Algebraic Topology, summer term 2025

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Contents

CHAPTER 1

Homology theory

1. Chain complexes

DEFINITION 1.1. A *chain complex* is a sequence of abelian groups, $(C_n)_{n\in\mathbb{Z}}$, together with homomorphisms $d_n\colon C_n \to C_{n-1}$ for $n \in \mathbb{Z}$, such that $d_{n-1} \circ d_n = 0$.

Let R be an associative ring with unit 1_R. A *chain complex of R-modules* can analoguously be defined as a sequence of R-modules $(C_n)_{n\in\mathbb{Z}}$ with R-linear maps $d_n\colon C_n \to C_{n-1}$ with $d_{n-1} \circ d_n = 0$.

DEFINITION 1.2.

- The d_n are *differentials* or *boundary operators*.
- The $x \in C_n$ are called *n-chains*.
- Is $x \in C_n$ and $d_n x = 0$, then x is an *n-cycle*.

$$Z_n(C) := \{x \in C_n | d_n x = 0\}.$$

- If $x \in C_n$ is of the form $x = d_{n+1}y$ for some $y \in C_{n+1}$, then x is an *n-boundary*.

$$B_n(C) := Im(d_{n+1}) = \{d_{n+1}y, y \in C_{n+1}\}.$$

Note that the cycles and boundaries form subgroups of the chains. As $d_n \circ d_{n+1} = 0$, we know that the image of d_{n+1} is a subgroup of the kernel of d_n and thus

$$B_n(C) \subset Z_n(C).$$

We'll often drop the subscript n from the boundary maps and we'll just write C_* for the chain complex.

DEFINITION 1.3. The abelian group $H_n(C) := Z_n(C)/B_n(C)$ is the *nth homology group of the complex* C_*.

Notation: We denote by $[c]$ the equivalence class of a $c \in Z_n(C)$.
If $c, c' \in C_n$ satisfy that $c - c'$ is a boundary, then c is *homologous* to c'. That's an equivalence relation.

Examples:

1) Consider

$$C_n = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

and let d_1 be the multiplication with $N \in \mathbb{N}$, then

$$H_n(C) = \begin{cases} \mathbb{Z}/N\mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2) Take $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ and

$$d_n = \begin{cases} \mathrm{id}_{\mathbb{Z}} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

What is the homology of this chain complex?

3) Consider $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ again, but let all boundary maps be trivial. What is the homology of this chain complex?

Definition 1.4. Let C_* and D_* be two chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence of homomorphisms $f_n\colon C_n \to D_n$ such that $d_n^D \circ f_n = f_{n-1} \circ d_n^C$ for all n, *i.e.*, the diagram

$$\begin{CD} C_n @>{d_n^C}>> C_{n-1} \\ @V{f_n}VV @VV{f_{n-1}}V \\ D_n @>{d_n^D}>> D_{n-1} \end{CD}$$

commutes for all n.

Such an f sends cycles to cycles and boundaries to boundaries. We therefore obtain an induced map

$$H_n(f)\colon H_n(C) \to H_n(D)$$

via $H_n(f)_*[c] = [f_n c]$.

There is a chain map from the chain complex mentioned in Example 1) to the chain complex D_* that is concentrated in degree zero and has $D_0 = \mathbb{Z}/N\mathbb{Z}$. Note, that $H_0(f)$ is an isomorphism on zeroth homology groups.

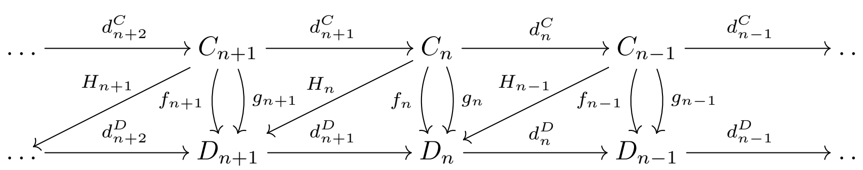
Are there chain maps between the complexes from Examples 2) and 3)?

Lemma 1.5. *If $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ are two chain maps, then $H_n(g) \circ H_n(f) = H_n(g \circ f)$ for all n.*

When do two chain maps induce the same map on homology?

Definition 1.6. A *chain homotopy* H between two chain maps $f, g\colon C_* \to D_*$ is a sequence of homomorphisms $(H_n)_{n\in\mathbb{Z}}$ with $H_n\colon C_n \to D_{n+1}$ such that for all n

$$d_{n+1}^D \circ H_n + H_{n-1} \circ d_n^C = f_n - g_n.$$



If such an H exists, then f and g are *(chain) homotopic*: $f \simeq g$.

We will later see geometrically defined examples of chain homotopies.

Proposition 1.7.
(a) *Being chain homotopic is an equivalence relation.*
(b) *If f and g are homotopic, then $H_n(f) = H_n(g)$ for all n.*

Proof. (a) If H is a homotopy from f to g, then $-H$ is a homotopy from g to f. Each f is homotopic to itself with $H = 0$. If f is homotopic to g via H and g is homotopic to h via K, then f is homotopic to h via $H + K$.

(b) We have for every cycle $c \in Z_n(C_*)$:

$$H_n(f)[c] - H_n(g)[c] = [f_n c - g_n c] = [d_{n+1}^D \circ H_n(c)] + [H_{n-1} \circ d_n^C(c)] = 0.$$

□

Definition 1.8. Let $f\colon C_* \to D_*$ be a chain map. We call f a *chain homotopy equivalence*, if there is a chain map $g\colon D_* \to C_*$ such that $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$. The chain complexes C_* and D_* are then *chain homotopically equivalent*.

Note, that such chain complexes have isomorphic homology. However, chain complexes with isomorphic homology do not have to be chain homotopically equivalent. (Can you find a counterexample?)

