TOWARDS AN UNDERSTANDING OF RAMIFIED EXTENSIONS OF STRUCTURED RING SPECTRA

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Abstract. We propose topological Hochschild homology as a tool for measuring ramification of maps of structured ring spectra. We determine second order topological Hochschild homology of the p-local integers. For the tamely ramified extension of the map from the connective Adams summand to p-local complex topological K-theory we determine the relative topological Hochschild homology and show that it detects the tame ramification of this extension. We also determine relative topological Hochschild homology for the complexification map from connective real to complex topological K-theory and for some quotient maps with commutative quotients.

1. Introduction

For a G-Galois extension of number fields $K \subset L$ the associated extension of rings of integers $\mathcal{O}_K \to \mathcal{O}_L$ will not be unramified in general. Greither shows in [Gr92, Chapter 0, Theorem 4.1] that the ramification of such an extension can be detected with the help of the map

$$h\colon \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \to \prod_G \mathcal{O}_L. \tag{1}$$

Here, h is defined as $h(b_1 \otimes b_2) = (b_1 g(b_2))_{g\in G}$ for $b_1, b_2 \in \mathcal{O}_L$. The extension is unramified if h is an isomorphism. For more general extensions of commutative rings this still gives an adequate notion of ramification. The Hochschild homology of $\mathcal{O}_L$ over $\mathcal{O}_K$ is an invariant that behaves differently depending on whether the extension is tamely or wildly ramified. For instance $\mathsf{HH}^{\mathbb{Z}}(\mathbb{Z}[i])$ is a square-zero extension of $\mathbb{Z}[i]$ with additive two-torsion in positive odd degrees.

In the following we consider cohomology theories with a multiplicative structure that can be represented by a commutative monoid object in one of the symmetric monoidal categories of spectra, for instance the one presented in [EKMM97]. The representing objects are called commutative ring spectra. Examples of such cohomology theories are singular cohomology with coefficients in a commutative ring, topological (real or complex) K-theory, and several cobordism theories. There is an analogue of Hochschild homology in the context of ring spectra, topological Hochschild homology. It was defined by Bökstedt [Bö∞_1] and a published account can for instance be found in [EKMM97, Chapter IX].

Let A be a commutative ring spectrum and let B be a commutative A-algebra with an action of a finite group G via maps of commutative A-algebras. Then the extension $A \to B$ is called unramified [R08, (4.1.2)], if the map

$$h\colon B \wedge_A B \to \prod_G B \tag{2}$$

is an equivalence. Here, h is the analogue of (1) in the context of spectra.

Rognes shows [R08, 9.2.6, proof of 9.1.2] that the condition for B to be unramified over A ensures that the map from B to relative topological Hochschild homology of B over A,

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$\mathsf{THH}^A(B)$, is a weak equivalence. Thus the failure of the map

$$B \to \mathsf{THH}^A(B)$$

to be a weak equivalence is a measure of the ramification of the extension $A \to B$. It also makes sense to study $\mathsf{THH}^A(B)$ in more general situations, for instance in the absence of a group action.

Algebraic K-theory of an ordinary commutative ring R, $K(R)$, contains a lot of arithmetic information about R, such as the Picard group of R, its Brauer group and its units. Trace methods have been useful for studying $K(R)$: There are trace maps

$$\begin{tikzcd} K(R) \arrow[r, "trc"] \arrow[dr, "tr"'] & TC(R) \arrow[d] \\ & \mathsf{THH}(R) \end{tikzcd}$$

that allow us to approximate $K(R)$ by invariants that are easier to compute, by topological Hochschild homology, $\mathsf{THH}(R)$, and by topological cyclic homology, $TC(R)$. Trace methods work also well for connective commutative ring spectra, *i.e.*, commutative ring spectra whose homotopy groups are concentrated in non-negative degrees.

Galois extensions of commutative S-algebras in the sense of Rognes [R08, 4.1.3] are unramified. A prominent example is given by the complexification map from real to complex periodic K-theory, $c\colon KO \to KU$. Here, complex conjugation on complex vector bundles induces a C_2-action on KU by maps of commutative KO-algebra spectra. But a result of Akhil Mathew [Ma16, Theorem 6.17] tells us that finite Galois extensions of a connective spectrum are purely algebraic. So taking the connective cover of the complexification map

$$\begin{tikzcd} ko \arrow[r, "c"] \arrow[d, "j"'] & ku \arrow[d, "j"] \\ KO \arrow[r, "c"] & KU \end{tikzcd}$$

does *not* yield a C_2-Galois extension $ko \to ku$ because algebraically

$$ko_* = \mathbb{Z}[\eta, y, w]/2\eta, \eta^3, \eta y, y^2 - 4w \to \mathbb{Z}[u] \cong ku_*$$

is certainly not étale. (Here, the degrees are $|\eta| = 1$, $|y| = 4$, $|w| = 8$ and $|u| = 2$.)

For a commutative A-algebra B we denote by $\mathsf{THH}^{[n],A}(B)$ the higher order topological Hochschild homology of B as a commutative A-algebra, *i.e.*,

$$\mathsf{THH}^{[n],A}(B) = B \otimes \mathbb{S}^n$$

where $(-) \otimes \mathbb{S}^n$ denotes the tensor with the n-sphere in the category of commutative A-algebras. This can be viewed as the realization of the simplicial commutative A-algebra whose q-simplices are given by

$$\coprod_{x \in \mathbb{S}^n_q} B,$$

where the coproduct is the smash product over A.

Higher THH of the Eilenberg Mac Lane spectra of local number rings also detects ramification [DLR18], but after we take coefficients in the residue field we cannot see the difference anymore between tame and wild ramification in higher THH. We offer some partial results towards calculations of higher THH with unreduced coefficients. We calculate second order THH of the p-local integers:

$$\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[x_1, x_2, \ldots]/p^n x_n, x_n^p = p x_{n+1}.$$

See Theorem 2.1.

It is possible to determine $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$ additively using that $H\mathbb{Z}_{(p)}$ can be constructed as a Thom spectrum of a double-loop map and applying the methods of [BCS10, Sch11]. Blumberg, Cohen and Schlichtkrull identify $\mathsf{THH}(\mathbb{Z}_{(p)})$ with $H\mathbb{Z}_{(p)} \wedge \Omega\mathbb{S}^3\langle 3\rangle_+$ [BCS10, Theorem 3.8]. See [Kl18, Corollary 1.1] for a calculation of $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$. However, this views $H\mathbb{Z}_{(p)}$ as an E_2-spectrum and not as a commutative S-algebra, so with this method the multiplicative structure of $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$ cannot be determined. The multiplicative structure is essential if one aims at a calculation of $\mathsf{THH}^{[n]}_*(\mathbb{Z}_{(p)})$ for larger n.

We study the examples of the connective covers of the Galois extensions [R08] $KO \to KU$ and $L_p \to KU_p$. In the latter case, the connective cover behaves like an extension of the corresponding rings of integers. We test ramification with relative (higher) topological Hochschild homology and for $\ell \to ku_{(p)}$ we see that it looks like tame ramification (see Theorem 4.1): $\mathsf{THH}^{\ell}_*(ku_{(p)})$ is a square zero extension of $\pi_* ku_{(p)}$ of bounded u-exponent. We also determine relative THH of the complexification map $c\colon ko \to ku$ (see Theorem 5.2).

Working with structured ring spectra means working in a derived setting, so quotient maps can be thought of as extensions. We offer some calculations of relative THH in situations where we kill generators of homotopy groups. We consider a version of $ku/(p, v_1)$ and quotients of the form R/x where x is a regular element in $\pi_*(R)$ where R is a commutative ring spectrum such that R/x is still commutative.

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2. Second order THH of the p-local integers

This section consists of a proof of the following somewhat surprising result. In the context of the current paper, this calculation is a starting point for comparing with future calculations for other rings of integers. See Remark 2.4 for a discussion of the fact that the answer agrees with topological Hochschild cohomology.

Theorem 2.1. *For all primes p:*

$$\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[x_1, x_2, \ldots]/p^n x_n, x_n^p - p x_{n+1}$$

with $|x_n| = 2p^n$.

The entire section is devoted to proving this result For all primes p the exact sequence

$$\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}, \mathbb{F}_p) \xrightarrow{\delta} \Sigma\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}) \tag{3}$$

is a sequence of $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$-modules; in particular, δ is a module map. Furthermore, from [DLR18] we have that

$$\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}, \mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(y) \otimes \Lambda_{\mathbb{F}_p}(z), \tag{4}$$

where $|y| = 2p$ and $|z| = 2p+1$. We denote the generator $\gamma_{p^i}(y)$ in the divided power algebra $\Gamma_{\mathbb{F}_p}(y)$ in degree $2p^{i+1}$ by y_{p^i} and if $t = t_0 + t_1 p + \cdots + t_n p^n$ is the p-adic expansion of t, then we set $y_t = y_1^{t_0} y_p^{t_1} \ldots y_{p^n}^{t_n}$ with $y_t^p = 0$.

