Exercises in Algebraic Topology (master)

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Summer term 2024

Exercise sheet no 9 due: 7th of June 2024, 11:45h in H2

1 ((Co)homology with coefficients) (2 + 2 points)

(1) Calculate $H^m(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ and $H^m(\mathbb{R}P^n;\mathbb{Z})$ for all $m \geqslant 0$ and $n \geqslant 1$.

(2) Let X be a finite CW complex. Show that $H_n(X;k)$ is a k vector space for every field k. Is $\dim_{\mathbb{Q}} H_n(X;\mathbb{Q}) = \dim_{\mathbb{R}} H_n(X;\mathbb{R})$? What about $\dim_{\mathbb{Q}} H_n(X;\mathbb{Q})$ and $\dim_{\mathbb{F}_p} H_n(X;\mathbb{F}_p)$ for a prime p?

2 (Explicit cap products) (1 + 2 points)

a) Let $\alpha \in H^1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$ and $a \in H_1(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$ be generators. What is $\alpha \cap a$?

b) Take the meridian $b \subset \mathbb{S}^1 \times \mathbb{S}^1 =: T$ and consider the class $\beta \in H^1(T)$ dual to $[b] \in H_1(T)$. We know that $H_2(T) \cong \mathbb{Z}$ and we denote the generator by σ. Show that $\beta \cap \sigma$ can be represented by the longitude $a \subset T$.

3 (Relative variant of the cap-product) (2 points)

Let A and B be subspaces of a topological space X such that the inclusion $S_*^{\mathfrak{U}}(A \cup B) \hookrightarrow S_*(A \cup B)$ induces an isomorphism in homology (with $\mathfrak{U} = \{A, B\}$). Show that there is a variant of the cap-product

$$\cap \colon H^q(X,A) \otimes H_n(X, A \cup B) \to H_{n-q}(X,B).$$

4 (Cap products and de Morgan) (1 + 2 + 2 points)

a) Show the following variant of excision: If (X,A) is a pair of spaces and if $Y \subset X$ with $\mathring{Y} \cup \mathring{A} = X$, then

$$H_*(Y, Y \cap A) \cong H_*(X,A).$$

b) Use this to show the following de Morgan isomorphisms for homology: If X_1, X_2 are open in $X_1 \cup X_2$, then there are isomorphisms

$$j_1 \colon H_*(X_1, X_1 \cap X_2) \cong H_*(X_1 \cup X_2, X_2),$$
$$j_2 \colon H_*(X_2, X_1 \cap X_2) \cong H_*(X_1 \cup X_2, X_1).$$

c) Let now A_1, A_2 be open in $A_1 \cup A_2 \subset X$. Show that the following diagram commutes:

$$\begin{array}{ccccc}
H^q(X,A_2) \otimes H_n(X, A_1 \cup A_2) & & \xrightarrow{\quad\cap\quad} & & H_{n-q}(X,A_1) \\
\downarrow{\scriptstyle (-1)^q i^* \otimes \delta} & & & & \downarrow{\scriptstyle \delta} \\
H^q(A_1, A_1 \cap A_2) \otimes H_{n-1}(A_1 \cup A_2, A_2) & \xrightarrow[\mathrm{id} \otimes j_1^{-1}]{\cong} & H^q(A_1, A_1 \cap A_2) \otimes H_{n-1}(A_1, A_1 \cap A_2) & \xrightarrow{\cap} & H_{n-q-1}(A_1)
\end{array}$$

Here, the δ's are suitable connecting homomorphisms and i is an inclusion.