Exercises in Algebraic Topology (master)

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Summer term 2024

Exercise sheet no 7 due: 17th of May 2024, 11:45h in H2

1 (Tensors and Tor) (2 + 2 + 2 points)

a) Is the abelian group $\mathbb{Q}$ free?

b) Let n, m be natural numbers larger than one. What is $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$?

c) Let A be a finitely generated abelian torsion group. What is $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z})$?

2 (Right-exactness) (3 + 1 points)

(1) Show that for every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of abelian groups and any abelian group D, the sequence

$$A \otimes D \xrightarrow{f \otimes \mathrm{id}} B \otimes D \xrightarrow{g \otimes \mathrm{id}} C \otimes D \longrightarrow 0$$

is exact.

(2) Prove that for a split-exact sequence $0 \to A \to B \to C \to 0$, the sequence

$$0 \longrightarrow A \otimes D \xrightarrow{f \otimes \mathrm{id}} B \otimes D \xrightarrow{g \otimes \mathrm{id}} C \otimes D \longrightarrow 0$$

is exact and splits.

3 (How bad can it be?) (1 point)

Give an example of a chain complex $(C_*.d)$ with trivial homology, such that the chain complex $C_* \otimes \mathbb{Z}/2\mathbb{Z}$ has non-vanishing homology in every degree.

4 (Same for R-modules?) (1 + 2 points)

(1) Assume that R is a commutative ring with unit. If you don't know what R-modules are, then look up the definition. Let M and N be two R-modules. Define $M \otimes_R N$ as a quotient of $M \otimes N$.

(2) Can you define Tor for R-modules in the same way as we did for $R = \mathbb{Z}$? What is different? What happens if R is a field?