Definition 1.9. If C_* and C'_* are chain complexes, then their *direct sum*, $C_* \oplus C'_*$, is the chain complex with

$$(C_* \oplus C'_*)_n = C_n \oplus C'_n = C_n \times C'_n$$

with differential $d = d_\oplus$ given by

$$d_\oplus(c, c') = (dc, dc').$$

Similarly, if $(C_*^{(j)}, d^{(j)})_{j \in J}$ is a family of chain complexes, then we can define their direct sum as follows:

$$(\bigoplus_{j \in J} C_*^{(j)})_n := \bigoplus_{j \in J} C_n^{(j)}$$

as abelian groups and the differential $d_\oplus$ is defined via the property that its restriction to the jth summand is $d^{(j)}$.

2. Singular homology

Let $v_0, \ldots, v_n$ be $n+1$ points in $\mathbb{R}^{n+1}$. Consider the convex hull

$$K(v_0, \ldots, v_n) := \{\sum_{i=0}^n t_i v_i | \sum_{i=0}^n t_i = 1, t_i \geqslant 0\}.$$

Definition 2.1. If the vectors $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent, then $K(v_0, \ldots, v_n)$ is the *simplex generated by* $v_0, \ldots, v_n$. We denote such a simplex by $\mathrm{simp}(v_0, \ldots, v_n)$.

Example. The standard topological n-simplex is $\Delta^n := \mathrm{simp}(e_0, \ldots, e_n)$. Here, e_i is the vector in $\mathbb{R}^{n+1}$ that has a 1 in coordinate $i+1$ and is zero in all other coordinates. The first examples are: Δ^0 is the point e_0, Δ^1 is the line segment between e_0 and e_1, Δ^2 is a triangle in $\mathbb{R}^3$ and Δ^3 is homeomorphic to a tetrahedron.

The coordinate description of the n-simplex is

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum t_i = 1, t_i \geqslant 0\}.$$

We consider Δ^n as $\Delta^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2} \subset \ldots$.

The boundary of Δ^1 consists of two copies of Δ^0, the boundary of Δ^2 consists of three copies of Δ^1. In general, the boundary of Δ^n consists of $n+1$ copies of Δ^{n-1}.

We need the following *face maps* for $0 \leqslant i \leqslant n$

$$d_i = d_i^{n-1} \colon \Delta^{n-1} \hookrightarrow \Delta^n; (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

The image of d_i^{n-1} in Δ^n is the face that is opposite to e_i. It is the simplex generated by $e_0, \ldots, e_{i-1}$, $e_{i+1}, \ldots, e_n$.

Draw the examples of the faces in Δ^1 and Δ^2!

Lemma 2.2. *Concerning the composition of face maps, the following rule holds:*

$$d_i^{n-1} \circ d_j^{n-2} = d_j^{n-1} \circ d_{i-1}^{n-2}, \quad 0 \leqslant j < i \leqslant n.$$

Example: face maps for Δ^0 and composition into Δ^2: $d_2 \circ d_0 = d_0 \circ d_1$.

Proof. Both expressions yield

$$d_i^{n-1} \circ d_j^{n-2}(t_0, \ldots, t_{n-2}) = (t_0, \ldots, t_{j-1}, 0, \ldots, t_{i-2}, 0, \ldots, t_{n-2}) = d_j^{n-1} d_{i-1}^{n-2}(t_0, \ldots, t_{n-2}).$$

□

Let X be an arbitrary topological space, $X \neq \varnothing$.

Definition 2.3. A *singular n-simplex in X* is a continuous map $\alpha \colon \Delta^n \to X$.

Note, that α just has to be continuous, not smooth or anything!

Definition 2.4. Let $S_n(X)$ be the free abelian group generated by all singular n-simplices in X. We call $S_n(X)$ the *nth singular chain module of X*.

Elements of $S_n(X)$ are finite sums $\sum_{i\in I}\lambda_i\alpha_i$ with $\lambda_i = 0$ for almost all $i \in I$ and $\alpha_i \colon \Delta^n \to X$.

For all $n \geqslant 0$ there are non-trivial elements in $S_n(X)$, because we assumed that $X \neq \varnothing$: we can always take an $x_0 \in X$ and the constant map $\kappa_{x_0} \colon \Delta^n \to X$ as α. By convention, we define $S_n(\varnothing) = 0$ for all $n \geqslant 0$.

If we want to define maps from $S_n(X)$ to some abelian group then it suffices to define such a map on generators.

Example. What is $S_0(X)$? A continuous $\alpha\colon \Delta^0 \to X$ is determined by its value $\alpha(e_0) =: x_\alpha \in X$, which is a point in X. A singular 0-simplex $\sum_{i\in I}\lambda_i\alpha_i$ can thus be identified with the formal sum of points $\sum_{i\in I}\lambda_i x_{\alpha_i}$. For instance if you count the zeroes and poles of a meromorphic function with multiplicities then this gives an element in $S_0(X)$. In algebraic geometry a divisor is an element in $S_0(X)$.

Definition 2.5. We define $\partial_i \colon S_n(X) \to S_{n-1}(X)$ on generators

$$\partial_i(\alpha) = \alpha \circ d_i^{n-1}$$

and call it *the ith face of* α.

On $S_n(X)$ we therefore get $\partial_i(\sum_j \lambda_j\alpha_j) = \sum_j \lambda_j(\alpha_j \circ d_i^{n-1})$.

Lemma 2.6. *The face maps on* $S_n(X)$ *satisfy*

$$\partial_j \circ \partial_i = \partial_{i-1} \circ \partial_j, \quad 0 \leqslant j < i \leqslant n.$$

Proof. The proof follows from the one of Lemma 2.2. □

Definition 2.7. We define the boundary operator on singular chains as $\partial\colon S_n(X) \to S_{n-1}(X)$, $\partial = \sum_{i=0}^n (-1)^i\partial_i$.