By the Tor spectral sequence,

$$\mathsf{Tor}^{\mathsf{THH}_*(\mathbb{Z}_{(p)})}_{*,*}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \Rightarrow \mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$$

we know that $\mathsf{THH}^{[2]}_s(\mathbb{Z}_{(p)})$ is finite p-torsion for positive s because

$$\mathsf{THH}_*(\mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}_{(p)}, & * = 0, \\ \mathbb{Z}_{(p)}/i, & * = 2i-1, \\ 0, & \text{otherwise.} \end{cases}$$

By (4) and using the notation introduced below it, this implies that there are integers $a_1, a_2, \ldots$ such that

$$\mathsf{THH}^{[2]}_s(\mathbb{Z}_{(p)}) \begin{cases} 0, & 2p \not| s, \\ \mathbb{Z}/p^{a_t}\{\tilde{y}_t\}, & s = 2pt, \end{cases}$$

where the $\tilde{y}_t$ are generators of the given cyclic groups which are sent to the corresponding generators y_t in $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)}, \mathbb{F}_p)$. We will show

Lemma 2.2. *The function $a\colon \mathbb{N} \to \mathbb{N}$ factors over the p-adic valuation $v\colon \mathbb{N} \to \mathbb{N}$, $a_t = b_{v(t)}$, with $b\colon \mathbb{N} \to \mathbb{N}$ a* strictly *increasing function with positive values and $b_0 = 1$.*

Proof. To this end we use induction on the following statement $\mathbf{P(n)}$ for positive integers n. The generators $\tilde{y}_t$ are chosen inductively.

$\mathbf{P(n)}$: For positive integers s, t such that $v(s)$, $v(t)$ are less than n the following properties hold:

(1) If $v(s) = v(t)$, then $a_s = a_t$,
(2) If $v(s) > v(t)$, then $a_s > a_t$,
(3) If $s = s_0 + s_1 p + \cdots + s_{n-1}p^{n-1}$ is the p-adic expansion of s (so that $0 \leqslant s_0, \ldots, s_n < p$), then $\tilde{y}_s = \tilde{y_1}^{s_0} \ldots \tilde{y}_{p^{n-1}}^{s_{n-1}}$,
(4) If $n > v$, then $\tilde{y}_{p^v} = p^{a_{p^v} - a_{p^{v-1}}} \tilde{y}^p_{p^{v-1}}$.

We will repeatedly be considering the cofiber sequence (3). In homotopy, the maps are trivial except in degrees of the form $2pt$ (for varying t) in which case they are

$$0 \longrightarrow \mathbb{F}_p\{zy_{t-1}\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_t\} \longrightarrow 0$$

forcing all the a_t to be positive. For any generator w, $\mathbb{F}_p\{w\}$ denotes the graded vector space generated by w. Here r is multiplicative and δ is a $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$-module map. By the surjectivity of r we have that the y_t's can be lifted to integral classes.

Establishing $\mathbf{P(1)}$. Let $t = 1$. The sequence

$$0 \longrightarrow \mathbb{F}_p\{z\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2p}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2p}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_1\} \longrightarrow 0.$$

shows that $a_1 > 0$ and by adjusting z up to a unit we may choose $\tilde{y}_1$ so that $\delta(z) = p^{a_1 - 1}\tilde{y}_1$ and $r(\tilde{y}_1) = y_1$. In the Tor-spectral sequence we only get a $\mathbb{Z}/p\mathbb{Z}$ in bidegree $(1, 2p-1)$ which survives and shows that $a_1 = 1$, and so $\delta(z) = \tilde{y}_1$.

If $1 < t < p$ the sequence

$$0 \longrightarrow \mathbb{F}_p\{zy_1^{t-1}\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_1^t\} \longrightarrow 0$$

gives that $\delta(zy_1^{t-1}) = \delta(z) \cdot \tilde{y}_1^{t-1} = \tilde{y}_1^t \neq 0$, $p\tilde{y}_1^t = 0$ and $r(\tilde{y}_1^t) = y_1^t \neq 0$. The last point shows that $\tilde{y}_1^t$ is not divisible by p and hence we can choose it as our generator: $\tilde{y}_t = \tilde{y}_1^t$, and furthermore, this generator is killed by p, so $a_t = 1$.

If $t = t_0 + t_1 p$ with $0 < t_0 < p$, then the sequence

$$0 \longrightarrow \mathbb{F}_p\{zy_1^{t_0-1}y_{t_1p}\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_1^{t_0}y_{t_1p}\} \longrightarrow 0$$

gives that $\delta(zy_1^{t_0-1}y_{t_1p}) = \delta(z) \cdot \tilde{y}_1^{t_0-1}\tilde{y}_{t_1p} = \tilde{y}_1^{t_0}\tilde{y}_{t_1p} \neq 0$, $p\tilde{y}_1^{t_0}\tilde{y}_{t_1p} = 0$ and $r(\tilde{y}_1^{t_0}\tilde{y}_{t_1p}) = {y_1}^{t_0}y_{t_1p} \neq 0$ for any choice of a lift $\tilde{y}_{t_1p}$. The last point shows that $\tilde{y}_1^{t_0}\tilde{y}_{t_1p}$ is not divisible by p and hence we can choose it as our generator: $\tilde{y}_t = \tilde{y}_1^{t_0}\tilde{y}_{t_1p}$, and furthermore, this generator is killed by p, so $a_t = 1$.

Note that we may reconsider our choice of $\tilde{y}_{t_1p}$ later, and so the exact choice of $\tilde{y}_t$ may still change within these bounds, but the choices of $\tilde{y}_1, \ldots, \tilde{y}_{p-1}$ remain fixed from now on. Hence $\mathbf{P(1)(1)} - \mathbf{P(1)(3)}$ are established and as $\mathbf{P(1)(4)}$ is vacuous we have shown $\mathbf{P(1)}$.

Establishing $\mathbf{P(n+1)}$. Now, assume $\mathbf{P(n)}$. First, consider the case $t = p^n$. For $\mathbf{P(n+1)(4)}$ we only have to show that

$$\tilde{y}_{p^n} = p^{a_{p^n} - a_{p^{n-1}}} \tilde{y}^p_{p^{n-1}},$$

and that $a_{p^n} > a_{p^{n-1}}$. Consider the sequence

$$0 \longrightarrow \mathbb{F}_p\{zy_1^{p-1} \dots y_{p^{n-1}}^{p-1}\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2p^n}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2p^n}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_{p^n}\} \longrightarrow 0.$$

Firstly, by induction we have that

$$\begin{aligned}
\delta(zy_1^{p-1} \dots y_{p^{n-1}}^{p-1}) &= \tilde{y}_1 \tilde{y}_1^{p-1} \dots \tilde{y}_{p^{n-1}}^{p-1} \\
&= p^{a_p - a_1} \tilde{y}_p \tilde{y}_p^{p-1} \dots \tilde{y}_{p^{n-1}}^{p-1} \\
&= p^{a_p - a_1} p^{a_{p^2} - a_p} \tilde{y}_{p^2} \tilde{y}_{p^2}^{p-1} \dots \tilde{y}_{p^{n-1}}^{p-1} = \dots \\
&= p^{a_{p^{n-1}} - 1} \tilde{y}^p_{p^{n-1}} \neq 0.
\end{aligned}$$

Secondly,

$$p\delta(zy_1^{p-1} \cdots y_{p^{n-1}}^{p-1}) = pp^{a_{p^{n-1}}-1} \tilde{y}^p_{p^{n-1}} = p^{a_{p^{n-1}}} \tilde{y}^p_{p^{n-1}} = 0.$$

Together this shows that (up to a unit) $\delta(zy_1^{p-1} \cdots y_{p^{n-1}}^{p-1}) = p^{a_{p^n}-1} \tilde{y}_{p^n}$, and that $\tilde{y}^p_{p^{n-1}} = p^{a_{p^n} - a_{p^{n-1}}} \tilde{y}_{p^n}$, and since $y^p_{p^{n-1}} = 0$ that $a_{p^n} > a_{p^{n-1}}$.

Now, for $\mathbf{P(n+1)(1)}$ and $\mathbf{P(n+1)(2)}$, consider a general t with $v(t) = n$ and write $t = t_n p^n + sp^{n+1}$ with $0 < t_n < p$. The exact sequence

$$\mathbb{F}_p\{zy_1^{p-1} \dots y_{p^{n-1}}^{p-1} y_{p^n}^{t_n - 1} y_{sp^{n+1}}\} \xrightarrow{\delta} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{p} \mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)}) \xrightarrow{r} \mathbb{F}_p\{y_{p^n}^{t_n} y_{sp^{n+1}}\}$$

gives that

$$\begin{aligned}
\delta(zy_1^{p-1} \dots y_{p^{n-1}}^{p-1} y_{p^n}^{t_n-1} y_{sp^{n+1}}) &= \tilde{y}_1 \tilde{y}_1^{p-1} \dots \tilde{y}_{p^{n-1}}^{p-1} \tilde{y}_{p^n}^{t_n-1} \tilde{y}_{sp^{n+1}} \\
&= p^{a_{p^n}-1} \tilde{y}^{t_n}_{p^n} \tilde{y}_{sp^{n+1}} \neq 0,
\end{aligned}$$

but $p\delta(zy_1^{p-1} \dots y_{p^{n-1}}^{p-1} y_{p^n}^{t_n-1} y_{sp^{n+1}}) = p^{a_{p^n}} \tilde{y}^{t_n}_{p^n} \tilde{y}_{sp^{n+1}} = 0$ and $r(\tilde{y}^{t_n}_{p^n} \tilde{y}_{sp^{n+1}}) = y^{t_n}_{p^n} y_{sp^{n+1}} \neq 0$. Again, the last point shows that $\tilde{y}^{t_n}_{p^n} \tilde{y}_{sp^{n+1}}$ is not divisible by p, and so we may choose $\tilde{y}_t = \tilde{y}^{t_n}_{p^n} \tilde{y}_{sp^{n+1}}$, and furthermore that this generator is annihilated by $p^{a_{p^n}}$, but not by $p^{a_{p^n}-1}$, so that $a_t = a_{p^n}$.

