Exercises in Algebraic Topology (master)

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Summer term 2024

Exercise sheet no 13 due: 5th of July 2024, 11:45h in H2

1 (Inverse limits) (2 + 2 + 2 points)

a) Consider the short exact sequence of inverse systems

$$0 \to \{p^i\mathbb{Z}\} \to \{\mathbb{Z}\} \to \{\mathbb{Z}/p^i\mathbb{Z}\} \to 0.$$

Determine the inverse limits and the $\lim^1$-terms.

b) Let k be a commutative ring with unit. Show that the inverse limit of the inverse system $\{k[x]/x^n\}_{n\geqslant 1}$ is isomorphic to the formal power series ring $k[[x]]$.

c) Let $\{A_i\}_{i\in\mathbb{N}_0}$ be an inverse system of abelian groups such that the structure maps $A_{i+1} \to A_i$ are monomorphisms. Define a topology on $A = A_0$ by declaring the sets $\{a + A_i\}$ to be open for $a \in A$ and $i \geqslant 0$. (The A_i are viewed as subsets of A via the monomorphisms.) Show that the inverse limit of the A_i is trivial if A is Hausdorff. When does the $\lim^1$-term vanish?

2 (Bockstein homomorphisms) (2 points)

The short exact sequences

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0, \quad 0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

give rise to short exact sequences of cochain complexes

$$0 \to S^*(X;\mathbb{Z}) \to S^*(X;\mathbb{Z}) \to S^*(X;\mathbb{Z}/p\mathbb{Z}) \to 0, \quad 0 \to S^*(X;\mathbb{Z}/p\mathbb{Z}) \to S^*(X;\mathbb{Z}/p^2\mathbb{Z}) \to S^*(X;\mathbb{Z}/p\mathbb{Z}) \to 0$$

and we get corresponding long exact sequences of cohomology groups. Let $\tilde{\beta}\colon H^n(X;\mathbb{Z}/p\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z})$ be the connecting homomorphism for the first sequence, let $\beta\colon H^n(X;\mathbb{Z}/p\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}/p\mathbb{Z})$ be the one for the second sequence and let $\rho_*\colon H^{n+1}(X;\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}/p\mathbb{Z})$ be induced by the reduction mod p. Then β is called the *Bockstein homomorphism* and $\tilde{\beta}$ is the *integral Bockstein*.

Prove that the diagram

$$\begin{array}{ccc} H^n(X;\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\tilde{\beta}} & H^{n+1}(X;\mathbb{Z}) \\ & \searrow^{\beta} & \downarrow{\rho_*} \\ & & H^{n+1}(X;\mathbb{Z}/p\mathbb{Z}) \end{array}$$

commutes.

3 (Odds and ends on ring structures and duality) (2 + 2 + 2 points)

(1) Let T^n be the n-torus, *i.e.*, the n-fold product of $\mathbb{S}^1$. Calculate the cohomology ring $H^*(T^n)$.
(2) Show that there is an additive isomorphism $H^*(\mathbb{S}^2 \times \mathbb{S}^4) \cong H^*(\mathbb{C}P^3)$ but that the corresponding graded cohomology rings are *not* isomorphic.
(3) Let M be a connected compact 3-manifold. Show that $H_1(M)$ cannot be finite if M is non-orientable.