Lemma 2.8. *The map* ∂ is *a boundary operator,* i.e., $\partial \circ \partial = 0$.

Proof. We calculate

$$\begin{aligned}\partial \circ \partial = (\sum_{j=0}^{n-1}(-1)^j\partial_j) \circ (\sum_{i=0}^{n}(-1)^i\partial_i) &= \sum\sum(-1)^{i+j}\partial_j \circ \partial_i \\ &= \sum_{0\leqslant j<i\leqslant n}(-1)^{i+j}\partial_j \circ \partial_i + \sum_{0\leqslant i\leqslant j\leqslant n-1}(-1)^{i+j}\partial_j \circ \partial_i \\ &= \sum_{0\leqslant j<i\leqslant n}(-1)^{i+j}\partial_{i-1} \circ \partial_j + \sum_{0\leqslant i\leqslant j\leqslant n-1}(-1)^{i+j}\partial_j \circ \partial_i = 0.\end{aligned}$$

□

We therefore obtain the *singular chain complex*, $S_*(X)$,

$$\ldots \longrightarrow S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} \ldots \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \longrightarrow 0.$$

We abbreviate $Z_n(S_*(X))$ by $Z_n(X)$, $B_n(S_*(X))$ by $B_n(X)$ and $H_n(S_*(X))$ by $H_n(X)$.

Definition 2.9. For a space X, $H_n(X)$ is the *nth singular homology group of* X.

Note that $Z_0(X) = S_0(X)$.

As an example of a 1-cycle consider a 1-chain $c = \alpha + \beta + \gamma$ where $\alpha, \beta, \gamma \colon \Delta^1 \to X$ such that $\alpha(e_1) = \beta(e_0)$, $\beta(e_1) = \gamma(e_0)$ and $\gamma(e_1) = \alpha(e_0)$ and calculate that $\partial c = 0$.

We need to understand how continuous maps of topological spaces interact with singular chains and singular homology.

Let $f\colon X \to Y$ be a continuous map.

Definition 2.10. The map $f_n = S_n(f)\colon S_n(X) \to S_n(Y)$ is defined on generators $\alpha\colon \Delta^n \to X$ as

$$f_n(\alpha) = f \circ \alpha : \Delta^n \xrightarrow{\alpha} X \xrightarrow{f} Y.$$

Lemma 2.11. *For any continuous $f\colon X \to Y$ we have*

$$\begin{array}{ccc} S_n(X) & \xrightarrow{f_n} & S_n(Y) \\ \Big\downarrow{\scriptstyle \partial^X} & & \Big\downarrow{\scriptstyle \partial^Y} \\ S_{n-1}(X) & \xrightarrow{f_{n-1}} & S_{n-1}(Y), \end{array}$$

i.e., *$(f_n)_n$ is a chain map and hence induces a map $H_n(f)\colon H_n(X) \to H_n(Y)$.*

Proof. By definition

$$\partial^Y(f_n(\alpha)) = \sum_{i=0}^{n}(-1)^i(f\circ\alpha)\circ d_i = \sum_{i=0}^{n}(-1)^i f\circ(\alpha\circ d_i) = f_{n-1}(\partial^X\alpha).$$

□

Of course, the identity map on X induces the identity map on $H_n(X)$ for all $n \geqslant 0$ and if we have a composition of continuous maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

then $S_n(g\circ f) = S_n(g)\circ S_n(f)$ and $H_n(g\circ f) = H_n(g)\circ H_n(f)$. In categorical language, this says precisely that $S_n(-)$ and $H_n(-)$ are functors from the category of topological spaces and continuous maps into the category of abelian groups. Taking all $S_n(-)$ together turns $S_*(-)$ into a functor from topological spaces and continuous maps into the category of chain complexes with chain maps as morphisms.

One implication of Lemma 2.11 is that homeomorphic spaces have isomorphic homology groups:

$$X \cong Y \Rightarrow H_n(X) \cong H_n(Y) \text{ for all } n \geqslant 0.$$

Our first (not too exciting) calculation is the following:

Proposition 2.12. *The homology groups of a one-point space* pt *are trivial but in degree zero,*

$$H_n(\mathrm{pt}) \cong \begin{cases} 0, & \text{if } n > 0, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. For every $n \geqslant 0$ there is precisely one continuous map $\alpha\colon \Delta^n \to \mathrm{pt}$, namely the constant map. We denote this map by κ_n. Then the boundary of κ_n is

$$\partial\kappa_n = \sum_{i=0}^{n}(-1)^i\kappa_n\circ d_i = \sum_{i=0}^{n}(-1)^i\kappa_{n-1} = \begin{cases} \kappa_{n-1}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

For all n we have $S_n(\mathrm{pt}) \cong \mathbb{Z}$ generated by κ_n and therefore the singular chain complex looks as follows:

$$\ldots \xrightarrow{\partial=0} \mathbb{Z} \xrightarrow{\partial=\mathrm{id}_{\mathbb{Z}}} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z}.$$

□

3. H_0 and H_1

Before we calculate anything, we define a map.

Proposition 3.1. *For any topological space X there is a homomorphism $\varepsilon\colon H_0(X) \to \mathbb{Z}$ with $\varepsilon \neq 0$ for $X \neq \varnothing$.*

Proof. If $X \neq \varnothing$, then we define $\varepsilon(\alpha) = 1$ for any $\alpha\colon \Delta^0 \to X$, thus $\varepsilon(\sum_{i\in I}\lambda_i\alpha_i) = \sum_{i\in I}\lambda_i$ on $S_0(X)$. As only finitely many λ_i are non-trivial, this is in fact a finite sum.

We have to show that this map is well-defined on homology, *i.e.*, that it vanishes on boundaries. One possibility is to see that ε can be interpreted as the map on singular chains that is induced by the projection map of X to a one-point space.

