RELATIVE LODAY CONSTRUCTIONS AND APPLICATIONS TO HIGHER THH-CALCULATIONS

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ABSTRACT. We define a relative version of the Loday construction for a sequence of commutative S-algebras $A \to B \to C$ and a pointed simplicial subset $Y \subset X$. We use this to construct several spectral sequences for the calculation of higher topological Hochschild homology and apply those for calculations in some examples that could not be treated before.

1. INTRODUCTION

Considering relative versions of (co)homology theories is crucial for obtaining calculational and structural results. We work in the setting of commutative S-algebras (see [EKMM]) and these can be tensored with simplicial sets or topological spaces. Important invariants of a commutative S-algebra A are the homotopy groups of $A \otimes X$ for suitable spaces or simplicial sets X. We call this the Loday construction of A with respect to X and denote it by $\mathcal{L}_X(A)$.

An important special case is the higher order topological Hochschild homology of A,

$$\mathsf{THH}^{[n]}_*(A) = \pi_*(A \otimes S^n),$$

where S^n is the n-sphere. For $n = 1$ this reduces to ordinary topological Hochschild homology of A, $\mathsf{THH}(A)$, which receives a trace map from the algebraic K-theory of A, $K(A)$, and can be used via the construction of topological cyclic homology to obtain an approximation of $K(A)$. Another important case is torus homology of A, $\pi_*(A \otimes T^n)$ for the n-torus T^n, which receives an n-fold iterated trace map from the iterated algebraic K-theory of A. The hope is that the homotopy fixed points with respect to the torus action of torus homology will reveal so-called red-shift phenomena [AuR]. Positive results in that direction were obtained by Rognes and Veen [V], and in ongoing work Ausoni and Dundas [AD] extend these results to all chromatic layers.

One strength of the construction of $A \otimes X$ is that it is functorial in both X and A, which allows us to study the homotopy type of $A \otimes X$ by iteratively constructing X out of smaller spaces. This iterative method is for instance heavily used in Veen's work [V] and in [DLR], [BLPRZ]. As spheres are the building blocks of CW complexes, the calculation of $\mathsf{THH}^{[n]}_*(A)$ is crucial for understanding $\pi_*(\mathcal{L}_X(A))$ for CW-complexes X. The aim of this paper is to develop new tools for the calculation of higher order topological Hochschild homology by using the extra flexibility that is gained by a relative approach.

For a sequence of morphisms in the category of commutative S-algebras $A \to B \to C$ and for a pair of pointed simplicial sets (X, Y) with Y a subcomplex of X, we consider a relative version of the Loday construction, $\mathcal{L}_{(X,Y)}(A, B; C)$: This relative version places C over the basepoint, B over all points of Y that are not the basepoint and A over the complement of Y in X.

We define this relative Loday construction and show some of its properties in Section 2. Section 3 exploits this relative structure and other geometric observations to establish several

weak equivalences (juggling formulae) which relate higher THH groups with respect to the sphere spectrum as the ground ring and those with respect to other commutative S-algebra spectra as ground rings, and also relate higher THH-groups THH$^{[m]}$ for $m = n$ and $n - 1$. We use these juggling formulae to construct several spectral sequences for the calculation of higher topological Hochschild homology and apply these spectral sequences to obtain calculations for (higher) topological Hochschild homology that were not known before. In Section 4 we determine higher order relative THH of certain Thom spectra and the higher order Shukla homology of $\mathbb{F}_p$ with respect to pointed commutative $\mathbb{F}_p$-monoid algebras. We gain additive results about THH$^E(H\mathbb{F}_p)$ and THH$^{[2]}(E; H\mathbb{F}_p)$ for $E = ko, ku, tmf$ at $p = 2$ and $E = \ell$ the Adams summand for p an odd prime. Furthermore, we show a splitting result for higher THH of the form THH$^{[n],Hk}(HA)$, where k is any commutative ring and A is any commutative k-algebra.

