Exercises in Algebraic Topology (master)

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Summer term 2025

Exercise sheet no 10 due: 17th of June 2025, 13:45h in H3

1 ((Co)homology with coefficients) (2 + 1 + 1 points)

(1) Calculate $H^m(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ and $H^m(\mathbb{R}P^n; \mathbb{Z})$ for all $m \geqslant 0$ and $n \geqslant 1$.
(2) Let X be an arbitrary topological space. Show that $H_n(X; k)$ is a k-vector space for every field k.
(3) Is $H^n(X; k)$ the dual vector space of $H_n(X; k)$?

2 (Explicit cap products) (2 + 2 points)

(1) Let $\alpha \in H^1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ and $a \in H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ be generators. What is $\alpha \cap a$?
(2) Take the meridian $b \subset \mathbb{S}^1 \times \mathbb{S}^1 =: T$ and consider the class $\beta \in H^1(T)$ dual to $[b] \in H_1(T)$. We know that $H_2(T) \cong \mathbb{Z}$ and we denote the generator by σ. Show that $\beta \cap \sigma$ can be represented by the longitude $a \subset T$.

3 (Variants of the cap-product) (2 + 2 + 2 points)

(1) Let A and B be subspaces of a topological space X such that the inclusion $S_*^{\mathfrak{U}}(A \cup B) \hookrightarrow S_*(A \cup B)$ induces an isomorphism in homology (with $\mathfrak{U} = \{A, B\}$). Show that there is a variant of the cap-product
$$\cap \colon H^q(X, A) \otimes H_n(X, A \cup B) \to H_{n-q}(X, B).$$
(2) Show the following variant of excision: If (X, A) is a pair of spaces and if $Y \subset X$ with $\mathring{Y} \cup \mathring{A} = X$, then
$$H_*(Y, Y \cap A) \cong H_*(X, A).$$
(3) Use (2) to show the following de Morgan isomorphisms for homology: If X_1, X_2 are open in $X_1 \cup X_2$, then there are isomorphisms
$$j_1 \colon H_*(X_1, X_1 \cap X_2) \cong H_*(X_1 \cup X_2, X_2),$$
$$j_2 \colon H_*(X_2, X_1 \cap X_2) \cong H_*(X_1 \cup X_2, X_1).$$