Lastly, by $\mathbf{P(n)(3)}$, we have that if $s = s_0 + s_1 p + \dots + s_{n-1} p^{n-1}$ is the p-adic expansion of s, then $\tilde{y}_s = \tilde{y}_1^{s_0} \dots \tilde{y}^{s_{n-1}}_{p^{n-1}}$. If $t = s + s_n p^n$, then $r(\tilde{y}_s \tilde{y}^{s_n}_{p^n}) = y_t$, so we can choose $\tilde{y}_t = \tilde{y}_s \tilde{y}^{s_n}_{p^n}$ as desired in $\mathbf{P(n+1)(3)}$. $\square$

Background on Bocksteins. Let (C_*, ∂) be a complex of free abelian groups and assume $\alpha \in C_n$ has the property that $\alpha \otimes 1$ is a cycle in $C_* \otimes \mathbb{F}_p$. That the Bockstein $\beta_{i-1}[\alpha \otimes 1]$ is defined and equal to zero for some $i \geqslant 2$, means that there exist $\gamma \in C_n$ and cycle $\delta \in C_{n-1}$ so that $\partial(\alpha + p\gamma) = p^i \delta$, and in that case, $\beta_i[\alpha \otimes 1] = [\delta \otimes 1]$.

Assume we have a short exact sequence of complexes of free abelian groups $0 \to B_* \to C_* \to A_* \to 0$. Choosing a section in each degree, we may assume $C_n = A_n \oplus B_n$ for all n. Suppose we have $a \in A_n$ and $b \in B_n$ so that $[a + b]$ represents a cycle in $C_* \otimes \mathbb{F}_p$ with $\beta_{i-1}([(a + b) \otimes 1]) = [0] \in H_{n-1}(C_* \otimes \mathbb{F}_p)$. As above, there exist $c \in A_n$, $d \in B_n$, $e \in A_{n-1}$, $f \in B_{n-1}$ with $e + f \in C_{n-1}$ a cycle with $\partial(a + b + p(c + d)) = p^i(e + f)$, and in that case $\beta_i([(a + b) \otimes 1]) = [(e + f) \otimes 1]$. Then if $[e \otimes 1] \neq [0] \in H_{n-1}(A_* \otimes \mathbb{F}_p)$, we get that $\beta_i([(a + b) \otimes 1]) \neq [0]$, since $[(e + f) \otimes 1] \mapsto [e \otimes 1] \neq [0]$ by the homomorphism induced by the projection $C_* \to A_*$.

More generally, consider a filtered complex C_* of free abelian groups. Assume we have a chain $a \in E(C_*)^0_{s,t}$ in the associated spectral sequence such that $[a \otimes 1]$ survives to $E(C_* \otimes \mathbb{F}_p)^\infty_{s,t}$ in the mod p spectral sequence. If we know that the class $[(a+b) \otimes 1] \in H_{s+t}(C_* \otimes \mathbb{F}_p)$ with $b \in F_{s-1}(C_*)$

which $[a\otimes 1]$ represents in $E(C_*\otimes \mathbb{F}_p)^\infty_{s,t}$ satisfies $\beta_{i-1}([(a+b)\otimes 1]) = [0] \in H_{s+t-1}(C_*\otimes \mathbb{F}_p)$, but that $d^0(a\otimes 1) = p^i(e\otimes 1)$ and $[e\otimes 1] \neq [0] \in E(C_*\otimes \mathbb{F}_p)^1_{s,t-1}$, then $\beta_i([(a+b)\otimes 1]) \neq [0]$.

The p-order of the multiplicative generators. We will calculate $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$ by studying its Hurewicz image in $H_*(\mathsf{THH}^{[2]}(\mathbb{Z}_{(p)});\mathbb{F}_p)$, using the model

$$\mathsf{THH}^{[2]}(\mathbb{Z}_{(p)}) \simeq B(H\mathbb{Z}_{(p)}, \mathsf{THH}(\mathbb{Z}_{(p)}), H\mathbb{Z}_{(p)}).$$

We use the filtration by simplicial skeleta. We denote $H_*(H\mathbb{Z}_{(p)};\mathbb{F}_p)$ by $\bar{\mathcal{A}}$, and by Bökstedt,

$$H_*(\mathsf{THH}(\mathbb{Z}_{(p)});\mathbb{F}_p) \cong \bar{\mathcal{A}} \otimes \mathbb{F}_p[x_{2p}] \otimes \Lambda[x_{2p-1}],$$

where the augmentation $\mathsf{THH}(\mathbb{Z}_{(p)}) \to H\mathbb{Z}_{(p)}$ induces the projection $\bar{\mathcal{A}}\otimes\mathbb{F}_p[x_{2p}]\otimes\Lambda[x_{2p-1}] \to \bar{\mathcal{A}}$ sending x_{2p} and x_{2p-1} to zero. We get that

$$E^1_{*,*} \cong B(\bar{\mathcal{A}}, \bar{\mathcal{A}}\otimes \mathbb{F}_p[x_{2p}]\otimes\Lambda[x_{2p-1}], \bar{\mathcal{A}})$$

is isomorphic to

$$B(\bar{\mathcal{A}},\bar{\mathcal{A}},\bar{\mathcal{A}}) \otimes B(\mathbb{F}_p, \mathbb{F}_p[x_{2p}], \mathbb{F}_p) \otimes B(\mathbb{F}_p, \Lambda[x_{2p-1}], \mathbb{F}_p),$$

and so its homology is

$$E^2_{*,*} \cong \bar{\mathcal{A}} \otimes \Lambda[y_{2p+1}] \otimes \Gamma[y_{2p}]$$

with $y_{2p+1} = 1\otimes x_{2p}\otimes 1$ and $y_{2p} = 1\otimes x_{2p-1}$ and $y_{2p}^{(a)} = 1\otimes x_{2p-1}^{\otimes a}\otimes 1$.

The dimensions in each total degree in the E^2-term account for p-torsion of rank 1 in each positive dimension divisible by $2p$, and from knowing $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)};\mathbb{F}_p)$ [DLR18, Theorem 3.1] we get that this agrees with the abutment of the spectral sequence, so it has to collapse at E^2.

We use this to prove Theorem 2.1. By Lemma 2.2, the only remaining problem is to determine the order of the p-torison in each dimension divisible by $2p$.

Lemma 2.3. *The p-torsion in* $\mathsf{THH}^{[2]}_{2pt}(\mathbb{Z}_{(p)})$ *is precisely* $\mathbb{Z}_{(p)}/pt \cong \mathbb{Z}_{(p)}/p^{v_p(t)+1}$.

Proof. We know from Lemma 2.2 that for $t = p^a m$, $(p,m) = 1$, in dimension $2pt$ the order of the torsion is divisible by p^{a+1}. We will use the general observation about Bocksteins above for

$$C_* = C_*(\Omega^\infty(\mathsf{THH}^{[2]}(\mathbb{Z}_{(p)}));\mathbb{Z}) = C_*(\Omega^\infty B(H\mathbb{Z}_{(p)}, \mathsf{THH}(\mathbb{Z}_{(p)}), H\mathbb{Z}_{(p)});\mathbb{Z}),$$

filtered by simplicial skeleta of the bar construction, to get that the torsion is exactly p^{a+1}.

Fixing a t, we have two quasi-isomorphisms (letting s vary)

$$\begin{aligned}
&C_s(\Omega^\infty B_t(H\mathbb{Z}_{(p)}, \mathsf{THH}(\mathbb{Z}_{(p)}), H\mathbb{Z}_{(p)});\mathbb{Z})\\
\to &C_{s+t}(\Delta^t \times \Omega^\infty B_t(H\mathbb{Z}_{(p)}, \mathsf{THH}(\mathbb{Z}_{(p)}), H\mathbb{Z}_{(p)}), \partial\Delta^t \times \Omega^\infty B_t(H\mathbb{Z}_{(p)}, \mathsf{THH}(\mathbb{Z}_{(p)}), H\mathbb{Z}_{(p)});\mathbb{Z})\\
\to &E^0_{t,s}
\end{aligned}$$

and we call their composition φ.