One can also show the claim directly: Let $S_0(X) \ni c = \partial b$ be a boundary and write $b = \sum_{i\in I} \nu_i\beta_i$ with $\beta_i \colon \Delta^1 \to X$. Then we get

$$\partial b = \partial \sum_{i\in I} \nu_i\beta_i = \sum_{i\in I} \nu_i(\beta_i \circ d_0 - \beta_i \circ d_1) = \sum_{i\in I} \nu_i\beta_i \circ d_0 - \sum_{i\in I} \nu_i\beta_i \circ d_1$$

and hence

$$\varepsilon(c) = \varepsilon(\partial b) = \sum_{i\in I} \nu_i - \sum_{i\in I} \nu_i = 0.$$

□

We said that $S_0(\varnothing)$ is zero, so $H_0(\varnothing) = 0$ and in this case we define ε to be the zero map.

If $X \neq \varnothing$, then any $\alpha \colon \Delta^0 \to X$ can be identified with its image point, so the map ε on $S_0(X)$ counts points in X with multiplicities.

Proposition 3.2. *If X is a path-connected, non-empty space, then $\varepsilon \colon H_0(X) \cong \mathbb{Z}$.*

Proof. As X is non-empty, there is a point $x \in X$ and the constant map κ_x with value x is an element in $S_0(X)$ with $\varepsilon(\kappa_x) = 1$. Therefore ε is surjective. For any other point $y \in X$ there is a continuous path $\omega \colon [0,1] \to X$ with $\omega(0) = x$ and $\omega(1) = y$. We define $\alpha_\omega \colon \Delta^1 \to X$ as

$$\alpha_\omega(t_0, t_1) = \omega(1 - t_0).$$

Then

$$\partial(\alpha_\omega) = \partial_0(\alpha_\omega) - \partial_1(\alpha_\omega) = \alpha_\omega(e_1) - \alpha_\omega(e_0) = \alpha_\omega(0,1) - \alpha_\omega(1,0) = \kappa_y - \kappa_x,$$

and the two generators κ_x, κ_y are homologous. This shows that ε is injective. □

From now on we will identify paths w and their associated 1-simplices α_w.

Corollary 3.3. *If X is of the form $X = \bigsqcup_{i\in I} X_i$ such that the X_i are non-empty and path-connected, then*

$$H_0(X) \cong \bigoplus_{i\in I} \mathbb{Z}.$$

In this case, the zeroth homology group of X is the free abelian group generated by the path-components.

Proof. The singular chain complex of X splits as the direct sum of chain complexes of the X_i:

$$S_n(X) \cong \bigoplus_{i\in I} S_n(X_i)$$

for all n. Boundary summands ∂_i stay in a component, in particular,

$$\partial \colon S_1(X) \cong \bigoplus_{i\in I} S_1(X_i) \to \bigoplus_{i\in I} S_0(X_i) \cong S_0(X)$$

is the direct sum of the boundary operators $\partial \colon S_1(X_i) \to S_0(X_i)$ and the claim follows. □

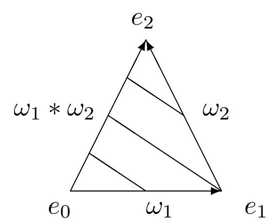
Next, we want to relate H_1 to the fundamental group. Let X be path-connected and $x \in X$.

Lemma 3.4. *Let $\omega_1, \omega_2, \omega$ be paths in X.*

(a) *Constant paths are null-homologous.*
(b) *If $\omega_1(1) = \omega_2(0)$, then $\omega_1 * \omega_2 - \omega_1 - \omega_2$ is a boundary. Here $\omega_1 * \omega_2$ is the concatenation of ω_1 followed by ω_2.*
(c) *If $\omega_1(0) = \omega_2(0), \omega_1(1) = \omega_2(1)$ and if ω_1 is homotopic to ω_2 relative to $\{0,1\}$, then ω_1 and ω_2 are homologous as singular 1-chains.*
(d) *Any 1-chain of the form $\bar{\omega} * \omega$ is a boundary. Here, $\bar{\omega}(t) := \omega(1-t)$.*

Proof. For a), consider the constant singular 2-simplex $\alpha(t_0, t_1, t_2) = x$ and c_x, the constant path on x. Then $\partial\alpha = c_x - c_x + c_x = c_x$.

For b), we define a singular 2-simplex $\beta\colon \Delta^2 \to X$ as follows.



We define β on the boundary components of Δ^2 as indicated and prolong it constantly along the sloped inner lines. Then

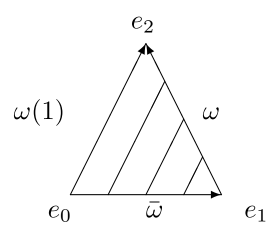
$$\partial\beta = \beta \circ d_0 - \beta \circ d_1 + \beta \circ d_2 = \omega_2 - \omega_1 * \omega_2 + \omega_1.$$

For c): Let $H\colon [0,1] \times [0,1] \to X$ a homotopy from ω_1 to ω_2. As we have that $H(0,t) = \omega_1(0) = \omega_2(0)$, we can factor H through the quotient $[0,1] \times [0,1]/\{0\} \times [0,1] \cong \Delta^2$ with induced map $h\colon \Delta^2 \to X$. Then

$$\partial h = h \circ d_0 - h \circ d_1 + h \circ d_2.$$

The first summand is null-homologous, because it's constant (with value $\omega_1(1) = \omega_2(1)$), the second one is ω_2 and the last is ω_1, thus $\omega_1 - \omega_2$ is null-homologous.

For d): Consider $\gamma\colon \Delta^2 \to X$ as indicated below.



□

Definition 3.5. Let $h\colon \pi_1(X,x) \to H_1(X)$ be the map, that sends the homotopy class of a closed path ω, $[\omega]_{\pi_1}$, to its homology class $[\omega] = [\omega]_{H_1}$. This map is called the *Hurewicz-homomorphism*.