In the following we work in the setting of [EKMM] and we use the model structure on commutative S-algebras from [EKMM, chapter VII]. Let A be a commutative S-algebra. As the category of commutative A-algebras is equivalent to the category of commutative S-algebras under A, we obtain a model category structure on the category of commutative A-algebras. In particular, a commutative A-algebra B is cofibrant if its unit map $A \to B$ is a cofibration of commutative S-algebras.

2. The relative Loday construction

Higher topological Hochschild homology of a commutative ring spectrum A is a special case of the Loday construction, or the internal tensor product, which sends A and a pointed simplicial set X to a commutative simplicial ring spectrum $\mathcal{L}_X(A) = A \otimes X$, which is a commutative augmented A-algebra. This is a ring spectrum version of the Loday construction defined by Pirashvili in [P] for commutative rings, which sends a commutative ring R and a pointed simplicial set X to the commutative augmented simplicial R-algebra $R \otimes X$.

For a cofibrant commutative S-algebra A this construction is homotopy invariant as a functor of X, that is: if one works with homotopy equivalent simplicial sets, we get homotopy equivalent augmented simplicial commutative A-algebras; in particular, this is true if one works with two simplicial models for the same space.

Let X be a pointed simplicial set. Since all boundary maps in a pointed simplicial set send the basepoint to the basepoint, given an A-module C (and in particular, a commutative A-algebra C) we can also study Loday constructions with coefficients $\mathcal{L}_X(A; C)$ which replaces the copy of A over the base point by a copy of C. We now define a relative version of this.

Definition 2.1. Let A be a commutative S-algebra, B a commutative A-algebra, and C a commutative B-algebra, with maps $A \xrightarrow{f} B \xrightarrow{g} C$. Let X be a pointed simplicial set and Y be a pointed simplicial subset. Then we can define

$$(\mathcal{L}_{(X,Y)}(A, B; C))_n := \left(\bigwedge_{X_n \setminus Y_n} A \right) \wedge \left(\bigwedge_{Y_n \setminus *} B \right) \wedge C,$$

where $\wedge$ is the smash product, which is the coproduct in the category of commutative ring spectra. We call this the *nth simplicial degree of the relative Loday construction of A and B with coefficients in C on (X, Y)*. In our paper we will only consider pointed simplicial sets that are finite in every degree. In the infinite case one defines the $\wedge$-product over an infinite set as the direct limit of the coproducts over finite subsets of the indexing set.

The structure maps of the relative Loday construction use the fact that the smash product is the coproduct in the category of commutative S-algebras and they are given as follows: Let $\varphi \in \Delta([m], [n])$ and let φ^* denote the induced map on X and Y:

$$\varphi^* \colon Y_n \to Y_m, \quad \varphi^* \colon X_n \to X_m.$$

Note that $X_n \setminus Y_n$ is not a subcomplex of X_n, so φ^* might send elements in here to Y_m. We get an induced map

$$\varphi^* \colon \left(\bigwedge_{X_n \setminus Y_n} A\right) \wedge \left(\bigwedge_{Y_n \setminus *} B\right) \wedge C \to \left(\bigwedge_{X_m \setminus Y_m} A\right) \wedge \left(\bigwedge_{Y_m \setminus *} B\right) \wedge C$$

by looking at the map induced on the coproduct by the following maps:

- If for $x \in X_n \setminus Y_n$ the image $\varphi^*(x)$ is in $Y_m \setminus *$, then for the smash factor A mapping into the coproduct by inclusion in the x-coordinate $A \xrightarrow{i_x} (\mathcal{L}_{(X,Y)}(A, B; C))_n$, we look at the composition

$$A \xrightarrow{\;f\;} B \xrightarrow{\;i_{\varphi^*(x)}\;} (\mathcal{L}_{(X,Y)}(A, B; C))_m.$$

- If $x \in X_n \setminus Y_n$ is sent to the basepoint in Y_m under φ^*, then for the smash factor A mapping into the coproduct by inclusion in the x-coordinate $A \xrightarrow{i_x} (\mathcal{L}_{(X,Y)}(A, B; C))_n$, we take the composition

$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \xrightarrow{\;i_{\varphi^*(x)}\;} (\mathcal{L}_{(X,Y)}(A, B; C))_m.$$