We know by Bökstedt that additively $\mathsf{THH}(\mathbb{Z}_{(p)}) \simeq H\mathbb{Z}_{(p)} \vee \Sigma^{2p-1}H\mathbb{F}_p \vee \cdots$, so we can map

$$\begin{aligned}
&S^0 \wedge K(\mathbb{F}_p, 2p-1)^{\wedge t} \wedge S^0 = S^0 \wedge (\Omega^\infty(\Sigma^{2p-1}H\mathbb{F}_p))^{\wedge t} \wedge S^0\\
\to &\Omega^\infty H\mathbb{Z}_{(p)} \wedge (\Omega^\infty(\mathsf{THH}(\mathbb{Z}_{(p)})))^{\wedge t} \wedge \Omega^\infty H\mathbb{Z}_{(p)} \to \Omega^\infty(H\mathbb{Z}_{(p)} \wedge (\mathsf{THH}(\mathbb{Z}_{(p)})^{\wedge t} \wedge H\mathbb{Z}_{(p)}).
\end{aligned}$$

We call this composition ψ. It induces

$$\psi_*:\ C_*(K(\mathbb{F}_p, 2p-1)^{\wedge t};\mathbb{Z}) \to C_*(\Omega^\infty(H\mathbb{Z}_{(p)} \wedge \mathsf{THH}(\mathbb{Z}_{(p)})^{\wedge t} \wedge H\mathbb{Z}_{(p)});\mathbb{Z}),$$

so composing we get a map of complexes

$$\varphi\circ\psi_*:\ C_*(K(\mathbb{F}_p, 2p-1)^{\wedge t};\mathbb{Z}) \to E^0_{t,*}.$$

On the Eilenberg Mac Lane space $K(\mathbb{F}_p, 2p-1)$, we have a $2p$-chain with integer coefficients $\tilde{x}_{2p}$ so that $[\tilde{x}_{2p}]$ (mod p) generates $H_{2p}(K(\mathbb{F}_p, 2p-1);\mathbb{F}_p) \cong \mathbb{F}_p$ and $\partial\tilde{x}_{2p} = p\tilde{x}_{2p-1}$ for a chain $\tilde{x}_{2p-1}$ so that $[\tilde{x}_{2p-1}]$ (mod p) generates $H_{2p-1}(K(\mathbb{F}_p, 2p-1);\mathbb{F}_p) \cong \mathbb{F}_p$. For these elements, $\beta_1([\tilde{x}_{2p}]) = [\tilde{x}_{2p-1}]$. Note that these elements map to generators of the stable homology in the correct dimensions. Thus, $\varphi \circ \psi_*(\tilde{x}_{2p}) \otimes 1$ can be taken as a representative of x_{2p}, and $\varphi \circ \psi_*(\tilde{x}_{2p-1}) \otimes 1$ can be taken as a representative of x_{2p-1}, and we still have $d^0(\varphi \circ \psi_*(\tilde{x}_{2p})) = p\varphi \circ \psi_*(\tilde{x}_{2p-1})$ in $E^0_{1,*}$. And more generally, for any $a, b \geqslant 0$, in $E^0_{a+b+1,*}$ we also have

$$d^0(\varphi \circ \psi_*(\tilde{x}_{2p-1}^{\wedge a} \wedge \tilde{x}_{2p} \wedge \tilde{x}_{2p-1}^{\wedge b})) = p\varphi \circ \psi_*(\tilde{x}_{2p-1}^{\wedge(a+b+1)}).$$

We know that the class $(\varphi \circ \psi_*(\tilde{x}_{2p-1}^{\wedge a} \wedge \tilde{x}_{2p} \wedge \tilde{x}_{2p-1}^{\wedge b})) \otimes 1$ represents the class $1 \otimes x_{2p-1}^{\otimes a} \otimes x_{2p} \otimes x_{2p-1}^{\otimes b} \otimes 1$ which survives to $E^2_{a+b+1,*}$ and therefore to $E^\infty_{a+b+1,*}$, and similarly for $(\varphi \circ \psi_*(\tilde{x}_{2p-1}^{\wedge(a+b+1)})) \otimes 1$ and $1 \otimes x_{2p-1}^{\otimes a+b+1} \otimes 1$.

And so, if $t = p^a m$ with $(p, m) = 1$,

$$d^0(\sum_{i=0}^{t-1}(-1)^i(\varphi\circ\psi_*(\tilde{x}_{2p-1}^{\wedge i}\wedge\tilde{x}_{2p}\wedge\tilde{x}_{2p-1}^{\wedge t-1-i}))\otimes 1 = pt\cdot(\varphi\circ\psi_*(\tilde{x}_{2p-1}^{\wedge(t)}))\otimes 1 = p^{a+1}m\cdot(\varphi\circ\psi_*(\tilde{x}_{2p-1}^{\wedge(t)}))\otimes 1.$$

The mod p homology class which is the image under the Hurewicz map of $z\gamma_{t-1}(y)$ can be expressed as

$$(1 \otimes x_{2p} \otimes 1)(1 \otimes x_{2p-1}^{\otimes n-1} \otimes 1)$$

via the bar construction and it is represented by $(\sum_{i=0}^{t-1}(-1)^i(\varphi \circ \psi_*(\tilde{x}_{2p-1}^{\wedge i} \wedge \tilde{x}_{2p} \wedge \tilde{x}_{2p-1}^{\wedge t-1-i}) \otimes 1$. From Lemma 2.2 we have a lower bound on the order of the torsion and hence $\beta_a(z\gamma_{t-1}(y)) = [0]$ and by the d^0 calculation above $\beta_{a+1}(z\gamma_{t-1}(y)) = \gamma_t(y)$ up to a unit.

This result is a result on stable mod p homology rather than on stable mod p homotopy, but since we are applying it to the images under the Hurewicz map of the two stable mod p homotopy classes of an Eilenberg Mac Lane space of rank 1 p-torsion, the Bockstein operators have to do the same on the mod p homotopy. □

Proof of Theorem 2.1. Set $x_n = \tilde{y}_{p^n}$. Then we get the p-order of these elements from Lemma 2.3 and we worked out the multiplicative relations in Lemma 2.2. □

Remark 2.4. Mike Hill noticed that $\mathsf{THH}^{[2]}_*(\mathbb{Z}_{(p)})$ is abstractly isomorphic to $\mathsf{THH}^*(\mathbb{Z}_{(p)})$: the calculation of $\mathsf{THH}^*(\mathbb{Z}_{(p)})$ is due to Franjou and Pirashvili [FP98]. We are not sure whether this is a coincidence or whether (for some commutative S-algebras) there is a duality between $\mathsf{THH}^{[2]}_*$ and topological Hochschild cohomology. Note, however, that $\mathsf{THH}^{[2]}_*(\mathbb{F}_p)$ is an exterior algebra over $\mathbb{F}_p$ on a class in degree three whereas $\mathsf{THH}^*(\mathbb{F}_p)$ is much larger:

$$\mathsf{THH}^*(\mathbb{F}_p) \cong \mathbb{F}_p[e_0, e_1, \ldots]/(e_0^p, e_1^p, \ldots), \quad |e_i| = 2p^i$$

[FLS94, 7.3], [Bö∞_2], so there is no isomorphism of these groups in general.

3. Greenlees' approach to THH

There is a relative version of the cofiber sequence from [G16, Lemma 7.1] already mentioned in [DLR18]. We make it explicit for later use. Here and elsewhere S denotes the sphere spectrum.

Lemma 3.1. *Let R be a commutative S-algebra and let $C \to B \to k$ be a sequence of maps of commutative R-algebras. Then there is a cofiber sequence of commutative k-algebras*

$$B \wedge^L_C k \to \mathsf{THH}^R(C, k) \to \mathsf{THH}^R(B, k).$$

The proof is obtained from the one of [G16, Lemma 7.1] by replacing the sphere spectrum by R.

Remark 3.2. Note that there are two cofiber sequences for any such sequence $C \to B \to k$, because we can forget the commutative R-algebra structures on C and B and consider them as commutative S-algebras. This gives a commutative diagram of cofiber sequences

$$\begin{tikzcd} B \wedge_C k \arrow[r] \arrow[d, equal] & \mathsf{THH}(C,k) \arrow[r] \arrow[d] & \mathsf{THH}(B,k) \arrow[d] \\ B \wedge_C k \arrow[r] & \mathsf{THH}^R(C,k) \arrow[r] & \mathsf{THH}^R(B,k), \end{tikzcd}$$

so $B \wedge_C k$ measures the difference of the absolute and also of the relative THH-terms of C and B.

Let us abbreviate $B \wedge_C^L k$ by A. Lemma 3.1 provides an equivalence

$$\mathsf{THH}^R(B,k) \simeq \mathsf{THH}^R(C,k) \wedge_A^L k$$

and thus we get a spectral sequence whose E^2-term is

$$\mathsf{Tor}_{*,*}^{A_*}(\mathsf{THH}_*^R(C,k), k_*)$$

which converges to $\mathsf{THH}_*^R(B,k)$.

We will consider the following examples.

- Let ℓ denote the Adams summand of p-local connective topological complex K-theory, $ku_{(p)}$, for some odd prime p. For

  $$R = \ell \to C = \ell \to B = ku_{(p)} \to k$$

  with $k = H\mathbb{Z}_{(p)}$ or $k = H\mathbb{F}_p$ we obtain calculations for $\mathsf{THH}_*^\ell(ku_{(p)}, k)$. We determine $\mathsf{THH}_*^\ell(ku_{(p)})$ by different means.
- The complexification map from real to complex topological K-theory $c\colon ko \to ku$ is a map of commutative S-algebras. Wood's theorem displays the ko-module ku as the cofiber of the Hopf map $\eta\colon \Sigma ko \to ko$. Consequently, the ku-module $ku \wedge_{ko} ku$ is the cofiber of $\eta\colon \Sigma ku \to ku$, and the resulting short exact sequences

  $$0 \to \pi_{2m} ku \to \pi_{2m}(ku \wedge_{ko} ku) \to \pi_{2m-1}(\Sigma ku) \to 0$$

  are split via the multiplication map on ku, because the map $ku \to ku \wedge_{ko} ku$ above is induced by the unit map of ku as a commutative ko-algebra so we get

  $$\pi_{2m}(ku \wedge_{ko} ku) \cong \pi_{2m} ku \oplus \pi_{2m-2}(ku).$$

  We will determine the ku_*-algebra structure of $\pi_*(ku \wedge_{ko} ku)$ in Lemma 5.1. This is the input for the Tor-spectral sequence computing $\mathsf{THH}_*^{ko}(ku)$ and we will identify $\mathsf{THH}_*^{ko}(ku)$ in Theorem 5.2.

