Exercises in Algebraic Topology (master)

Prof. Dr. Birgit Richter

Summer term 2025

Exercise sheet no 1 You don't hand this one in; we'll talk about this sheet in the tutorials.

1 (Basics about abelian groups) Let A and B be two abelian groups. We denote by $\mathsf{Hom}(A, B)$ the set of group homomorphisms from A to B.

a) Show that $\mathsf{Hom}(A, B)$ is an abelian group.

b) Construct an explicit isomorphism $\varphi\colon \mathsf{Hom}(\mathbb{Z}, A) \cong A$ for all abelian groups A.

c) Let $n > 1$ be a natural number. Describe $\mathsf{Hom}(\mathbb{Z}/n\mathbb{Z}, A)$ as a subgroup of A. What is $\mathsf{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ or $\mathsf{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q})$?

2 (Disks and spheres)

Let n be an arbitrary integer and let $\mathbb{D}^n$ be the chain complex whose only non-trivial entries are in degrees n and $n-1$ with $\mathbb{D}^n_n = \mathbb{D}^n_{n-1} = \mathbb{Z}$. Its only non-trivial boundary operator is the identity.

Similarly, let $\mathbb{S}^n$ be the chain complex whose only non-trivial entry is in degree n with $\mathbb{S}^n_n = \mathbb{Z}$.

a) Assume that $(C_* d)$ is an arbitrary chain complex. Describe the abelian group of chain maps from $\mathbb{D}^n$ to C_*, $\mathsf{Ch}(\mathbb{D}^n, C_*)$, and from $\mathbb{S}^n$ to C_*, $\mathsf{Ch}(\mathbb{S}^n, C_*)$, in terms of subobjects of C_n.

b) What are the homology groups of $\mathbb{D}^n$ and $\mathbb{S}^n$?

c) Prove or disprove the following claim: A chain complex C_* is trivial if and only if $\mathsf{Ch}(\mathbb{D}^n, C_*) = 0$ for all $n \in \mathbb{Z}$.

3 (Right or wrong?)

Let $f_*\colon C_* \to C'_*$ be a chain map and assume that f_n is a monomorphism for all n. Do we then know that $H_n(f_*)$ is also a monomorphism? What about epis and isos?

4 (Too much to ask for?)

a) What are the homology groups of the chain complex

$$C_* = \ldots \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \longrightarrow \ldots ?$$

b) Is there a chain homotopy from the identity of C_* to the zero map, *i.e.*, can there be maps $s_n\colon C_n \to C_{n+1}$ with $d \circ s + s \circ d = \mathrm{id}_{C_n}$ for all $n \in \mathbb{Z}$?

5 (Lego)

Let $(A_n)_{n\in\mathbb{Z}}$ be an arbitrary family of finitely generated abelian groups. Is there a chain complex F_* with F_n free abelian for all $n \in \mathbb{Z}$ and with $H_n(F_*) \cong A_n$? (Recall the structure theorem for finitely generated abelian groups for this.)