- If for $x \in X_n \setminus Y_n$, the element $\varphi^*(x)$ is in $X_m \setminus Y_m$, then for the smash factor A mapping into the coproduct by inclusion in the x-coordinate $A \xrightarrow{i_x} (\mathcal{L}_{(X,Y)}(A, B; C))_n$ we consider

$$i_{\varphi^*(x)} \colon A \to (\mathcal{L}_{(X,Y)}(A, B; C))_m.$$

- Similarly, if a $y \in Y_n \setminus *$ is sent to the basepoint, then for the smash factor B mapping into the coproduct by inclusion in the y-coordinate $B \xrightarrow{i_y} (\mathcal{L}_{(X,Y)}(A, B; C))_n$, we look at the composition

$$B \xrightarrow{\;g\;} C \xrightarrow{\;i_{\varphi^*(y)}\;} (\mathcal{L}_{(X,Y)}(A, B; C))_m.$$

- If $y \in Y_n \setminus *$ is sent to $Y_m \setminus *$, then for the smash factor B mapping into the coproduct by inclusion in the y-coordinate $B \xrightarrow{i_y} (\mathcal{L}_{(X,Y)}(A, B; C))_n$, we take

$$i_{\varphi^*(y)} \colon B \to (\mathcal{L}_{(X,Y)}(A, B; C))_m.$$

- For the base point $*$ which has to map to itself by φ^*, then for the smash factor C mapping into the coproduct by inclusion in the $*$-coordinate $C \xrightarrow{i_*} (\mathcal{L}_{(X,Y)}(A, B; C))_n$, we consider

$$i_{\varphi^*(*)} \colon C \to (\mathcal{L}_{(X,Y)}(A, B; C))_m.$$

As the multiplication maps on A, B and C are associative and commutative and as the maps f and g are morphisms of commutative S-algebras, this gives the relative Loday construction the structure of a simplicial spectrum.

Lemma 2.2. *The relative Loday construction, $\mathcal{L}_{(X,Y)}(A, B; C)_\bullet$, is a simplicial augmented commutative C-algebra spectrum.*

Proof. The multiplication

$$\mathcal{L}_{(X,Y)}(A, B; C)_n \wedge_C \mathcal{L}_{(X,Y)}(A, B; C)_n \to \mathcal{L}_{(X,Y)}(A, B; C)_n$$

is defined coordinatewise and is therefore compatible with the simplicial structure. Hence we obtain a simplicial commutative augmented C-algebra structure on $\mathcal{L}_{(X,Y)}(A, B; C)_\bullet$. $\qquad\square$

Remark 2.3. We could use the smash product to denote the coproduct in the category of commutative algebras over a general commutative ring spectrum, k. In this case, we will add a superscript to the notation,

$$\mathcal{L}^k_{(X,Y)}(A, B; C)_\bullet.$$

If $B = C$ then the simplicial subset Y of X does not have to be pointed; then the construction will be called $\mathcal{L}^k_{(X,Y)}(A, B)_\bullet$, and is a simplicial k-algebra.

One can also define a version of the relative Loday construction if C is a B-module, rather than a commutative B-algebra. In that case, we let $\mathcal{L}^k_{(X,Y)}(A, B; C)_\bullet = \mathcal{L}^k_{(X,Y)}(A, B; B)_\bullet \wedge_B C$, and obtain a simplicial B-module spectrum.

Example 2.4. As an explicit example of a pointed simplicial subcomplex we consider $\partial\Delta_2 \subset \Delta_2$ whose basepoint $* \in \Delta([n], [2])$ is the constant map with value 0. Note that the number of elements in $\Delta([n], [m])$ is $\binom{n+m+1}{n+1}$.