  We will also use the cofiber sequences of commutative k-algebras

  $$ku \wedge_{ko} k \to ku \to \mathsf{THH}^{ko}(ku, k)$$

  for $k = H\mathbb{Z}_{(2)}$ and $k = H\mathbb{F}_2$ and we will calculate THH of ku over ko with coefficients in $H\mathbb{Z}_{(2)}$ and $H\mathbb{F}_2$ (see Proposition 5.3).
- We propose $ku_{(p)} \wedge_\ell H\mathbb{F}_p$ as a model for $ku/(p, v_1)$ and use the sequence

  $$S \to H\mathbb{F}_p \to ku_{(p)} \wedge_\ell H\mathbb{F}_p \to H\mathbb{F}_p$$

  for calculating its THH with coefficients in $H\mathbb{F}_p$ (Proposition 6.2).
- In Section 7 we determine relative topological Hochschild homology of quotient maps $R \to R/x$.

4. Relative THH of $ku_{(p)}$ as a commutative ℓ-algebra

Let p be an odd prime. On the level of coefficients, the map from the connective Adams summand to p-local connective topological complex K-theory is $\ell_* = \mathbb{Z}_{(p)}[v_1] \to \mathbb{Z}_{(p)}[u] = (ku_{(p)})_*$, $v_1 \mapsto u^{p-1}$. The corresponding p-complete periodic extension is a C_{p-1}-Galois extension [R08]. However, the connective extension is not unramified, but it is a topological analogue of a tamely ramified extension. Rognes defined a notion of THH-étale extensions in [R08, 9.2.1]: A map of commutative S-algebras $A \to B$ is *formally* THH*-étale*, if the canonical map from B to $\mathsf{THH}^A(B)$ is an equivalence. For instance, Galois extensions are formally THH-étale [R08, 9.2.6]. We will show that the map $\ell \to ku_{(p)}$ is not formally THH-étale by determining $\mathsf{THH}^\ell(ku_{(p)})$. Rognes mentions in [R08, p. 59] that $ku_{(p)} \to \mathsf{THH}^\ell(ku_{(p)})$ is a $K(1)$-local equivalence and Sagave showed in [S14] that the map $\ell \to ku_{(p)}$ is log-étale. Ausoni proved that the p-completed extension even satisfies Galois descent for THH and algebraic K-theory [Au05, Theorem 1.5]:

$$\mathsf{THH}(ku_p)^{hC_{p-1}} \simeq \mathsf{THH}(\ell_p), \quad K(ku_p)^{hC_{p-1}} \simeq K(\ell_p).$$

The tame ramification is visible in THH:

Theorem 4.1.

$$\mathsf{THH}^\ell_*(ku_{(p)}) \cong (ku_{(p)})_* \rtimes ((ku_{(p)})_*/u^{p-2})\langle y_0, y_1, \ldots\rangle,$$

where $(ku_{(p)})_* \rtimes M$ denotes a square-zero extension of $(ku_{(p)})_*$ by a $(ku_{(p)})_*$-module M. The degree of y_i is $2pi+3$.

Proof. We can apply the Bökstedt spectral sequence with π_* as the homology theory because $(ku_{(p)})_*$ is projective over ℓ_*. The E^2-page consists of

$$E^2_{s,t} = \mathsf{HH}^{\ell_*}_{s,t}((ku_{(p)})_*, (ku_{(p)})_*).$$

As an ℓ_*-algebra $(ku_{(p)})_*$ is isomorphic to $\ell_*[u]/(u^{p-1} - v_1)$. From [LL92] we know that we can use the following complex in order to calculate Hochschild homology:

$$\ldots \xrightarrow{\Delta(u)} \Sigma^{2p}(ku_{(p)})_* \xrightarrow{0} \Sigma^{2p-2}(ku_{(p)})_* \xrightarrow{\Delta(u)} \Sigma^{2}(ku_{(p)})_* \xrightarrow{0} (ku_{(p)})_*,$$

where $\Delta(u) = (p-1)u^{p-2}$. As $(p-1)$ and v_1 are units in ℓ_*, this yields:

$$\mathsf{HH}^{\ell_*}_i((ku_{(p)})_*, (ku_{(p)})_*) = \begin{cases} (ku_{(p)})_*, & \text{if } i = 0, \\ \Sigma^{2mp-2m+2}(ku_{(p)})_*/u^{p-2}, & \text{if } i = 2m+1, m \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

As $\mathsf{THH}^\ell(ku_{(p)})$ is an augmented commutative $ku_{(p)}$-algebra, we know that $ku_{(p)}$ splits off $\mathsf{THH}^\ell(ku_{(p)})$. Therefore the copy of the homotopy groups of $ku_{(p)}$ in the zero column of the spectral sequence has to survive and cannot be hit by any differentials. For degree reasons, there are no other possible non-trivial differentials and the spectral sequence collapses at the E^2-page.

In every fixed total degree there is only one term in the E^2-page contributing to this degree: If we consider an element u^{k_1} in homological degree $2m_1+1$ and another element u^{k_2} in homological degree $2m_2+1$ for $m_1 \neq m_2$, then their total degrees are $2m_1p + 2k_1 + 3$ and $2m_2p + 2k_2 + 3$. These degrees can only be equal if $2p(m_1 - m_2) = 2(k_2 - k_1)$. Thus p has to divide $k_2 - k_1 \neq 0$. But $0 \leqslant k_1, k_2 \leqslant p-3$, so this cannot happen.

Thus there are no additive extensions and therefore additively we get the desired result.

As $\mathsf{THH}^\ell_*(ku_{(p)})$ is an augmented graded commutative $(ku_{(p)})_*$-algebra and as everything in the augmentation ideal is concentrated in odd degrees there cannot be any non-trivial multiplication of any two elements in the augmentation ideal.

The spectral sequence is a spectral sequence of $(ku_{(p)})_*$-modules and elements of the form $u^k \cdot \Sigma^{2mp-2m+2}u^m$ are cycles, thus the copy of $(ku_{(p)})_*$ in homological degree zero acts on $ku_{(p)})_*/u^{p-2}y_m$ in the standard way. $\square$

Remark 4.2. For Galois extensions of non-connective commutative ring spectra we would like to have a good notion of rings of integers. In the above case $ku_{(p)}$ behaves like the ring of integers of $KU_{(p)}$, and similarly for the connective Adams summand. The result for relative THH corresponds to the one of ordinary rings of integers [LM00]. In other cases, taking the connective cover does not seem to give good results.

For coefficients in $H\mathbb{Z}_{(p)}$ and $H\mathbb{F}_p$ we obtain a rather different result.

Proposition 4.3.
$$\mathsf{THH}_*^\ell(ku_{(p)}, H\mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\varepsilon u) \otimes \Gamma_{\mathbb{Z}_{(p)}}(\varphi^0 u)$$
and also
$$\mathsf{THH}_*^\ell(ku_{(p)}, H\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u).$$

Proof. We consider the sequence of comutative ℓ-algebras
$$R = \ell \to C = \ell \to B = ku_{(p)} \to k$$
with $k = H\mathbb{Z}_{(p)}$ and $k = H\mathbb{F}_p$. In both cases we can identify $\mathsf{THH}^\ell(ku_{(p)}, k)$ with $ku_{(p)} \wedge_\ell k$ and get a Tor-spectral sequence
$$\mathsf{Tor}_{*,*}^{\pi_*(ku_{(p)}\wedge_\ell k)}(\pi_* k, \pi_* k) \Rightarrow \mathsf{THH}_*^\ell(ku_{(p)}, k).$$
For $k = H\mathbb{Z}_{(p)}$ homological algebra tells us that
$$\mathsf{Tor}_{*,*}^{\mathbb{Z}_{(p)}[u]/u^{p-1}}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\varepsilon u) \otimes \Gamma_{\mathbb{Z}_{(p)}}(\varphi^0 u).$$
Here, $|\varepsilon u| = 3$ and $|\varphi^0 u| = 2p$. There are no differentials in this spectral sequence for degree reasons and there are no multiplicative extensions, hence we get the claim.

For $k = H\mathbb{F}_p$ the same method gives
$$\mathsf{THH}_*^\ell(ku_{(p)}, H\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u).$$
□

Remark 4.4. At the prime 3 we get
$$\mathsf{THH}_*^{[0],\ell}(ku_{(3)}, H\mathbb{F}_3) \cong \mathbb{F}_3[u]/u^2$$
hence with the methods of [DLR18] we can deduce that
$$\mathsf{THH}^{[0],\ell}(ku_{(3)}, H\mathbb{F}_3) \simeq H\mathbb{F}_3 \vee \Sigma^2 H\mathbb{F}_3$$
as an augmented commutative $H\mathbb{F}_3$-algebra and that we can calculate higher THH as iterated Tor-algebras. Hence we get
$$\mathsf{THH}_*^{[n+1],\ell}(ku_{(3)}, H\mathbb{F}_3) \cong \mathsf{Tor}_{*,*}^{\mathsf{THH}_*^{[n],\ell}(ku_{(3)},H\mathbb{F}_3)}(\mathbb{F}_3, \mathbb{F}_3)$$
for all $n \geqslant 0$.

