Exercises in Algebraic Topology (master)

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Exercise sheet no 12 due: 28th of June 2024, 11:45h in H2

1 (Cup product pairing) (2 + 2 points)

a) What are the cup product pairings on $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}P^2$?

b) What can you say about the symmetry of the cup pairing if the dimension of the manifold is $4n$ or $4n+2$?

2 (Transfer or Umkehr maps) (2 + 2 points)

Assume that M is an m-dimensional and N is an n-dimensional topological manifold and both are connected, compact, closed and oriented with fundamental classes $[M]$ and $[N]$. Assume that $f\colon M \to N$ is continuous. Define the transfer or Umkehr maps $f^!\colon H^{m-p}(M) \to H^{n-p}(N)$ and $f_!\colon H_{n-p}(N) \to H_{m-p}(M)$ as

$$f^!(\alpha) = (\mathsf{PD}_N^{-1} \circ H_p(f) \circ \mathsf{PD}_M)(\alpha) \text{ and } f_!(a) = (\mathsf{PD}_M \circ H^p(f) \circ \mathsf{PD}_N^{-1})(a):$$

$$\begin{tikzcd} H^{m-p}(M) \arrow[r, dotted, "f^!"] \arrow[d, "\mathsf{PD}_M"'] & H^{n-p}(N) \arrow[d, "\mathsf{PD}_N"] \\ H_p(M) \arrow[r, "H_p(f)"] & H_p(N) \end{tikzcd} \qquad \begin{tikzcd} H_{n-p}(N) \arrow[r, dotted, "f_!"] & H_{m-p}(M) \\ H^p(N) \arrow[u, "\mathsf{PD}_N"'] \arrow[r, "H^p(f)"] & H^p(M). \arrow[u, "\mathsf{PD}_M"] \end{tikzcd}$$

So these map go 'the wrong way' – and this is why they are sometimes also called *wrong-way maps*.

a) Show that $f^!(f^*(\alpha) \cup \beta) = \alpha \cup f^!(\beta)$ and also $f_!(\alpha \cap a) = f^*(\alpha) \cap f_!(a)$.

b) Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a map of degree $n > 1$. What is the effect of $f^!$ and $f_!$ in degree 1?

3 (Easy applications of duality) (2 + 2 + 2 points)

(1) Prove that for $n \geqslant 1$ every homotopy equivalence $f\colon \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ must be orientation preserving.

(2) Let $n > m \geqslant 1$ and show that every continuous $f\colon \mathbb{R}P^n \to \mathbb{R}P^m$ induces $\pi_1(f) = 0$.

(3) Let $m \geqslant 2$ and let M be a compact, connected, oriented m-manifold without boundary and assume that $f\colon \mathbb{S}^m \to M$ is a continuous map with $\deg(f) \neq 0$. Show that for all $0 \leqslant i \leqslant m$

$$H_i(M; \mathbb{Q}) \cong H_i(\mathbb{S}^m; \mathbb{Q}).$$