We describe the effect of the maps $\varphi\colon [1] \to [2]$, $\varphi(0) = 0$, $\varphi(1) = 2$ and $\psi\colon [2] \to [1]$, $\psi(1) = \psi(0) = 0$ and $\psi(2) = 1$.

$$
\varphi \quad
\begin{array}{ccc}
 & & 2 \\
1 & \nearrow & 1 \\
0 & \longrightarrow & 0,
\end{array}
\qquad\qquad
\psi \quad
\begin{array}{ccc}
2 & & \\
1 & \searrow & 1 \\
0 & \longrightarrow & 0.
\end{array}
$$

In $\mathcal{L}_{(\Delta_2, \partial\Delta_2)}(A, B; C)_2$ there is only one copy of A because

$$\Delta_2[2] \setminus \partial\Delta_2([2]) = \Delta([2], [2]) \setminus \partial\Delta_2([2]) = \{\mathrm{id}_{[2]}\}.$$

Thus

$$\mathcal{L}_{(\Delta_2, \partial\Delta_2)}(A, B; C)_2 = A \wedge \left(\bigwedge_{\partial\Delta_2([2]) \setminus *} B \right) \wedge C.$$

As $\partial\Delta_2[1] = \Delta_2[1]$ we get

$$\mathcal{L}_{(\Delta_2, \partial\Delta_2)}(A, B; C)_1 = \left(\bigwedge_{\partial\Delta_2([1]) \setminus *} B \right) \wedge C.$$

The map $\psi^*\colon \Delta_2([1]) \to \Delta_2([2])$ sends the six elements of $\Delta_2([1])$ injectively to six elements in $\Delta_2([2])$, so on the Loday construction we only use the unit maps of A and B to fill in the gaps. In particular, the identity of $[2]$ is not in the image of ψ and we get as ψ^* on the Loday construction

$$\left(\bigwedge_{\partial\Delta_2([1]) \setminus *} B \right) \wedge C \xrightarrow{\cong} S \wedge \left(\bigwedge_{\Delta_2([2]) \setminus (\{\mathrm{id}_{[2]}\} \cup \mathrm{im}(\psi^*))} S \right) \wedge \left(\bigwedge_{\mathrm{im}(\psi^*) \setminus *} B \right) \wedge C$$

$$\eta_A \wedge \left(\bigwedge_{\Delta_2([2]) \setminus (\{\mathrm{id}_{[2]}\} \cup \mathrm{im}(\psi^*))} \eta_B \right) \wedge \left(\bigwedge_{\mathrm{im}(\psi^*) \setminus *} \mathrm{id}_B \right) \wedge \mathrm{id}_C \Bigg\downarrow$$

$$A \wedge \left(\bigwedge_{\partial\Delta_2([2]) \setminus *} B \right) \wedge C.$$

In contrast to this, the map $\varphi^*\colon \Delta_2([2]) \to \Delta_2([1])$ is surjective. The preimage of the basepoint under φ^* is just the basepoint, so we get the identity on the C-factor in $A \wedge \left(\bigwedge_{\partial\Delta_2([2]) \setminus *} B \right) \wedge C$, but we have to use the map $f\colon A \to B$ and several instances of the multiplication on B to get to $\left(\bigwedge_{\partial\Delta_2([1]) \setminus *} B \right) \wedge C$ because there is a fiber of cardinality three and a fiber of cardinality two.

Example 2.5. If in Definition 2.1 we take $A = B$, then $\mathcal{L}_{(X,Y)}(A, A; C)_\bullet = \mathcal{L}_X(A; C)_\bullet$. If in addition $A = C$ then we obtain $\mathcal{L}_{(X,Y)}(A, A; A)_\bullet = \mathcal{L}_X(A)_\bullet$.

Example 2.6. If we work relative to A, *i.e.*, if we consider $\mathcal{L}_{(X,Y)}^A(A, B; C)_\bullet$, then the A-factors disappear because we smash over A and we get

$$\mathcal{L}_{(X,Y)}^A(A, B; C)_\bullet \cong \mathcal{L}_Y^A(B; C)_\bullet.$$

Definition 2.7. For k a commutative S-algebra, A a commutative k-algebra, and C a commutative A-algebra, we define *higher topological Hochschild homology of order n of A with coefficients in C* by $\mathsf{THH}^{[n],k}(A; C) := \mathcal{L}_{S^n}^k(A; C)_\bullet$. Here, S^n is a pointed simplicial model of the n-sphere. Note that if we take C to be just an A-module, this definition still makes sense but $\mathsf{THH}^{[n],k}(A; C)$ will be an A-module rather than a C-algebra (and therefore also an A-algebra).