Witold Hurewicz: 1904–1956 `https://en.wikipedia.org/wiki/Witold_Hurewicz` (Mayan pyramids are dangerous, at least for mathematicians.)

Lemma 3.4 ensures that h is well-defined and

$$h([\omega_1][\omega_2]) = h([\omega_1 * \omega_2]) = [\omega_1] + [\omega_2] = h([\omega_1]) + h([\omega_2]);$$

thus h is a homomorphism.

Note that for a closed path ω we have that $[\bar{\omega}] = -[\omega]$ in $H_1(X)$.

Definition 3.6. Let G be an arbitrary group, then its *abelianization*, G_{ab}, is $G/[G,G]$.

Recall that $[G,G]$ is the commutator subgroup of G. That is the smallest subgroup of G containing all commutators $ghg^{-1}h^{-1}, g, h \in G$. It is a normal subgroup of G: If $c \in [G,G]$, then for any $g \in G$ the element $gcg^{-1}c^{-1}$ is a commutator and also by the closure property of subgroups the element $gcg^{-1}c^{-1}c = gcg^{-1}$ is in the commutator subgroup.

Proposition 3.7. *The Hurewicz homomorphism factors through the abelianization of $\pi_1(X,x)$ and induces an isomorphism*

$$\pi_1(X,x)_{\mathrm{ab}} \cong H_1(X)$$

for all path-connected X.

$$\begin{array}{ccc} \pi_1(X,x) & \xrightarrow{\quad h \quad} & H_1(X) \\ \downarrow p & \overset{\cong}{\underset{h_{\mathrm{ab}}}{\nearrow}} & \\ \pi_1(X,x)_{\mathrm{ab}} = \pi_1(X,x)/[\pi_1(X,x),\pi_1(X,x)] & & \end{array}$$

Proof. We will construct an inverse to h_{ab}. For any $y \in X$ we choose a path u_y from x to y. For $y = x$ we take u_x to be the constant path on x. Let α be an arbitrary singular 1-simplex and $y_i = \alpha(e_i)$. Define $\phi\colon S_1(X) \to \pi_1(X,x)_{\mathrm{ab}}$ on generators as $\phi(\alpha) = [u_{y_0} * \alpha * \bar{u}_{y_1}]$ and extend ϕ linearly to all of $S_1(X)$, keeping in mind that the composition in π_1 is written multiplicatively.

We have to show that ϕ is trivial on boundaries, so let $\beta\colon \Delta^2 \to X$. Then

$$\phi(\partial\beta) = \phi(\beta \circ d_0 - \beta \circ d_1 + \beta \circ d_2) = \phi(\beta \circ d_0)\phi(\beta \circ d_1)^{-1}\phi(\beta \circ d_2).$$

Abbreviating $\beta \circ d_i$ with α_i we get as a result

$$[u_{y_1} * \alpha_0 * \bar{u}_{y_2}][u_{y_0} * \alpha_1 * \bar{u}_{y_2}]^{-1}[u_{y_0} * \alpha_2 * \bar{u}_{y_1}] = [u_{y_0} * \alpha_2 * \bar{u}_{y_1} * u_{y_1} * \alpha_0 * \bar{u}_{y_2} * u_{y_2} * \bar{\alpha_1} * \bar{u}_{y_0}].$$

Here, we've used that the image of ϕ is abelian. We can reduce $\bar{u}_{y_1} * u_{y_1}$ and $\bar{u}_{y_2} * u_{y_2}$ and are left with $[u_{y_0} * \alpha_2 * \alpha_0 * \bar{\alpha_1} * \bar{u}_{y_0}]$ but $\alpha_2 * \alpha_0 * \bar{\alpha_1}$ is the closed path tracing the boundary of β and therefore it is null-homotopic in X. Thus $\phi(\partial\beta) = 0$ and ϕ passes to a map

$$\phi\colon H_1(X) \to \pi_1(X,x)_{\mathrm{ab}}.$$

The composition $\phi \circ h_{\mathrm{ab}}$ evaluated on the class of a closed path ω gives

$$\phi \circ h_{\mathrm{ab}}[\omega]_{\pi_1} = \phi[\omega]_{H_1} = [u_x * \omega * \bar{u}_x]_{\pi_1}.$$

But we chose u_x to be constant, thus $\phi \circ h_{\mathrm{ab}} = \mathrm{id}$.

If $c = \sum \lambda_i \alpha_i$ is a cycle, then $h_{\mathrm{ab}} \circ \phi(c)$ is of the form $[c + D_{\partial c}]$ where the $D_{\partial c}$-part comes from the contributions of the u_{y_i}. The fact that $\partial(c) = 0$ implies that the summands in $D_{\partial c}$ cancel off and thus $h_{\mathrm{ab}} \circ \phi = \mathrm{id}_{H_1(X)}$. □

Note, that abelianization doesn't change anything for abelian groups, *i.e.*, whenever we have an abelian fundamental group, we know that $H_1(X) \cong \pi_1(X,x)$.

Corollary 3.8. *Knowledge of π_1 gives*

$$\begin{aligned} H_1(\mathbb{S}^n) &= 0, \ \textit{for } n > 1, \\ H_1(\mathbb{S}^1) &\cong \mathbb{Z}, \\ H_1(\underbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}_{n}) &\cong \mathbb{Z}^n, \\ H_1(\mathbb{S}^1 \vee \mathbb{S}^1) &\cong (\mathbb{Z} * \mathbb{Z})_{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}, \\ H_1(\mathbb{R}P^n) &\cong \begin{cases} \mathbb{Z}, & n = 1, \\ \mathbb{Z}/2\mathbb{Z}, & n > 1, \end{cases} \\ H_1(F_g) &\cong \mathbb{Z}^{2g}, \ \textit{for } g \geqslant 1, \\ H_1(K) &\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

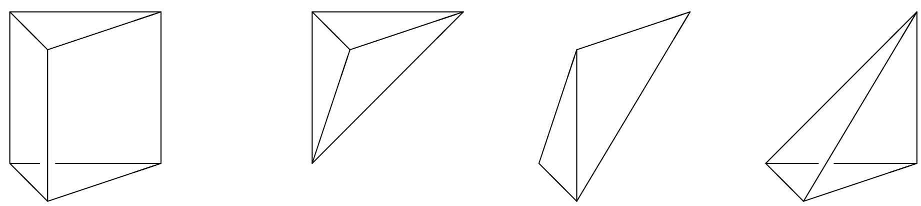
In the last case, K denotes the Klein bottle.