Using Greenlees' spectral sequence [G16, Lemma 3.1] one can actually deduce that this is true at all odd primes. See [AuD∞] for related arguments.

5. Relative THH of the complexification map

The graded commutative ring ko_* is $\mathbb{Z}[\eta, y, w]/\langle 2\eta, \eta y, \eta^3, y^2 - 4w\rangle$ with $|\eta| = 1$, $|y| = 4$ and w is the Bott class in degree 8. The complexification map $c\colon ko \to ku$ induces a map $c_*\colon ko_* \to ku_* = \mathbb{Z}[u]$ and it sends η to zero, y to $2u^2$ and the Bott class w to u^4.

Note that the homotopy fixed points of ku with respect to complex conjugation are not equivalent to ko. The homotopy fixed points spectral sequence yields generators in negative degrees in the homotopy groups of ku^{hC_2} [R08, 5.3].

The published version of this paper unfortunately contained a mistake in a calculation that resulted in erroneous statements in Lemma 5.1 and Theorem 5.2. We are grateful to Eva Höning who discovered the mistake.

Lemma 5.1. *There is an isomorphism of graded commutative augmented ku_*-algebras*

$$(ku \wedge_{ko} ku)_* \cong ku_*[s]/(s^2 - su)$$

with $|s| = 2$, where the ku_*-algebra structure on $(ku \wedge_{ko} ku)_*$ is from the left and the augmentation is given by the multiplication $m\colon ku \wedge_{ko} ku \to ku$ and by $s \mapsto 0$.

Proof. Smashing Wood's cofiber sequence $ko \xrightarrow{\iota} ku \xrightarrow{\jmath} \Sigma^2 ko$ with ku (from the left) over ko gives a split exact sequence (with unit isomorphism $ku \cong ku \wedge_{ko} ko$ suppressed)

$$\begin{tikzcd} 0 \arrow[r] & \pi_* ku \arrow[r, "(1\wedge\iota)_*"] & \pi_*(ku \wedge_{ko} ku) \arrow[l, bend right, "m_*"'] \arrow[r, "(1\wedge\jmath)_*"] & \pi_* \Sigma^2 ku \arrow[r] & 0. \end{tikzcd}$$

Let $u \in \pi_2 ku$ be the generator with $\jmath_*(u) = \Sigma^2 2 \in \pi_2 \Sigma^2 ko \cong \mathbb{Z}$. Let u_l and u_r be the images of u in $\pi_2(ku \wedge_{ko} ku)$ induced by the left and right inclusion of ku in $ku \wedge_{ko} ku$. If s is the unique element in $\pi_2(ku \wedge_{ko} ku)$ with $(1\wedge\jmath)_* s = -\Sigma^2 1$ and $m_* s = 0$, then $(1\wedge\jmath)_*(u_r + 2s) = 1 \cdot \jmath_* u - 2 = 0$ and $m_*(u_r + 2s) = u$. Since also $(1 \wedge \jmath)_* u_l = 0$ and $m_* u_l = u$ we must have $u_r + 2s = u_l$.

In $ku_* \otimes_{ko_*} ku_*$, and hence also in $\pi_4(ku \wedge_{ko} ku)$, we have that $2u_r^2 - 2u_l^2 = 0$. As $\pi_*(ku \wedge_{ko} ku)$ is torsion free, we get $u_r^2 - u_l^2 = 0$ and therefore

$$u_r^2 - u_l^2 = (u_l - 2s)^2 - u_l^2 = 4s^2 - 4su_l = 0.$$

Again, since there is no torsion, this yields $s^2 - su_l = 0$. □

Theorem 5.2. *The Tor spectral sequence*

$$E^2_{*,*} = \mathsf{Tor}^{(ku \wedge_{ko} ku)_*}_{*,*}(ku_*, ku_*) \Rightarrow \mathsf{THH}^{ko}_*(ku)$$

collapses at the E^2-page and $\mathsf{THH}^{ko}_*(ku)$ is a square zero extension of ku_*:

$$\mathsf{THH}^{ko}_*(ku) \cong ku_* \rtimes (ku_*/u)\{y_0, y_1, \ldots\}$$

with $|y_j| = (1 + |u|)(2j + 1) = 3(2j + 1)$.

Proof. Lemma 5.1 implies that the E^2-term of the Tor spectral sequence is

$$E^2_{*,*} = \mathsf{Tor}^{(ku \wedge_{ko} ku)_*}_{*,*}(ku_*, ku_*) = \mathsf{Tor}^{ku_*[s]/(s^2 - su)}_{*,*}(ku_*, ku_*).$$

We have a periodic free resolution of ku_* as a module over $ku_*[s]/(s^2 - su)$

$$\ldots \xrightarrow{s} \Sigma^4 ku_*[s]/(s^2 - su) \xrightarrow{s-u} \Sigma^2 ku_*[s]/(s^2 - su) \xrightarrow{s} ku_*[s]/(s^2 - su).$$

Tensoring this down to ku_* yields

$$\ldots \xrightarrow{0} \Sigma^4 ku_* \xrightarrow{-u} \Sigma^2 ku_* \xrightarrow{0} ku_*.$$

As ku_* splits off $\mathsf{THH}^{ko}_*(ku)$, the zero column has to survive and cannot be hit by differentials and hence all differentials are trivial.

For the E^∞-term we therefore get $E^\infty_{0,*} \cong ku_*$, $E^\infty_{2j,*} = 0$ for $j > 0$, and $E^\infty_{2j+1,*} \cong (ku_*/u)\{y_j\}$ for y_j in bidegree $(2j+1, 4j+2)$ if $j > 0$. So even total degrees occur only in $E^\infty_{0,*}$ and odd total degrees occur only in at most one bidegree and we do not need to worry about additive extensions. As y_j corresponds to $\Sigma^{4j+2} 1 \in \Sigma^{4j+2} ku_*$, the action of $u^i \in ku_*$ on y_j is $u^i y_j$ and this is trivial for $i \geqslant 1$ in $(ku_*/u)\{y_j\}$ so $u^i y_j$ is zero in $E^\infty_{2j+1,4j+2}$. But $E^\infty_{2j+1,4j+2}$ has all the elements of total degree $6j + 3$ in the entire E^∞-term, so in fact the element in $\mathsf{THH}^{ko}_*(ku)$ that y_j represents is killed by multiplication by u^i for any $i \geqslant 1$. Thus we have no nontrivial products of the u^i, $i \geqslant 1$, and the odd dimensional elements.

Since the elements of $\mathsf{THH}^{ko}_*(ku)$ represented by the y_i are all in odd degrees, if there were nonzero products among them they would have to be elements in $E^\infty_{0,*} \cong ku_*$. But elements in ku_* are not killed by multiplying by u, whereas the elements represented by the y_j are. So there can be no such nontrivial products. □

We consider the sequence of commutative ko-algebras $R = ko \to C = ko \to B = ku$ with $k = H\mathbb{F}_2$ or $k = H\mathbb{Z}_{(2)}$ and, (since $\mathsf{THH}^{ko}(ko, k) \simeq k$), we get cofiber sequences of commutative k-algebras

$$ku \wedge_{ko} k \to k \to \mathsf{THH}^{ko}(ku, k).$$

This yields a Tor-spectral sequence

$$E^2_{s,t} = \mathsf{Tor}^{\pi_*(ku \wedge_{ko} k)}_{s,t}(k_*, k_*) \Rightarrow \mathsf{THH}^{ko}_{s+t}(ku, k). \tag{5}$$

Wood's cofiber sequence identifies ku as the cone on η: $\Sigma ko \to ko$. Thus we get a cofiber sequence

$$\Sigma k \to k \to ku \wedge_{ko} k$$

and $\pi_*(ku \wedge_{ko} k) \cong \pi_*(k \vee \Sigma^2 k) \cong \Lambda_{\pi_* k}(x_2)$ where x_2 is a generator of degree two.

