Exercises in Algebraic Topology (master)

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Exercise sheet no 13 due: 8th of July 2025, 13:45h in H3

1 (Cup product pairing) (2 points)
What are the cup product pairings on $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}P^2$?

2 (Transfer or Umkehr maps) (2 points)
Assume that M is an m-dimensional and N is an n-dimensional topological manifold and both are connected, compact, closed and oriented with fundamental classes $[M]$ and $[N]$. Assume that $f\colon M \to N$ is continuous. Define the transfer or Umkehr maps $f^!\colon H^{m-p}(M) \to H^{n-p}(N)$ and $f_!\colon H_{n-p}(N) \to H_{m-p}(M)$ as

$$f^!(\alpha) = (\mathsf{PD}_N^{-1} \circ H_p(f) \circ \mathsf{PD}_M)(\alpha) \text{ and } f_!(a) = (\mathsf{PD}_M \circ H^p(f) \circ \mathsf{PD}_N^{-1})(a):$$

$$\begin{array}{ccc} H^{m-p}(M) & \overset{f^!}{\dashrightarrow} & H^{n-p}(N) \\ {\scriptstyle \mathsf{PD}_M}\downarrow & & \downarrow{\scriptstyle \mathsf{PD}_N} \\ H_p(M) & \xrightarrow{H_p(f)} & H_p(N) \end{array} \qquad \begin{array}{ccc} H_{n-p}(N) & \overset{f_!}{\dashrightarrow} & H_{m-p}(M) \\ \uparrow{\scriptstyle \mathsf{PD}_N} & & {\scriptstyle \mathsf{PD}_M}\uparrow \\ H^p(N) & \xrightarrow{H^p(f)} & H^p(M). \end{array}$$

Sometimes these maps are also called *wrong-way maps*.

Show that $f^!(f^*(\alpha) \cup \beta) = \alpha \cup f^!(\beta)$ and also $f_!(\alpha \cap a) = f^*(\alpha) \cap f_!(a)$.

3 (Easy applications of duality) (2 + 2 + 2 points)

(1) Prove that for $n \geqslant 1$ every homotopy equivalence $f\colon \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ must be orientation preserving.
(2) Let $n > m \geqslant 1$ and show that every continuous $f\colon \mathbb{R}P^n \to \mathbb{R}P^m$ induces $\pi_1(f) = 0$.
(3) Let $m \geqslant 2$ and let M be a compact, connected, oriented m-manifold without boundary and assume that $f\colon \mathbb{S}^m \to M$ is a continuous map with $\deg(f) \neq 0$. Show that for all $0 \leqslant i \leqslant m$

$$H_i(M; \mathbb{Q}) \cong H_i(\mathbb{S}^m; \mathbb{Q}).$$

4 (Inverse limits) (2 + 2 points)
a) Consider the short exact sequence of inverse systems

$$0 \to \{p^i\mathbb{Z}\} \to \{\mathbb{Z}\} \to \{\mathbb{Z}/p^i\mathbb{Z}\} \to 0.$$

Determine the inverse limits and the $\lim^1$-terms.

b) Let k be a commutative ring with unit. Show that the inverse limit of the inverse system $\{k[x]/x^n\}_{n\geqslant 1}$ is isomorphic to the formal power series ring $k[[x]]$.