Exercises in Algebraic Topology (master)

Prof. Dr. Birgit Richter

Summer term 2024

Exercise sheet no 2 due: 12th of April 2024, 11:45h in H2

1 (Cones) (2 + 2 points)

Let $f\colon A_* \to B_*$ be a chain map. The *mapping cone* of f, $C(f)$, is a chain complex with $C(f)_n = A_{n-1} \oplus B_n$ and whose differential is $D(a,b) = (-da, db - f(a))$.

a) Prove that this *is* a chain complex.

b) Show that f_* is null-homotopic if and only if f_* extends over $C(\mathrm{id}_{A_*})$.

2 (Exactness) (2 points)

Let C_* be an arbitrary chain complex and let p be a prime. Is it always true that the sequence of chain complexes

$$0 \longrightarrow C_* \xrightarrow{p\cdot} C_* \xrightarrow{\pi} C_*/pC_* \longrightarrow 0$$

is exact? Give a proof or a counterexample.

3 (Induced maps) (2 + 2 points)

a) Let X and Y be topological spaces. Is every chain map $f_*\colon S_*(X) \to S_*(Y)$ induced by a map of topological spaces?

b) Let $p\colon \tilde{X} \to X$ be a covering map. We know that the induced map on fundamental groups is a monomorphism. Is that also true for $H_1(p)$? Here you can use that for path-connected spaces X, $H_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$.

4 (Klein bottle and surfaces) (2 + 2 points)

a) Let F_g denote the closed orientable connected surface of genus g. Use the Seifert van Kampen theorem to determine the fundamental group of F_g.

b) Do the same for the Klein bottle, K.

We'll use these results together with the Hurewicz isomorphism to determine the first homology groups in these cases.