For $k = H\mathbb{F}_2$ and $H\mathbb{Z}_{(2)}$ we can deduce with [DLR18, 2.1] that as a commutative augmented k-algebra $ku \wedge_{ko} k$ is weakly equivalent to the square-zero extension $k \vee \Sigma^2 k$. Thus

$$\mathsf{THH}^{ko}(ku, k) \simeq k \wedge_{k \vee \Sigma^2 k} k$$

and the spectral sequence (5) reduces to

$$E^2_{s,t} = \mathsf{Tor}^{\pi_* k[x_2]/x_2^2}_{s,t}(\pi_* k, \pi_* k) \Rightarrow \mathsf{THH}^{ko}_{s+t}(ku, k).$$

But $\mathsf{Tor}^{\pi_* k[x_2]/x_2^2}_{s,t}(\pi_* k, \pi_* k) \cong \Lambda_{\pi_* k}(\varepsilon x_2) \otimes \Gamma_{\pi_* k}(\varphi^0 x_2)$ with $|\varepsilon x_2| = 3$, $|\varphi^0 x_2| = 6$, and we know from [BLPRZ15] combined with the methods from [DLR18, Section 3] that there cannot be any differentials in this spectral sequence. Hence we obtain

Proposition 5.3.

$$\mathsf{THH}^{ko}_*(ku, H\mathbb{Z}_{(2)}) \cong \Lambda_{\mathbb{Z}_{(2)}}(\varepsilon x_2) \otimes \Gamma_{\mathbb{Z}_{(2)}}(\varphi^0 x_2)$$

and

$$\mathsf{THH}^{ko}_*(ku, H\mathbb{F}_2) \cong \Lambda_{\mathbb{F}_2}(\varepsilon x_2) \otimes \Gamma_{\mathbb{F}_2}(\varphi^0 x_2).$$

Over $\mathbb{F}_2$ we can also iterate the calculation and obtain

$$\mathsf{THH}^{[n+1],ko}_*(ku, H\mathbb{F}_2) \cong \mathsf{Tor}^{\mathsf{THH}^{[n],ko}_*(ku, H\mathbb{F}_2)}_{*,*}(\mathbb{F}_2, \mathbb{F}_2).$$

Remark 5.4. To the eyes of THH with coefficients in $H\mathbb{F}_p$ coefficients (for $p = 2$ resp. $p = 3$) the extensions $ko \to ku$ and $\ell \to ku_{(3)}$ show a similar behaviour. This is analogous to the algebraic case: Hochschild homology homology of the 2-local Gaussian integers with coefficients in $\mathbb{F}_2$ is isomorphic to $\Lambda_{\mathbb{F}_2}(x_1) \otimes \Gamma_{\mathbb{F}_2}(x_2)$ and $\mathsf{HH}^{\mathbb{Z}_{(3)}}_*(\mathbb{Z}_{(3)}[\sqrt{3}], \mathbb{F}_3) \cong \Lambda_{\mathbb{F}_3}(x_1) \otimes \Gamma_{\mathbb{F}_3}(x_2)$. Thus Hochschild homology (and also higher Hochschild homology) with reduced coefficients doesn't distinguish tame from wild ramification either.

6. $ku_{(p)} \wedge_\ell H\mathbb{F}_p$ as a model for $ku/(p, v_1)$

John Greenlees asks in [G16, Example 8.4] for a commutative S-algebra model of $ku/(p, v_1)$. We suggest $ku/(p, v_1) = ku_{(p)} \wedge_\ell H\mathbb{F}_p$ which is a commutative S-algebra (even an augmented commutative $H\mathbb{F}_p$-algebra, which might not be what Greenlees had in mind) and satisfies $\pi_*(ku_{(p)} \wedge_\ell H\mathbb{F}_p) \cong \mathbb{F}_p[u]/u^{p-1}$.

Remark 6.1. Alternatively one could consider $ku/(p, v_1)$ defined by an iterated cofiber sequence. This is an A_∞-ring spectrum [A08, 3.7], hence an associative S-algebra, but we cannot expect any decent level of commutativity without the price of getting something of the homotopy type of a generalized Eilenberg-Mac Lane spectrum: if $ku/(p, v_1)$ were a pseudo-H_2 spectrum, then it automatically splits as a wedge of suspensions of $H\mathbb{F}_p$'s [BMMS86, III.4.1]. In particular, an E_∞-structure (*i.e.*, a commutative S-algebra structure) would lead to such a splitting.

We determine $\mathsf{THH}(ku_{(p)} \wedge_\ell H\mathbb{F}_p, H\mathbb{F}_p)$.

Proposition 6.2. *Topological Hochschild homology of $ku_{(p)} \wedge_\ell H\mathbb{F}_p$ with coefficients in $H\mathbb{F}_p$ is*

$$\mathsf{THH}_*(ku_{(p)} \wedge_\ell H\mathbb{F}_p, H\mathbb{F}_p) \cong \mathbb{F}_p[\mu] \otimes \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u)$$

where $\mathbb{F}_p[\mu] = \mathsf{THH}_*(H\mathbb{F}_p)$.

Proof. Greenlees' cofiber sequence [G16, 7.1] yields an equivalence

$$\mathsf{THH}(ku_{(p)} \wedge_\ell H\mathbb{F}_p, H\mathbb{F}_p) \simeq H\mathbb{F}_p \wedge^L_{ku_{(p)} \wedge_\ell H\mathbb{F}_p} \mathsf{THH}(H\mathbb{F}_p).$$

Therefore, the Tor-spectral sequence has E^2-term

$$\mathsf{Tor}_{*,*}^{\mathbb{F}_p[u]/u^{p-1}}(\mathbb{F}_p, \mathsf{THH}_*(H\mathbb{F}_p)).$$

We use the standard periodic resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[u]/u^{p-1}$. As $\mathsf{THH}(H\mathbb{F}_p)$ has the same chromatic type as $H\mathbb{F}_p$, u acts by zero on $\mathsf{THH}_*(H\mathbb{F}_p) = \mathbb{F}_p[\mu]$ and hence the E^2-term is isomorphic to

$$\mathbb{F}_p[\mu] \otimes \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u).$$

As $\mathsf{THH}(ku_{(p)} \wedge_\ell H\mathbb{F}_p)$ is an augmented commutative $\mathsf{THH}(H\mathbb{F}_p)$-algebra, the $\mathbb{F}_p[\mu]$-factor splits off and hence there cannot be any differentials and multiplicative extensions. □

7. Killing regular generators in π_*R

Killing regular elements in the homotopy groups of a commutative S-algebra rarely gives rise to commutative quotients. However, there are some important examples for which we *do* get commutative quotients whose relative THH can be calculated.

Proposition 7.1. *Let R be a connective commutative S-algebra whose coefficients π_*R are concentrated in even degrees, with a nonzero divisor x of positive degree. If the canonical map $R \to R/x$ is a morphism of commutative S-algebras, then the Tor spectral sequence*

$$\mathsf{Tor}_{*,*}^{\pi_*(R/x \wedge_R R/x)}(R_*, R_*) \Rightarrow \mathsf{THH}^R_*(R/x)$$

collapses at the E^2-term. Its E^∞-term is isomorphic to $\Gamma_{\pi_*R/x}(\rho^0\varepsilon x)$ with $|\rho^0\varepsilon x| = |x| + 2$ and there are no additive extensions.

Proof. The defining cofiber sequence

$$\Sigma^{|x|}R \xrightarrow{x} R \longrightarrow R/x$$

gives, via a Tor-spectral sequence, that

$$\pi_*(R/x \wedge_R R/x) \cong \Lambda_{\pi_*(R)/x}(\varepsilon x)$$

with $|\varepsilon x| = |x| + 1$. In the spectral sequence for THH we have as an E^2-term

$$\mathsf{Tor}_{*,*}^{\Lambda_{\pi_*(R)/x}(\varepsilon x)}(\pi_*R/x, \pi_*R/x).$$

We consider the periodic resolution of π_*R/x

$$\ldots \xrightarrow{\varepsilon x} \Sigma^{2|x|+2}\Lambda_{\pi_*R/x}(\varepsilon x) \xrightarrow{\varepsilon x} \Sigma^{|x|+1}\Lambda_{\pi_*R/x}(\varepsilon x) \xrightarrow{\varepsilon x} \Lambda_{\pi_*R/x}(\varepsilon x)$$

and tensor it down to π_*R/x. As π_*R/x is concentrated in even degrees, the multiplication by εx induces the trivial map and hence our Tor-terms are the homology of the complex

$$\ldots \xrightarrow{\varepsilon x = 0} \Sigma^{2|x|+2}\pi_*R/x \xrightarrow{\varepsilon x = 0} \Sigma^{|x|+1}\pi_*R/x \xrightarrow{\varepsilon x = 0} \pi_*R/x$$

and this gives a divided power algebra $\Gamma_{\pi_*R/x}(\rho^0\varepsilon x)$ with a generator $\rho^0\varepsilon x$ in degree $|x|+2$. We have to show that there are no non-trivial differentials and no extension problems. The spectral sequence is a spectral sequence of π_*R/x-algebras because R/x is assumed to be a commutative R-algebra, hence $\mathsf{THH}^R(R/x)$ is a commutative R/x-algebra.

As we assumed that x has positive degree, we can split $\Gamma_{\pi_*R/x}(\rho^0\varepsilon x)$ as $\pi_*R/x \otimes_{\pi_0 R} \Gamma_{\pi_0 R}(\rho^0\varepsilon x)$. The π_*R/x-module generators are the π_0R-module generators in $\Gamma_{\pi_0R}(\rho^0\varepsilon x)$. These generators sit in bidegrees of the form $(n, n(|x|+1))$. A differential d^r on a generator in bidegree

$(n, n(|x|+1))$ is in bidegree $(n-r, n(|x|+1)+r-1)$. A general element in the spectral sequence come from a product of powers of generators times an element from R_*/x, hence we get that a potential target has a bidegree of the form

$$(\sum_i u_i n_i, (\sum_i u_i n_i)(|x|+1)+2m).$$

Comparing components of the bidegree gives the two equations

$$n-r=\sum_i u_i n_i \text{ and } n(|x|+1)+r-1=(|x|+1)(\sum_i u_i n_i)+2m.$$

We rewrite the second equation as

$$2m+1=(n-\sum_i u_i n_i)(|x|+1)+r.$$

Using that $n-\sum_i u_i n_i$ is r yields $2m+1=r(|x|+2)$, but the degree of x is even, so there can be no non-trivial differentials in this spectral sequence.

