Exercises in Algebraic Topology (master)

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Summer term 2025

Exercise sheet no 2 due: 15th of April 2025, 13:45h in H3

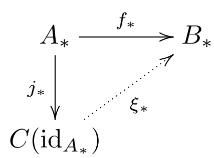
1 (Cones) (2 + 1 + 2 points)

Let $f\colon A_* \to B_*$ be a chain map. The *mapping cone* of f, $C(f)$, is a chain complex with $C(f)_n = A_{n-1} \oplus B_n$ and whose differential is $D(a, b) = (-da, db - f(a))$.

a) Prove that this *is* a chain complex.

b) Show that there are chain maps $j_*\colon B_* \to C(f)$ and $\varrho_*\colon C(f) \to \Sigma A_*$ where ΣA_* is the chain complex with $(\Sigma A_*)_n = A_{n-1}$ and $d^{\Sigma A_*} = -d^{A_*}$.

c) Prove that f_* is null-homotopic if and only if f_* extends over $C(\mathrm{id}_{A_*})$, *i.e.*, if and only if there is a chain map $\xi_*\colon C(\mathrm{id}_{A_*}) \to B_*$ with $\xi_* \circ j_* = f_*$.



2 (Exactness) (2 points)

Let C_* be an arbitrary chain complex and let p be a prime. Is it always true that the sequence of chain complexes

$$0 \longrightarrow C_* \xrightarrow{p\cdot} C_* \xrightarrow{\pi} C_*/pC_* \longrightarrow 0$$

is exact? Give a proof or a counterexample.

3 (Induced maps) (2 points)

Let X and Y be topological spaces. Is every chain map $f_*\colon S_*(X) \to S_*(Y)$ induced by a map of topological spaces?

4 (Klein bottle and surfaces) (3 + 2 points)

a) Let F_g denote the closed orientable connected surface of genus $g \geqslant 0$. Use the Seifert van Kampen theorem to determine the fundamental group of F_g for all g.

b) Do the same for the Klein bottle, K.

We'll use these results together with the Hurewicz isomorphism to determine the first homology groups in these cases.