^{\mathcal{I}}(D^{2n+1})(\mathbf{p})$ and $y \in \big(\overline{F}_0^{\mathcal{I}}(D^{2n+1})(\mathbf{q})$ is given by

$$d[f \otimes \varphi \otimes x \otimes y] = [f \circ \varphi \otimes x \otimes y] - [\varphi \otimes x \otimes y].$$

In particular, x and y are not affected by the boundary. So if we want to hit $[(12 \mid 34) \otimes x_{2n}^2 \otimes x_{2n}^2]$ with a differential on an element in bidgree $(1, 4n)$, this element had to be a k-linear combination of elements of the form $[f \otimes \varphi \otimes x_{2n}^r \otimes x_{2n}^s]$ as the morphisms in $\mathcal{I}$ cannot change the internal degrees of elements. Such linear combinations give a sum of elements with an even number of summands that cannot cancel to give just one copy of $[(12 \mid 34) \otimes x_{2n}^2 \otimes x_{2n}^2]$. □

If 2 happens to be invertible in k, but some other $m > 2$ isn't, then we can adapt the argument above and consider the element $[S_m]$ and get similarly:

Corollary 7.6. *If $\mathbb{Q} \not\subset k$, then there is an $m \geqslant 2$ such that*

$$H_{2mn}\Gamma^{\mathcal{I}}(\overline{F}_0^{\mathcal{I}}(D^{2n+1}))(\mathbf{2m}) \neq 0 \text{ and } H_{2mn}\mathsf{hocolim}_{\mathcal{I}}\Gamma^{\mathcal{I}}(\overline{F}_0^{\mathcal{I}}(D^{2n+1})) \neq 0.$$

8. Model category structures

Our basic objects are $\mathcal{I}$-chain complexes. For such diagram categories there is a *projective model structure* [Lur09, Proposition A.2.8.2.].

We recall the model structures on $\mathsf{Ch}(k)^{\mathcal{I}}$ from [RS20]. We equip $\mathsf{Ch}(k)$ with the projective model structure whose weak equivalences are the quasi-isomorphisms and whose fibrations are the chain maps that are degreewise surjections [Hov99, Theorem 2.3.11]. It has the inclusions $S^{q-1} \hookrightarrow D^q$ as generating cofibrations and the maps $0 \to D^q$ as generating acyclic cofibrations. This model structure is combinatorial.

A map $f\colon X_* \to Y_*$ in $\mathsf{Ch}(k)^{\mathcal{I}}$ is a (*positive*) *level equivalence* if $f(\mathbf{m})$ is a quasi-isomorphism for all $\mathbf{m}$ in $\mathcal{I}$ (resp. all $\mathbf{m}$ in $\mathcal{I}_+$). It is a (*positive*) *level fibration* if $f(\mathbf{m})$ is a fibration for all $\mathbf{m}$ in $\mathcal{I}$ (resp. all $\mathbf{m}$ in $\mathcal{I}_+$). A (*positive*) *level cofibration* is a map with the left lifting property with respect to all maps which are both (positive) level fibrations and equivalences.

These maps define a *level model category structure* and a *positive level model structures* on $\mathsf{Ch}(k)^{\mathcal{I}}$. Both model structures are proper and combinatorial [RS20, Proposition 4.2]. The cofibrations in these level model structures are the retracts of relative cell complexes built out of cells of the form $F^{\mathcal{I}}_{\mathbf{m}}(S^{q-1}) \to F^{\mathcal{I}}_{\mathbf{m}}(D^q)$ with $\mathbf{m}$ in $\mathcal{I}$ (resp. $\mathcal{I}_+$) and $q \in \mathbb{Z}$.

A map $X_* \to Y_*$ in $\mathsf{Ch}(k)^{\mathcal{I}}$ is an *$\mathcal{I}$-equivalence* if it induces a quasi-isomorphism $(X_*)_{h\mathcal{I}} \to (Y_*)_{h\mathcal{I}}$ on the homotopy colimit. The $\mathcal{I}$-equivalences are part of an *$\mathcal{I}$-model structure.* An $\mathcal{I}$-chain complex X_* is (*positive*) $\mathcal{I}$-fibrant if $\alpha_*\colon X_*(\mathbf{m}) \to X_*(\mathbf{n})$ is a quasi-isomorphism for all $\alpha \in \mathcal{I}(\mathbf{m},\mathbf{n})$ (resp. in $\mathcal{I}_+$). These model structure are left Bousfield localizations of the level model structure (the positive level model structure) on $\mathsf{Ch}(k)^{\mathcal{I}}$ [RS20, Proposition 4.4].