4. Homotopy invariance

We want to show that two continuous maps that are homotopic induce identical maps on the level of homology groups.

Heuristics: If $\alpha\colon \Delta^n \to X$ is a singular n-simplex and if f, g are homotopic maps from X to Y, then the homotopy from $f \circ \alpha$ to $g \circ \alpha$ starts on $\Delta^n \times [0,1]$. We want to translate this geometric homotopy into a chain homotopy on the singular chain complex. To that end we have to cut the prism $\Delta^n \times [0,1]$ into $(n+1)$-simplices. In low dimensions this is easy:

$\Delta^0 \times [0,1]$ is homeomorphic to Δ^1, $\Delta^1 \times [0,1] \cong [0,1]^2$ and this can be cut into two copies of Δ^2 and $\Delta^2 \times [0,1]$ is a 3-dimensional prism and that can be glued together from three tetrahedrons, *e.g.*, like



As you might guess now, we use $n+1$ copies of Δ^{n+1} to build $\Delta^n \times [0,1]$.

Definition 4.1. For $i = 0, \ldots, n$ define $p_i\colon \Delta^{n+1} \to \Delta^n \times [0,1]$ as

$$p_i(t_0, \ldots, t_{n+1}) = ((t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}), t_{i+1} + \ldots + t_{n+1}) \in \Delta^n \times [0,1].$$

On the standard basis vectors e_k we obtain

$$p_i(e_k) = \begin{cases} (e_k, 0), & \text{for } 0 \leqslant k \leqslant i, \\ (e_{k-1}, 1), & \text{for } k > i. \end{cases}$$

We obtain maps $P_i\colon S_n(X) \to S_{n+1}(X \times [0,1])$ via $P_i(\alpha) = (\alpha \times \mathrm{id}) \circ p_i$:

$$\Delta^{n+1} \xrightarrow{p_i} \Delta^n \times [0,1] \xrightarrow{\alpha \times \mathrm{id}} X \times [0,1].$$

For $k = 0, 1$ let $j_k\colon X \to X \times [0,1]$ be the inclusion $x \mapsto (x, k)$.

Lemma 4.2. *The maps P_i satisfy the following relations*

(a) $\partial_0 \circ P_0 = S_n(j_1)$,
(b) $\partial_{n+1} \circ P_n = S_n(j_0)$,
(c) $\partial_i \circ P_i = \partial_i \circ P_{i-1}$ *for* $1 \leqslant i \leqslant n$.
(d)

$$\partial_j \circ P_i = \begin{cases} P_i \circ \partial_{j-1}, & \textit{for } i \leqslant j-2 \\ P_{i-1} \circ \partial_j, & \textit{for } i \geqslant j+1. \end{cases}$$

Proof. Note that it suffices to check the corresponding claims for the p_i's and d_j's.

For the first two points, we note that on Δ^n we have

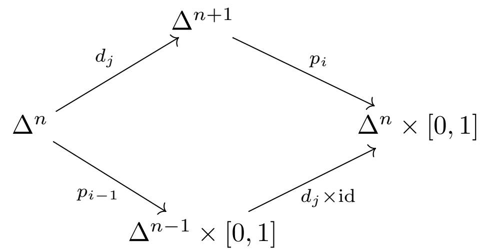
$$p_0 \circ d_0(t_0, \ldots, t_n) = p_0(0, t_0, \ldots, t_n) = ((t_0, \ldots, t_n), \sum t_i) = ((t_0, \ldots, t_n), 1) = j_1(t_0, \ldots, t_n)$$

and

$$p_n \circ d_{n+1}(t_0, \ldots, t_n) = p_n(t_0, \ldots, t_n, 0) = ((t_0, \ldots, t_n), 0) = j_0(t_0, \ldots, t_n).$$

For c), one checks that $p_i \circ d_i = p_{i-1} \circ d_i$ on Δ^n: both give $((t_0, \ldots, t_n), \sum_{j=i}^n t_j)$ on $(t_0, \ldots, t_n)$.

For d) in the case $i \geqslant j+1$, consider the following diagram



Checking coordinates one sees that this diagram commutes. The remaining case follows from a similar observation. □

Definition 4.3. We define $P\colon S_n(X) \to S_{n+1}(X \times [0,1])$ as $P = \sum_{i=0}^{n} (-1)^i P_i$.

Lemma 4.4. *The map P is a chain homotopy between $(S_n(j_0))_n$ and $(S_n(j_1))_n$*, i.e., $\partial \circ P + P \circ \partial = S_n(j_1) - S_n(j_0)$.

Proof. We take an $\alpha\colon \Delta^n \to X$ and calculate

$$\partial P\alpha + P\partial\alpha = \sum_{i=0}^{n}\sum_{j=0}^{n+1}(-1)^{i+j}\partial_j P_i \alpha + \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j} P_i \partial_j \alpha.$$

If we single out the terms involving the pairs of indices $(0,0)$ and $(n,n+1)$ in the first sum, we are left with

$$S_n(j_1)(\alpha) - S_n(j_0)(\alpha) + \sum_{(i,j)\neq(0,0),(n,n+1)} (-1)^{i+j}\partial_j P_i \alpha + \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j} P_i \partial_j \alpha.$$

Using Lemma 4.2 we see that only the first two summands survive. □

So, finally we can prove the main result of this section:

Theorem 4.5. *(Homotopy invariance)*
If $f, g\colon X \to Y$ are homotopic maps, then they induce the same map on homology.

