Exercises in Algebraic Topology (master)

Prof. Dr. Birgit Richter

Summer term 2025

Exercise sheet no 8 due: 3rd of June 2025, 13:45h in H3

1 (Tensors and Tor) (2 + 2 + 2 points)

a) Is the abelian group $\mathbb{Q}$ free?

b) Let n, m be natural numbers greater than one. What is $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$?

c) Let A be a finitely generated abelian torsion group. What is $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z})$?

2 (Right-exactness) (3 + 1 points)

(1) Show that for every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of abelian groups and any abelian group D, the sequence

$$A \otimes D \xrightarrow{f\otimes \mathrm{id}} B \otimes D \xrightarrow{g\otimes \mathrm{id}} C \otimes D \longrightarrow 0$$

is exact.

(2) Prove that for a split-exact sequence $0 \to A \to B \to C \to 0$, the sequence

$$0 \longrightarrow A \otimes D \xrightarrow{f\otimes \mathrm{id}} B \otimes D \xrightarrow{g\otimes \mathrm{id}} C \otimes D \longrightarrow 0$$

is exact and splits.

3 (How bad can it be?) (1 + 1 points)

(1) Give an example of a chain complex $(C_*.d)$ with trivial homology, such that the chain complex $C_* \otimes \mathbb{Z}/2\mathbb{Z}$ has non-vanishing homology in every degree.

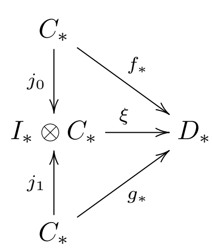
(2) Can you find a chain complex D_* with non-trivial homology such that the homology of $D_* \otimes \mathbb{Z}/2\mathbb{Z}$ is trivial?

4 (Chain homotopies and the interval I_*) (2 points)

Let $f_*, g_* \colon C_* \to D_*$ be two chain maps. Define a suitable differential for a chain complex I_* with

$$I_n = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = 0, \\ \mathbb{Z}, & n = 1, \\ 0, & n > 1 \end{cases}$$

such that chain maps $\xi \colon I_* \otimes C_* \to D_*$ that make the following diagram commute



correspond to chain homotopies between f_* and g_*. Here j_0 embeds C_* into $I_* \otimes C_*$ using the left copy of $\mathbb{Z} \oplus \mathbb{Z} = I_0$ and j_1 uses the right one.