Remark 8.1. One could try to work with the injective level structure instead. This exists and has levelwise cofibrations and weak equivalences. Even if k is a field, the projective and the injective model structures on $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1}$ don't agree, for instance $I_2(S^2)$ is not cofibrant in the projective model structure over $\mathbb{F}_2$ because $0 \to I_2(S^2)$ does not have the left lifting property with respect to the acyclic fibration $q\colon I_2(\mathbb{F}_2\{\Sigma_2\} \otimes D^3) \to I_2(D^3)$ when S^2 and D^3 carry the trivial Σ_2-action and $\mathbb{F}_2\{\Sigma_2\}$ the regular one. In Neisendorfer's work [Nei78] he considers chain complexes over a field of characteristik zero that are concentrated in degree $\geqslant 1$. But over a field the projective model structure on $\mathsf{Ch}(k)_{\geqslant 1}$ (degreewise condition on fibrations and weak equivalences) *does* agree with the injective model structure (degreewise condition on cofibrations and weak equivalences).

For the injective model structure it is less clear whether one can form a left Bousfield localization in order to get the $\mathcal{I}$-equivalences as weak equivalences. Also, the monoidal structure on $\mathcal{I}$-chains has correct homotopical behaviour with respect to the positive $\mathcal{I}$-model structure as above, as the following lemma shows.

Lemma 8.2. *If C_* and D_* are two $\mathcal{I}$-chain complexes that are cofibrant in the positive $\mathcal{I}$-model structure, then*

$$H_*(\mathsf{hocolim}_{\mathcal{I}}C_* \otimes \mathsf{hocolim}_{\mathcal{I}}D_*) \cong H_*\mathsf{hocolim}_{\mathcal{I}}(C_* \boxtimes D_*).$$

Note that the cofibrations in the positive $\mathcal{I}$-model structure agree with the ones in the positive projective model structure on $\mathsf{Ch}(k)^{\mathcal{I}}$.

Proof. The proof is analogous to the one of [SS13, Lemma 2.25]: We consider the colimit functor

$$\mathrm{colim}_{\mathcal{I}}\colon \mathsf{Ch}(k)^{\mathcal{I}} \to \mathsf{Ch}(k).$$

As $F_0^{\mathcal{I}}(S^0)$ is the constant functor on S^0, it is clear that $\mathrm{colim}_{\mathcal{I}} U^{\mathcal{I}} \cong S^0$.

As the symmetric monoidal structure on the category of chain complexes is closed, one immediately gets that

$$\mathrm{colim}_{\mathcal{I}} C_* \otimes \mathrm{colim}_{\mathcal{I}} D_* \cong \mathrm{colim}_{\mathcal{I}\times\mathcal{I}} C_* \overline{\otimes} D_*$$

where $C_*\overline{\otimes}D_* \colon \mathcal{I}\times\mathcal{I} \to \mathsf{Ch}(k)$ is the functor that sends a pair $(\mathbf{n}, \mathbf{m})$ to $C_*(\mathbf{n}) \otimes D_*(\mathbf{m})$. By the very definition of $C_* \boxtimes D_*$ as a left Kan extension of the functor $C_*\overline{\otimes}D_*$ along the functor

$$\sqcup \colon \mathcal{I}\times\mathcal{I} \to \mathcal{I},\ (\mathbf{n}, \mathbf{m}) \mapsto \mathbf{n} \sqcup \mathbf{m} = \mathbf{n} + \mathbf{m}$$

we get that

$$\mathrm{colim}_{\mathcal{I}}(C_* \boxtimes D_*) \cong \mathrm{colim}_{\mathcal{I}\times\mathcal{I}} C_* \overline{\otimes} D_*$$

and thus in total we obtain an isomorphism

$$\varphi\colon\ \mathrm{colim}_{\mathcal{I}} C_* \otimes \mathrm{colim}_{\mathcal{I}} D_* \cong \mathrm{colim}_{\mathcal{I}}(C_* \boxtimes D_*).$$

This isomorphism is natural in C_* and D_* and turns $\mathrm{colim}_{\mathcal{I}}$ into a strong symmetric monoidal functor.

By [Sch18, Proposition 2.54] we know that for cofibrant objects A_* the canonical projection map (3.1) $\pi_{A_*} \colon \mathsf{hocolim}_{\mathcal{I}} A_* \to \mathrm{colim}_{\mathcal{I}} A_*$ is a homology isomorphism. This result is an adaptation of [SaS12, Lemma 6.22].

