

Exercises in Algebraic Topology (master)

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Summer term 2024

Exercise sheet no 11

due: 21st of June 2024, 11:45h in H2

1 (Degree again) (1 + 2 + 2 + 2 points)

Let M and N be connected, oriented compact m -manifolds without boundary.

- (1) For $N = \mathbb{S}^m$ consider the degree as a map from the set of homotopy classes $[M, \mathbb{S}^m]$ to \mathbb{Z} . Show that this map is surjective.
- (2) Let $p: M \rightarrow N$ be an n -sheeted covering map. Show that p has degree $\pm n$.
- (3) Assume that $f: M \rightarrow N$ is a map of degree one between connected compact oriented manifolds. Show that $\pi_1(f): \pi_1(M) \rightarrow \pi_1(N)$ is surjective. Does this imply that $H_1(f)$ is surjective?
- (4) Let g_1, g_2 be greater or equal to one. Prove that a degree one map $f: F_{g_1} \rightarrow F_{g_2}$ exists if and only if $g_1 \geq g_2$.

2 (Exactness of direct limits) (2 points)

Prove the remaining two bits that establish that direct limits map short exact sequences of directed systems of R -modules to short exact sequences of R -modules (proof of Lemma 7.3).

3 (Compact support) (1 point)

If X is a path-connected, non-compact space, what is $H_c^0(X)$?

4 (3-manifolds) (2 + 1 + 1 points)

Let M be a compact connected 3-manifold without boundary. You can use that its first homology group is a finitely generated abelian group and is therefore of the form

$$H_1(M) \cong \mathbb{Z}^n \oplus T,$$

where T denotes the finite torsion part of $H_1(M)$.

- a) Determine $H_2(M)$ if M is orientable.
- b) Does $\pi_1(M)$ determine $H_*(M)$ in this case?
- c) What happens if we drop the assumption that M is orientable? Can you still say something about $H_2(M)$?