Linear Operads

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17th of April 2025

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The term *operad* was coined by Peter May in the setting of topological spaces. He used them to study iterated loop spaces.

Symmetric sequences and Schur functors

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Remarks:

We can view M as a functor $\Sigma^{op} \to \text{Vect}$, where Σ is the category whose objects are $\mathbb{N}_0$ and whose morphisms are

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Note that every $M(n)$ is a Σ_n-representation, so the theory of representations of the symmetric groups plays an important role in this topic.

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4. and their *composite* is
$$(M \circ M')(n) := \bigoplus_{\ell \geqslant 0} M(\ell) \otimes_{k[\Sigma_\ell]} (M')^{\odot \ell}(n).$$

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and this is

$$\bigoplus_{\ell \geqslant 0} M(\ell) \otimes_{k[\Sigma_\ell]} \left(\bigoplus_{p_1 + \ldots + p_\ell = n} M'(p_1) \otimes \ldots \otimes M'(p_\ell) \otimes_{k[\Sigma_{p_1} \times \ldots \times \Sigma_{p_\ell}]} k[\Sigma_n] \right)$$

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The first isomorphism of functors is rather straightforward, but the last one is more painful.

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Lemma

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2. Let A be a vector space. We consider the symmetric sequence $\mathbb{A} = (A, 0, 0, \ldots)$. We say that A is an *algebra over the operad* O, if $\mathbb{A}$ is a left O-module in Vect-Σ, *i.e.*, if there is a map $\theta_A \colon O \circ \mathbb{A} \to \mathbb{A}$ that is associative and the map $\eta \colon e \to O$ sits in a commuting diagram:

$$e \circ \mathbb{A} \xrightarrow{\eta \circ \mathrm{id}} O \circ \mathbb{A} \xrightarrow{\theta_A} \mathbb{A}, \qquad e \circ \mathbb{A} \xrightarrow{\cong} \mathbb{A}.$$

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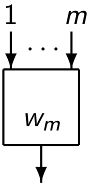

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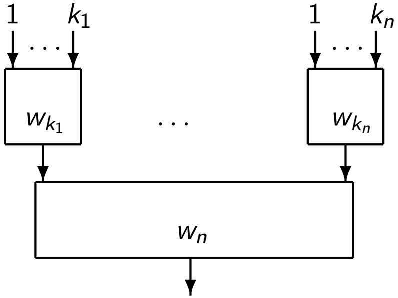

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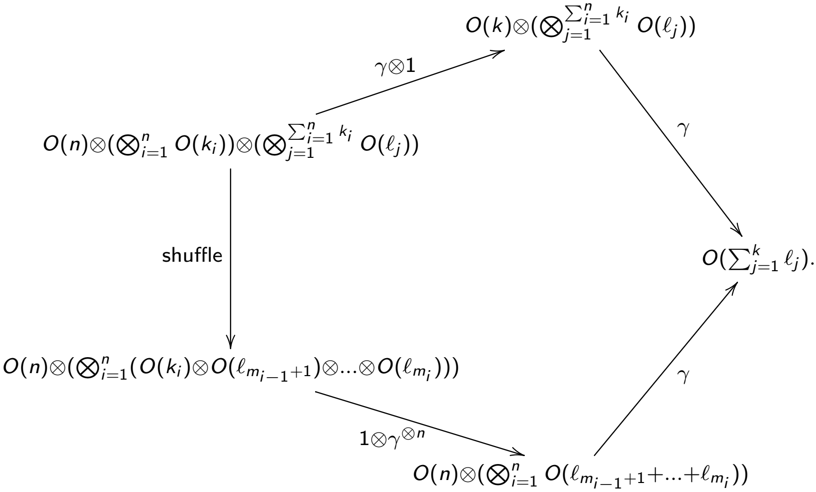

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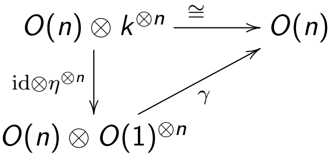

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We also need the permutation $\tau_1 \oplus \ldots \oplus \tau_n \in \Sigma_{k_1+\ldots+k_n}$ for $\tau_i \in \Sigma_{k_i}$ for $1 \leqslant i \leqslant n$, which is the concatenation of the τ_is.

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that have to satisfy several coherence conditions. In terms of the classical definition, you can define the $\circ_i$ map as

$$w \circ_i \nu := \gamma(w \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes \nu \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id})$$

where you insert the operation ν into the i-th spot of w.

