Exercises in Algebraic Topology (master)

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Summer term 2025

Exercise sheet no 12 due: 1st of July 2025, 13:45h in H3

1 (Degree again) (1 + 2 + 2 points)

Let M and N be connected, oriented compact m-manifolds without boundary.

(1) For $N = \mathbb{S}^m$ consider the degree as a map from the set of homotopy classes $[M, \mathbb{S}^m]$ to $\mathbb{Z}$. Show that this map is surjective.

(2) Assume that $f\colon M \to N$ is a map of degree one between connected compact oriented manifolds. Show that $\pi_1(f)\colon \pi_1(M) \to \pi_1(N)$ is surjective. Does this imply that $H_1(f)$ is surjective?

(3) Let g_1, g_2 be greater or equal to one. Prove that a degree one map $f\colon F_{g_1} \to F_{g_2}$ exists if and only if $g_1 \geqslant g_2$.

2 (Exactness of direct limits) (2 points)

Prove the remaining two bits that establish that direct limits map short exact sequences of directed systems of R-modules to short exact sequences of R-modules (proof of Lemma 7.3).

3 (Compact support) (1 + 2 points)

a) If X is a path-connected, non-compact space, what is $H^0_c(X)$?

b) Prove a version of suspension for cohomology with compact support, i. e., show that $H^{n+1}_c(X \times \mathbb{R}) \cong H^n_c(X)$ for all $n \geqslant 1$.

4 (3-manifolds) (2 + 1 + 1 points)

Let M be a compact connected 3-manifold without boundary. You can use that its first homology group is a finitely generated abelian group and is therefore of the form

$$H_1(M) \cong \mathbb{Z}^n \oplus T,$$

where T denotes the finite torsion part of $H_1(M)$.

a) Determine $H_2(M)$ if M is orientable.

b) Does $\pi_1(M)$ determine $H_*(M)$ in this case?

c) What happens if we drop the assumption that M is orientable? Can you still say something about $H_2(M)$?