The positive $\mathcal{I}$-model structure on $\mathsf{Ch}(k)^{\mathcal{I}}$ is monoidal [PS18, §3.4]. Hence in the diagram

$$\begin{tikzcd}
\mathsf{hocolim}_{\mathcal{I}} C_* \otimes \mathsf{hocolim}_{\mathcal{I}} D_* \arrow[d, "\pi_{C_*}\otimes\pi_{D_*}"'] & \mathsf{hocolim}_{\mathcal{I}}(C_* \boxtimes D_*) \arrow[d, "\pi_{C_*\boxtimes D_*}"'] \\
\mathrm{colim}_{\mathcal{I}} C_* \otimes \mathrm{colim}_{\mathcal{I}} D_* \arrow[r, "\varphi"] & \mathrm{colim}_{\mathcal{I}}(C_* \boxtimes D_*)
\end{tikzcd}$$

the map $\pi_{C_*\boxtimes D_*}$ is a homology isomorphism because the $\boxtimes$-product preserves cofibrancy. As colimits and homotopy colimits map cofibrant objects in $\mathsf{Ch}(k)^{\mathcal{I}}$ to cofibrant objects in the projective model structure on chain complexes, the pushout-product axiom in that model structure then implies that the map $\pi_{C_*} \otimes \pi_{D_*}$ is a quasi-isomorphism. $\square$

In the context of $\mathcal{I}$-spaces, *flatness* (as in [SS13, Definition 2.16]) actually suffices to obtain Lemma 8.2 and I assume that something analogous works for chain complexes, but I didn't check that.

If we want to find a cocommutative comonoid $C_*^{\mathcal{I}}(X; k)$ such that $\mathsf{hocolim}_{\mathcal{I}} C_*^{\mathcal{I}}(X; k)$ is equivalent to the E_∞-coalgebra of the chains on a connected space X, $S_*(X; k)$, then $C_*^{\mathcal{I}}(X; k)$ has to be connected as a cocommutative comonoid. Therefore we consider the following variant of the above model structures:

Lemma 8.3. *On $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1}$ there is a positive $\mathcal{I}$-model structure such that the weak equivalences are given by the $\mathsf{hocolim}_{\mathcal{I}}$-equivalences. It is the Bousfield localization of the positive level model structure on $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1}$ with respect to the set*

$$S = \{F^{\mathcal{I}}_{n+1}(C) \to F^{\mathcal{I}}_n(C), n \geqslant 1, C \in \{S^n, D^m, n \geqslant 1, m \geqslant 2\}. \quad \square$$

In particular, the $\mathcal{I}$-fibrancy condition still holds.

Remark 8.4. There is a Quillen equivalence between the positive projective level structure on $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1}$ and the projective level structure on $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$ where the left adjoint functor is the inclusion functor $J\colon \mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r} \hookrightarrow \mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1}$ and the right adjoint is the reduction functor $R\colon \mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1} \to \mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$. We will use these model structures and the corresponding $\mathcal{I}$-model structures interchangeably.

Lemma 8.5. *For all n the chain complex $\overline{F}_0^{\mathcal{I}}(D^{2n+1})$ is acyclic and is fibrant in the positive $\mathcal{I}$-model structure.*

Proof. As the chain complex D^{2n+1} is concentrated in degrees $2n+1$ and $2n$ the bicomplex for the homotopy colimit of $\overline{F}_0^{\mathcal{I}}(D^{2n+1})$ is also concentrated in internal degrees $2n+1$ and $2n$, thus in two rows. Its vertical homology is zero, hence so is the homology of the associated total complex.

As all maps in $\mathcal{I}_+$ induce the identity map, $\overline{F}_0^{\mathcal{I}}(D^{2n+1})$ is fibrant. $\square$

We consider the adjunction $\mathsf{ccCoalg}^{\mathcal{I}} \underset{\Gamma^{\mathcal{I}}}{\overset{U}{\rightleftarrows}} \mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$ from Proposition 4.8 and ask, if there can be a left-induced model structure on $\mathsf{ccCoalg}$. In such a structure the cofibrations and weak equivalences are determined by the left adjoint functor, here U. In contrast to right-induced model structures, left-induced ones are hard to establish. For criteria for the existence of such model structure and for several important examples see [HKRS17] (and a correction in [GKR20]).

Proof of Theorem 1.2. If there were a left-induced model structure on $\mathsf{ccCoalg}^{\mathcal{I}}$, then the functor $\Gamma^{\mathcal{I}}$ had to preserve acyclic fibrations: Assume that $f\colon A_* \to B_*$ is a cofibration in the left-induced model structure and assume that $g\colon X_* \to Y_*$ is an acyclic fibration in the level model structure on $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$. Let

$$\begin{tikzcd} A_* \arrow[r, "\alpha"] \arrow[d, "f"'] & \Gamma^{\mathcal{I}}(X_*) \arrow[d, "\Gamma^{\mathcal{I}}(g)"] \\ B_* \arrow[r, "\beta"] & \Gamma^{\mathcal{I}}(Y_*) \end{tikzcd}$$

be a commutative diagram in the category $\mathsf{ccCoalg}^{\mathcal{I}}$. By adjunction we obtain the commutative diagram

$$\begin{tikzcd} UA_* \arrow[r, "ad(\alpha)"] \arrow[d, "Uf"'] & X_* \arrow[d, "g"] \\ UB_* \arrow[r, "ad(\beta)"] & Y_* \end{tikzcd}$$

in $\mathcal{I}$-chain complexes. By definition Uf is a cofibration and by assumption g is an acyclic fibration, so we obtain a lift in the last diagram that translates to a lift in the first diagram.