We do not have additive extensions because the E^∞-term is free over $\pi_* R/x$. Thus as an $\pi_* R/x$-module we get that $\mathsf{THH}_*^R(R/x)$ is isomorphic to $\pi_* R/x \otimes_{\pi_0 R} \Gamma_{\pi_0 R}(\rho^0 \varepsilon x)$. ☐

Corollary 7.2. *If in addition to the assumptions in Proposition 7.1 we have that R/x is an Eilenberg-MacLane spectrum of a commutative ring k, then*

$$\mathsf{THH}^R(Hk, Hk) \simeq Hk \wedge_{Hk \vee \Sigma^{|x|+1} Hk} Hk$$

as augmented commutative Hk-algebras. In particular,

$$\mathsf{THH}_*^R(Hk) \cong \Gamma_k(\rho^0 \varepsilon x)$$

with $|\rho^0 \varepsilon x| = |x|+2$

Proof. Greenlees' cofiber sequence identifies $\mathsf{THH}^R(Hk)$ as

$$Hk \wedge^L_{Hk \wedge_R Hk} Hk$$

using the sequence of commutative ring spectra $R = R \to Hk = Hk$. The homotopy groups of $Hk \wedge_R Hk$ are isomorphic to $\Lambda_k(\varepsilon x)$ with $|\varepsilon x| = |x|+1$. Hence we know from [DLR18, Proposition 2.1] that

$$Hk \wedge_R Hk \sim Hk \vee \Sigma^{|x|+1} Hk$$

with the square zero multiplication as augmented commutative Hk-algebras. Therefore we get the first claim. This also shows that $\mathsf{THH}^R(Hk)$ can be modeled as the two-sided bar construction

$$B^{Hk}(Hk, Hk \vee \Sigma^{|x|+1} Hk, Hk)$$

and by [DLR18] we know that its homotopy groups are the homology groups of the algebraic bar construction $B^k(k, \Lambda(\varepsilon x), k)$. We know from [BLPRZ15, Proposition 2.3] that there is a quasi-isomorphism between $\Gamma_k(\rho^0 \varepsilon x)$ (with zero differential) and the differential graded commutative algebra associated to $B^k(k, \Lambda(\varepsilon x), k)$. ☐

Corollary 7.3. *If in addition to the assumptions of Corollary 7.2 the ring k is the field $\mathbb{F}_p$ we get*

$$\mathsf{THH}_*^{[n+1],R}(H\mathbb{F}_p, H\mathbb{F}_p) \cong \mathsf{Tor}_{*,*}^{\mathsf{THH}_*^{[n],R}(H\mathbb{F}_p, H\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$$

for all $n \geqslant 0$.

Remark 7.4. In the above statement one can consider a slightly more general case of any field of characteristic p.

Proposition 7.5. *Assume in addition to the requirements of Proposition 7.1 that there is a regular sequence $(x, y_1, \dots, y_n)$ in π_*R such that $R/(x, y_1, \dots, y_n)$ is Hk for some field k. Then*

$$\mathsf{THH}_*^R(R/x, Hk) \cong \Gamma_k(\rho^0\varepsilon x)$$

with $|\rho^0\varepsilon x| = |x| + 2$. If $k = \mathbb{F}_p$, then

$$\mathsf{THH}_*^{[n+1],R}(R/x, H\mathbb{F}_p) \cong \mathsf{Tor}_{*,*}^{\mathsf{THH}_*^{[n],R}(R/x,H\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$$

for all $n \geqslant 0$.

Proof. We consider the sequence of commutative S-algebras

$$R \to R \to R/x \to Hk.$$

Then $\pi_*(Hk \wedge_R R/x) \cong \Lambda_k(\varepsilon x)$ and as before we can conclude with [DLR18, 2.1] that $Hk \wedge_R R/x$ is equivalent to the square zero extension $Hk \vee \Sigma^{|\varepsilon x|} Hk$ in the homotopy category of commutative augmented Hk-algebras.

Greenlees' cofiber sequence identifies $\mathsf{THH}^R(R/x, Hk)$ as

$$Hk \wedge^L_{Hk \vee \Sigma^{|\varepsilon x|} Hk} Hk$$

and we know from [DLR18, BLPRZ15] that this gives $\mathsf{THH}_*^R(R/x, Hk) \cong \Gamma_k(\rho^0\varepsilon x)$.

Higher THH can be calculated using the Tor spectral sequence associated to the 2-sided bar construction: A simplicial model for $\mathsf{THH}^{[n+1],R}(R/x, H\mathbb{F}_p)$ is

$$B(H\mathbb{F}_p, \mathsf{THH}^{[n],R}(R/x, H\mathbb{F}_p), H\mathbb{F}_p)$$

and we know from the methods established in [DLR18] and [BLPRZ15] that these Tor-spectral sequences all collapse at the E^2-term with no non-trivial extensions. □

Examples 7.6. We end the section with some examples.

(1) Let R be an Eilenberg-MacLane spectrum HA with A a commutative ring and let x be regular in A. Then $\mathsf{THH}_*^{HA}(HA/x)$ is isomorphic to Shukla-homology of A/x over A, $SH_*^A(A/x)$. In this case we obtain

$$\mathsf{THH}_*^{HA}(HA/x) \cong SH_*^A(A/x) \cong \Gamma_{A/x}(\rho^0\varepsilon x)$$

with $|\rho^0\varepsilon x| = 2$. An explicit generator of $SH_{2m}^A(A/x)$ is given by

$$\sum_{i=0}^{m} (-1)^i \tau^{\otimes i} \otimes 1 \otimes \tau^{\otimes m-i}.$$

Here, we consider the resolution of A/x that is of the form $(A[\tau]/\tau^2, d(\tau) = x)$.
Higher order Shukla homology is crucial for determining higher order THH of $\mathbb{Z}/p^m\mathbb{Z}$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$, see [BHLPRZ19].

(2) Recall that the connective covers of the Morava E-theories, e_n, have coefficients

$$\pi_*(e_n) \cong W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u]$$

with $|u| = 2$, where $W\mathbb{F}_{p^n}$ denotes the Witt vectors over $\mathbb{F}_{p^n}$ and where the u_i are generators in degree zero. Hence $\pi_0(e_n) = W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]]$. The quotient $e_n/u = HW\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]]$ is a commutative S-algebra and the map $e_n \to e_n/u$ can be realized as a map of commutative S-algebras.

The residue field $H\mathbb{F}_{p^n}$ is the quotient $e_n/(u, u_1, \dots, u_{n-1}, p)$ and thus the results of Section 7 allow us to calculate $\mathsf{THH}_*^{e_n}(e_n/u, e_n/u)$ and $\mathsf{THH}_*^{e_n,[m]}(e_n/u, H\mathbb{F}_{p^n})$ for all $m \geqslant 1$.

(3) Lawson and Naumann show in [LN12] that $BP\langle 2\rangle$ at the prime two is a commutative S-algebra by identifying it with the 2-localized connective spectrum of topological modular forms together with a level three structure, $\mathsf{tmf}_1(3)_{(2)}$. They prove in [LN14, section 5] that there is a map of commutative S-algebras $\varrho\colon \mathsf{tmf}_1(3)_{(2)} \to ku_{(2)}$ and there is

a complex orientation of $\mathsf{tmf}_1(3)_{(2)}$ such that the effect of ϱ on homotopy groups is as follows [LN14, section 5]:

$$\pi_*(\mathsf{tmf}_1(3)_{(2)}) = \mathbb{Z}_{(2)}[a_1, a_3] \to \mathbb{Z}_{(2)}[u], \quad a_1 \mapsto u, \quad a_3 \mapsto 0.$$

Here the degree of a_i is $2i$.

With Propositions 7.1 and 7.5 we can determine $\mathsf{THH}_*^{\mathsf{tmf}_1(3)_{(2)}}(ku_{(2)})$ additively and we get explicit formulae for higher relative THH of $ku_{(2)} \cong \mathsf{tmf}_1(3)_{(2)}/a_3$ with respect to $\mathsf{tmf}_1(3)_{(2)}$ and with coefficients in $H\mathbb{F}_2 = \mathsf{tmf}_1(3)_{(2)}/(a_3, a_1, 2)$.

(4) The discretization map from ku to $H\mathbb{Z} = ku/u$ gives rise to another example of a regular quotient with a commutative S-algebra structure with residue field $H\mathbb{F}_p = ku/(u,p)$ for any prime p, and so does the map from the connective Adams summand ℓ to $H\mathbb{Z}_{(p)} = \ell/v_1$ with residue field $H\mathbb{F}_p = \ell/(v_1, p)$. Thus in these cases we can determine $\mathsf{THH}^{[n],ku}(H\mathbb{Z}, H\mathbb{Z}/p\mathbb{Z})$, $\mathsf{THH}^{[n],ku}(H\mathbb{Z}/p\mathbb{Z}, H\mathbb{Z}/p\mathbb{Z})$ for all primes and $\mathsf{THH}^{[n],\ell}(H\mathbb{Z}_{(p)}, H\mathbb{Z}/p\mathbb{Z})$, $\mathsf{THH}^{[n],\ell}(H\mathbb{Z}/p\mathbb{Z}, H\mathbb{Z}/p\mathbb{Z})$ for all odd primes and all n.

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