# Exercises in Algebraic Topology (master)

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**Exercise sheet no 1** You don't hand this one in; we'll do the exercises in the tutorials.

**1** (Basics about abelian groups) Let $A$ and $B$ be two abelian groups. We denote by $\mathsf{Hom}(A, B)$ the set of group homomorphisms from $A$ to $B$.

a) Show that $\mathsf{Hom}(A, B)$ is an abelian group.

b) Construct an explicit isomorphism $\varphi\colon \mathsf{Hom}(\mathbb{Z}, A) \cong A$ for all abelian groups $A$.

c) Let $n > 1$ be a natural number. Describe $\mathsf{Hom}(\mathbb{Z}/n\mathbb{Z}, A)$ as a subgroup of $A$. What is $\mathsf{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ or $\mathsf{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q})$?

**2** (Disks and spheres)

Let $n$ be an arbitrary integer and let $\mathbb{D}^n$ be the chain complex whose only non-trivial entries are in degrees $n$ and $n-1$ with $\mathbb{D}^n_n = \mathbb{D}^n_{n-1} = \mathbb{Z}$. Its only non-trivial boundary operator is the identity.

Similarly, let $\mathbb{S}^n$ be the chain complex whose only non-trivial entry is in degree $n$ with $\mathbb{S}^n_n = \mathbb{Z}$.

a) Assume that $(C_*d)$ is an arbitrary chain complex. Describe the abelian group of chain maps from $\mathbb{D}^n$ to $C_*$ and from $\mathbb{S}^n$ to $C_*$ in terms of subobjects of $C_n$.

b) What are the homology groups of $\mathbb{D}^n$ and $\mathbb{S}^n$?

c) Let $f_*\colon C_* \to C'_*$ be a chain map and assume that $f_n$ is a monomorphism for all $n$. Do we then know that $H_n(f_*)$ is also a monomorphism? What about epis and isos?

**3** (Too much to ask for?)

a) What are the homology groups of the chain complex

$$C_* = \ldots \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z}\cdot 2 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \ldots ?$$

b) Is there a chain homotopy from the identity of $C_*$ to the zero map, *i.e.*, can there be maps $s_n\colon C_n \to C_{n+1}$ with $d \circ s + s \circ d = \mathrm{id}_{C_n}$ for all $n \in \mathbb{Z}$?

**4** (Lego)

Let $(A_n)_{n\in\mathbb{Z}}$ be an arbitrary family of finitely generated abelian groups. Is there a chain complex $F_*$ with $F_n$ free abelian for all $n \in \mathbb{Z}$ and with $H_n(F_*) \cong A_n$? (Recall the structure theorem for finitely generated abelian groups for this.)