With Lemma 8.5 we obtain that the map $\varrho\colon \overline{F}_0^{\mathcal{I}}(D^{2n+1}) \to 0$ is an acyclic fibration in $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$ for every $n \geqslant 1$, so the induced map

$$\Gamma^{\mathcal{I}}(\varrho)\colon \Gamma^{\mathcal{I}}(\overline{F}_0^{\mathcal{I}}(D^{2n+1})) \to \Gamma^{\mathcal{I}}(0) = U^{\mathcal{I}}$$

had to be an acyclic fibration. But by Corollary 7.6 we have a non-trivial homology contribution in $\mathsf{hocolim}_{\mathcal{I}}\Gamma^{\mathcal{I}}(\overline{F}_0^{\mathcal{I}}(D^{2n+1}))$ whereas the homology of $\mathsf{hocolim}_{\mathcal{I}}U^{\mathcal{I}}$ is trivial in positive degrees. Hence the map $\Gamma^{\mathcal{I}}(\varrho)$ is not an acyclic fibration, so such a left-induced model structure cannot exist. $\square$

Remark 8.6. If we consider the projective level structure on $\mathsf{Ch}(k)^{\mathcal{I}}_{\geqslant 1,r}$, then we still obtain that there is no corresponding left-induced model structure on $\mathsf{ccCoalg}$: The map $\varrho\colon \overline{F}_0^{\mathcal{I}}(D^{2n+1}) \to 0$ is levelwise surjective, hence it is a projective level fibration. As the homology of $\overline{F}_0^{\mathcal{I}}(D^{2n+1})$ is trivial, ϱ is a projective level weak equivalence. Hence ϱ is also an acyclic fibration in this model structure. The same argument as in the proof of Theorem 1.2 then shows that such a left-induced model structure cannot exist.

If we consider symmetric sequences of chain complexes, $\mathsf{Ch}(k)^\Sigma$, then this problem cannot occur: Stover [Sto93, §9.10] gives an explicit description of $\mathsf{S}^\Sigma(X_*)$ that consists of cosets of the symmetric group and tensor powers of X_*. Hence if $X_* \in \mathsf{Ch}(k)^\Sigma_{\geqslant 1}$ is a reduced object that is acyclic, then $\mathsf{S}^\Sigma(X_*)$ is also acyclic. As the norm map is an isomorphism, we get that $\mathsf{S}^\Sigma(X_*) \cong \Gamma^\Sigma(X_*)$ and hence Γ^Σ preserves acyclic objects.

Remark 8.7. Of course, one could try to model the E_∞-coalgebra of singular chains on a space using different diagram categories than $\mathcal{I}$. If one thinks of Kronecker pairings, then it would be natural to try the opposite category, $\mathcal{I}^{op}$. The category $\mathsf{Ch}(k)^{I^{op}}$ is symmetric monoidal with respect to $\sqcup$, hence there is a corresponding Day convolution product on $\mathsf{Ch}(k)^{I^{op}}$, $\boxtimes$. One big problem with $\mathcal{I}^{op}$ is that the free $\mathcal{I}^{op}$-chain complexes don't give rise to Tate trivial objects: We still get that $F_1^{\mathcal{I}^{op}}(C_*)^{\boxtimes n} \cong F_n^{\mathcal{I}^{op}}(C_*^{\otimes n})$, but now the Σ_n-action isn't free on

$$F_n^{\mathcal{I}^{op}}(C_*^{\otimes n})(\mathbf{m}) = \bigoplus_{\mathcal{I}^{op}(\mathbf{n},\mathbf{m})} C_*^{\otimes n} = \bigoplus_{\mathcal{I}(\mathbf{m},\mathbf{n})} C_*^{\otimes n}.$$

If $f \in \mathcal{I}(\mathbf{m}, \mathbf{n})$ and $\mathrm{id} \neq \sigma \in \Sigma_n$, then $\sigma.f$ is equal to f if the support of σ is contained in $\mathbf{n} \setminus \mathrm{im}(f)$. Taking the product of sets instead of the concatenation is even worse.

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