Proof. Let $H\colon X \times [0,1] \to Y$ be a homotopy from f to g, *i.e.*, $H \circ j_0 = f$ and $H \circ j_1 = g$. Set $K_n := S_{n+1}(H) \circ P$. We claim that $(K_n)_n$ is a chain homotopy between $(S_n(f))_n$ and $(S_n(g))_n$. Note that H induces a chain map $(S_n(H))_n$. Therefore we get

$$\begin{aligned}\partial \circ S_{n+1}(H) \circ P + S_n(H) \circ P \circ \partial &= S_n(H) \circ \partial \circ P + S_n(H) \circ P \circ \partial \\ &= S_n(H) \circ (\partial \circ P + P \circ \partial) \\ &= S_n(H) \circ (S_n(j_1) - S_n(j_0)) = S_n(H \circ j_1) - S_n(H \circ j_0) \\ &= S_n(g) - S_n(f).\end{aligned}$$

Hence these two maps are chain homotopic and $H_n(g) = H_n(f)$ for all n. □

Corollary 4.6. *If two spaces X, Y are homotopy equivalent, then $H_*(X) \cong H_*(Y)$. In particular, if X is contractible, then*

$$H_*(X) \cong \begin{cases} \mathbb{Z}, & \text{for } * = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Examples. As $\mathbb{R}^n$ is contractible for all n, the above corollary gives that its homology is trivial but in degree zero where it consists of the integers.

As the Möbius strip is homotopy equivalent to $\mathbb{S}^1$, we know that their homology groups are isomorphic.

If you know about vector bundles: the zero section of a vector bundle induces a homotopy equivalence between the base and the total space, hence these two have isomorphic homology groups.

5. The long exact sequence in homology

A typical situation is that there is a subspace A of a topological space X and you might know something about A or X and want to calculate the homology of the other space using that partial information.

But before we can move on to topological applications we need some techniques about chain complexes. We need to know that a short exact sequence of chain complexes gives rise to a long exact sequence in homology.

Definition 5.1. Let A, B, C be abelian groups and

$$A \xrightarrow{f} B \xrightarrow{g} C$$

a sequence of homomorphisms. Then this sequence is *exact*, if the image of f is the kernel of g.

Definition 5.2. If

$$\ldots \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \xrightarrow{f_{i-1}} \ldots$$

is a sequence of homomorphisms of abelian groups (indexed over the integers), then this sequence is called *(long) exact*, if it is exact at every A_i, *i.e.*, the image of f_{i+1} is the kernel of f_i for all i.

An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence.*

Examples. The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is a short exact sequence.

If $\iota\colon U \to A$ is a monomorphism, then $0 \longrightarrow U \xrightarrow{\iota} A$ is exact. Similarly, an epimorphism $\varrho\colon B \to Q$ gives rise to an exact sequence $B \xrightarrow{\varrho} Q \longrightarrow 0$ and an isomorphism $\phi\colon A \cong A'$ sits in an exact sequence $0 \longrightarrow A \xrightarrow{\phi} A' \longrightarrow 0$.

A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact iff f is injective, the image of f is the kernel of g and g is an epimorphism. Another equivalent description is to view a sequence as above as a chain complex with vanishing homology groups. Homology measures the deviation from exactness.

Definition 5.3. If A_*, B_*, C_* are chain complexes and $f_*\colon A_* \to B_*$, $g\colon B_* \to C_*$ are chain maps, then we call the sequence

$$A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_*$$

exact, if the image of f_n is the kernel of g_n for all $n \in \mathbb{Z}$.

Thus such an exact sequence of chain complexes is a commuting double ladder

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n+1} \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
\vdots & & \vdots & & \vdots
\end{array}$$

in which every row is exact.

Example. Let p be a prime, then

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p^2} & & \downarrow \\
\mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathbb{Z}/p\mathbb{Z} & \xrightarrow{p} & \mathbb{Z}/p^2\mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

has exact rows and columns, in particular it is an exact sequence of chain complexes. Here, π denotes varying canonical projection maps.

Proposition 5.4. *If* $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ *is a short exact sequence of chain complexes, then there exists a homomorphism* $\delta\colon H_n(C_*) \to H_{n-1}(A_*)$ *for all* $n \in \mathbb{Z}$ *which is natural,* i.e., *if*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{f} & B_* & \xrightarrow{g} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{f'} & B'_* & \xrightarrow{g'} & C'_* & \longrightarrow & 0
\end{array}$$

is a commutative diagram of chain maps in which the rows are exact then $H_{n-1}(\alpha) \circ \delta = \delta \circ H_n(\gamma)$,

$$\begin{array}{ccc}
H_n(C_*) & \xrightarrow{\delta} & H_{n-1}(A_*) \\
{\scriptstyle H_n(\gamma)}\downarrow & & \downarrow{\scriptstyle H_{n-1}(\alpha)} \\
H_n(C'_*) & \xrightarrow{\delta} & H_{n-1}(A'_*)
\end{array}$$

The method of proof is an instance of a *diagram chase.* The homomorphism δ is called *connecting homomorphism.* The implicit claim in the proposition above is that δ is not always the zero map.

PROOF. We show the existence of a δ first and then prove that the constructed map satisfies the naturality condition.

a) Definition of δ:

Is $c \in C_n$ with $d(c) = 0$, then we choose a $b \in B_n$ with $g_n b = c$. This is possible because g_n is surjective. We know that $dg_n b = dc = 0 = g_{n-1} db$ thus db is in the kernel of g_{n-1}, hence it is in the image of f_{n-1}. Thus there is an $a \in A_{n-1}$ with $f_{n-1}a = db$. We have that $f_{n-2}da = df_{n-1}a = ddb = 0$ and as f_{n-2} is injective, this shows that a is a cycle.

