Exercises in Algebraic Topology (master)

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Summer term 2025

Exercise sheet no 7 due: 20th of May 2025, 13:45h in H3

1 (Euler characteristic) (2 + 2 + 2 + 2 points)

Let X be a finite CW complex. The *Euler characteristic* of X, $\chi(X)$, is then defined as

$$\chi(X) := \sum_{n \geqslant 0} (-1)^n \mathrm{rk}(H_n(X;\mathbb{Z})).$$

Here, rk denotes the rank of a finitely generated abelian group, *i.e.*, the number of its free summands. Then $\chi(X)$ is well-defined.

a) Let $c_n(X)$ denote the number of n-cells of X. Prove that

$$\chi(X) = \sum_{n \geqslant 0} (-1)^n c_n(X).$$

b) What is $\chi(X)$ for a oriented compact closed surface of genus g, F_g for $g \geqslant 0$?

c) What can you say about $\chi(X \cup Y)$ if X and Y are finite CW complexes and if $X \cap Y$ is a subcomplex of X and Y?

d) For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

2 (Moore spaces) (2 + 2 points)

Let G be an arbitrary finitely generated abelian group and assume $n \geqslant 1$.

a) Construct a path connected CW complex $M(G,n)$ whose reduced homology is concentrated in degree n with $\tilde{H}_n(M(G,n)) \cong G$. Such a space is called a *Moore space of type* (G,n).

b) Interpret $\mathbb{R}P^2$ as a Moore space.

3 (Non-orientable surfaces) (2 points)

For $g \geqslant 2$ consider a regular $2g$-gon $P_{2g} \subset \mathbb{R}^2$ with vertices $z_1, \ldots, z_{2g}$. We identify edges according to

$$(1-t)z_{2j-1} + tz_{2j} \sim tz_{2j+1} + (1-t)z_{2j}$$

(here the indices are to be read mod $2g$) and call the quotient $N_g = P_{2g}/\sim$ the *closed non-orientable surface of genus* g. You know N_2.

Calculate the homology of N_g using the cellular chain complex.