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$$\gamma(\sigma, \tau_1, \ldots, \tau_n) = (\tau_{\sigma^{-1}(1)} \oplus \ldots \oplus \tau_{\sigma^{-1}(n)}) \circ \sigma(k_1, \ldots, k_n).$$

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Beware of the antisymmetry condition if the characteristic is two! In finite characteristic you might want to model restricted Lie algebras. These need operadic algebras with divided powers. Free Lie algebras have many different bases, called Hall bases. Some of these are more amenable to generalizations to non-linear situations than others.

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1) The action maps are associative. For all $k = \sum_{i=1}^{n} k_i$, the diagram

$$
\begin{array}{ccc}
O(n) \otimes O(k_1) \otimes \ldots \otimes O(k_n) \otimes A^{\otimes k} & \xrightarrow{\gamma \otimes 1} & O(k) \otimes A^{\otimes k} \\
\downarrow{\scriptstyle \text{shuffle}} & & \\
O(n) \otimes O(k_1) \otimes A^{\otimes k_1} \otimes \ldots \otimes O(k_n) \otimes A^{\otimes k_n} & & \downarrow{\scriptstyle \theta_k} \\
\downarrow{\scriptstyle 1 \otimes \theta_{k_1} \otimes \ldots \otimes \theta_{k_n}} & & \\
O(n) \otimes A^{\otimes n} & \xrightarrow{\theta_n} & A
\end{array}
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There are many equivalent ways of saying what an algebra over an operad is. I'll mention one more.

A vector space A is an algebra over an operad O, if there is an action map $\theta_A\colon \widetilde{O}(A) \to A$ from the Schur functor for O and A to A such that the diagrams

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\begin{array}{ccccc}
\widetilde{O \circ O}(A) & \xrightarrow{\cong} & \widetilde{O}(\widetilde{O}(A)) & \xrightarrow{\widetilde{O}(\theta_A)} & \widetilde{O}(A) \\
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\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\widetilde{e}(A) & \xrightarrow{i_1} & \widetilde{O}(A) \\
& \searrow\!\!\!\!\!= & \downarrow{\scriptstyle \theta_A} \\
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Anti-symmetry and the Jacobi relation hold because they hold in the free Lie-algebra.

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Let O be a linear operad. The functor $\widetilde{O}\colon \mathrm{Vect} \to O\text{-algs}$, $V \mapsto \tilde{O}(V)$, is left adjoint to the forgetful functor $U\colon O\text{-algs} \to \mathrm{Vect}$.

Remarks A morphism of operads $\beta \colon O \to P$ induces a functor from the category of P-algebras to the category of O-algebras: if A is a P-algebra, then we can precompose its structure map $P \to End(A)$ with β to obtain $O \to End(A)$.

Every commutative and associative algebra is an associative algebra. This can be encoded by the map of operads

$$\mathrm{As} \to \mathrm{Com}, \text{ with } \mathrm{As}(n) = k[\Sigma_n] \to k = \mathrm{Com}(n), \sigma \mapsto 1 \text{ for all } \sigma \in \Sigma_n.$$

Every associative algebra A is a Lie algebra with the commutator bracket

$$[a, b] := ab - ba.$$

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So $\tilde{O}(V) = \bigoplus_{n \geqslant 0} O(n) \otimes_{k[\Sigma_n]} V^{\otimes n}$ is the free O-algebra generated by V.

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$$\hat{\mathcal{C}}(C) := \prod_{n \geqslant 0} (\mathcal{C}(n) \otimes C^{\otimes n})^{\Sigma_n}.$$

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Similarly, for a coalgebra C over a cooperad $\mathcal{C}$ we have linear coaction maps

$$C \to (\mathcal{C}(n) \otimes C^{\otimes n})^{\Sigma_n}$$

satisfying the dual axioms to those of an algebra over an operad.

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Spoiler: We will later see that for a so called quadratic operad there is a Koszul dual cooperad.

References:

- Jean-Louis Loday, La renaissance des opérades, Séminaire Bourbaki, Vol. 1994/95, Astérisque 237, 1996, Exp. No. 792, 3, 47–74.
- Jean-Louis Loday, Bruno Vallette, Algebraic operads. Grundlehren der mathematischen Wissenschaften 346. Springer, Heidelberg, 2012. xxiv+634 pp.
- Christophe Reutenauer, Free Lie algebras. London Mathematical Society Monographs. New Series, 7. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. xviii+269 pp.
- Birgit Richter, From Categories to Homotopy Theory, Cambridge Studies in Advanced Mathematics No 188, Cambridge University Press, Cambridge, 2020. x+390 pp.