We define $\delta[c] := [a]$.

$$B_n \ni b \stackrel{g_n}{\longmapsto} c \in C_n$$

$$A_{n-1} \ni a \stackrel{f_{n-1}}{\longmapsto} db \in B_{n-1}$$

In order to check that δ is well-defined, we assume that there are b and b' with $g_n b = g_n b' = c$. Then $g_n(b - b') = 0$ and thus there is an $\tilde{a} \in A_n$ with $f_n \tilde{a} = b - b'$. Define a' as $a - d\tilde{a}$. Then

$$f_{n-1}a' = f_{n-1}a - f_{n-1}d\tilde{a} = db - db + db' = db'$$

because $f_{n-1}d\tilde{a} = db - db'$. As f_{n-1} is injective, we get that a' is uniquely determined with this property. As a is homologous to a' we get that $[a] = [a'] = \delta[c]$, thus the latter is independent of the choice of b.

In addition, we have to make sure that the value stays the same if we add a boundary term to c, *i.e.*, take $c' = c + d\tilde{c}$ for some $\tilde{c} \in C_{n+1}$. Choose preimages of $c, \tilde{c}$ under g_n and g_{n+1}, *i.e.*, b and $\tilde{b}$ with $g_n b = c$ and $g_{n+1}\tilde{b} = \tilde{c}$. Then the element $b' = b + d\tilde{b}$ has boundary $db' = db$ and thus both choices will result in the same a.

Therefore $\delta\colon H_n(C_*) \to H_{n-1}(A_*)$ is well-defined.

b) We have to show that δ is natural with respect to maps of short exact sequences.

Let $c \in Z_n(C_*)$, then $\delta[c] = [a]$ for a $b \in B_n$ with $g_n b = c$ and an $a \in A_{n-1}$ with $f_{n-1}a = db$. Therefore, $H_{n-1}(\alpha)(\delta[c]) = [\alpha_{n-1}(a)]$.

On the other hand, we have

$$f'_{n-1}(\alpha_{n-1}a) = \beta_{n-1}(f_{n-1}a) = \beta_{n-1}(db) = d\beta_n b$$

and

$$g'_n(\beta_n b) = \gamma_n g_n b = \gamma_n c$$

and we can conclude that by the construction of δ

$$\delta[\gamma_n(c)] = [\alpha_{n-1}(a)]$$

and this shows $\delta \circ H_n(\gamma) = H_{n-1}(\alpha) \circ \delta$. □

With this auxiliary result at hand we can now prove the main result in this section:

PROPOSITION 5.5. *For any short exact sequence*

$$0 \longrightarrow A_* \stackrel{f}{\longrightarrow} B_* \stackrel{g}{\longrightarrow} C_* \longrightarrow 0$$

of chain complexes we obtain a long exact sequence of homology groups

$$\ldots \stackrel{\delta}{\longrightarrow} H_n(A_*) \stackrel{H_n(f)}{\longrightarrow} H_n(B_*) \stackrel{H_n(g)}{\longrightarrow} H_n(C_*) \stackrel{\delta}{\longrightarrow} H_{n-1}(A_*) \stackrel{H_{n-1}(f)}{\longrightarrow} \ldots$$

PROOF. a) Exactness at the spot $H_n(B_*)$:

We have $H_n(g) \circ H_n(f)[a] = [g_n(f_n(a))] = 0$ because the composition of g_n and f_n is zero. This proves that the image of $H_n(f)$ is contained in the kernel of $H_n(g)$.

For the converse, let $[b] \in H_n(B_*)$ with $[g_n b] = 0$. Then there is a $c \in C_{n+1}$ with $dc = g_n b$. As g_{n+1} is surjective, we find a $b' \in B_{n+1}$ with $g_{n+1}b' = c$. Hence

$$g_n(b - db') = g_n b - dg_{n+1}b' = dc - dc = 0.$$

Exactness gives an $a \in A_n$ with $f_n a = b - db'$ and $da = 0$ and therefore $f_n a$ is homologous to b and $H_n(f)[a] = [b]$ thus the kernel of $H_n(g)$ is contained in the image of $H_n(f)$.

b) Exactness at the spot $H_n(C_*)$:

Let $b \in H_n(B_*)$, then $\delta[g_n b] = 0$ because b is a cycle, so 0 is the only preimage under f_{n-1} of $db = 0$. Therefore the image of $H_n(g)$ is contained in the kernel of δ.

Now assume that $\delta[c] = 0$, thus in the construction of δ, the a is a boundary, $a = da'$. Then for a preimage of c under g_n, b, we have by the definition of a

$$d(b - f_n a') = db - df_n a' = db - f_{n-1} a = 0.$$

Thus $b - f_n a'$ is a cycle and $g_n(b - f_n a') = g_n b - g_n f_n a' = g_n b - 0 = g_n b = c$, so we found a preimage for $[c]$ and the kernel of δ is contained in the image of $H_n(g)$.

c) Exactness at $H_{n-1}(A_*)$:

Let c be a cycle in $Z_n(C_*)$. Again, we choose a preimage b of c under g_n and an a with $f_{n-1}(a) = db$. Then $H_{n-1}(f)\delta[c] = [f_{n-1}(a)] = [db] = 0$. Thus the image of δ is contained in the kernel of $H_{n-1}(f)$.

If $a \in Z_{n-1}(A_*)$ with $H_{n-1}(f)[a] = 0$. Then $f_{n-1} a = db$ for some $b \in B_n$. Take $c = g_n b$. Then by definition $\delta[c] = [a]$. □