Notation 2.8. As above, if k is the sphere spectrum it is omitted from the notation. Similarly, if $n = 1$ this may be omitted also and written as $\mathsf{THH}^k(A; C)$. If $C = A$, we may write $\mathsf{THH}^{[n],k}(A)$.

Remark 2.9. If k is a discrete commutative ring, A is a commutative k-algebra, and C is an A-algebra (or, as above, even just an A-module if we are willing to give up the multiplicative structure on the result), Pirashvili [P] defined *higher Hochschild homology of order n of A with coefficients in C* $\mathsf{HH}^{[n],k}(A; C)$ by doing the analogous Loday construction for discrete commutative rings rather than for commutative S-algebras. Note that if A is flat over k, then by Theorem 1.7 in [EKMM] in the case $n = 1$, and in general by the method of the proof there, we have

$$\mathsf{THH}_*^{[n],Hk}(HA; HC) \cong \mathsf{HH}_*^{[n],k}(A; C).$$

Proposition 2.10.
  (a) *For A, B, and as in Definition 2.1, C a commutative B-algebra, X a pointed simplicial set and Y a pointed simplicial subset, we get an isomorphism of augmented simplicial commutative C-algebras*

(2.10.1) $$\mathcal{L}_{(X,Y)}^k(A, B; C)_\bullet \cong \mathcal{L}_X^k(A; C)_\bullet \wedge_{\mathcal{L}_Y^k(A;C)_\bullet} \mathcal{L}_Y^k(B; C)_\bullet.$$

  (b) *For X_0 a common pointed simplicial subset of X_1 and X_2 and Y_0 a common pointed simplicial subset of Y_1 and Y_2 so that $Y_i \subseteq X_i$ for $i = 1, 2$ and $Y_0 = X_0 \cap Y_1 \cap Y_2$, we have an isomorphism of augmented simplicial commutative C-algebras*

(2.10.2) $$\mathcal{L}_{(X_1 \cup_{X_0} X_2, Y_1 \cup_{Y_0} Y_2)}^k(A, B; C)_\bullet \cong \mathcal{L}_{(X_1,Y_1)}^k(A, B; C)_\bullet \wedge_{\mathcal{L}_{(X_0,Y_0)}^k(A,B;C)_\bullet} \mathcal{L}_{(X_2,Y_2)}^k(A, B; C)_\bullet.$$

If $C = B$, then in both statements we can work in the unpointed setting, though the algebras in Equations (2.10.1) and (2.10.2) will no longer be commutative C-algebras and the isomorphisms we get will be isomorphisms of simplicial commutative k-algebras.

Proof. For the claim in (2.10.1) we have a levelwise isomorphism of simplicial spectra

$$(\mathcal{L}_X^k(A; C))_n \wedge_{(\mathcal{L}_Y^k(A;C))_n} (\mathcal{L}_Y^k(B; C))_n \cong (\mathcal{L}_{(X,Y)}^k(A, B; C))_n$$

given by the identification of coequalizers

$$\left(\left(\bigwedge_{X_n \backslash Y_n} A \right) \wedge \left(\bigwedge_{Y_n \backslash *} A \right) \wedge C \right) \bigwedge_{(\bigwedge_{Y_n \backslash *} A) \wedge C} \left(\left(\bigwedge_{Y_n \backslash *} B \right) \wedge C \right) \cong \left(\bigwedge_{X_n \backslash Y_n} A \right) \wedge \left(\bigwedge_{Y_n \backslash *} B \right) \wedge C.$$

Similarly, for (2.10.2) we have a levelwise isomorphism of simplicial spectra

$$(\mathcal{L}_{(X_1,Y_1)}^k(A, B; C))_n \wedge_{(\mathcal{L}_{(X_0,Y_0)}^k(A,B;C))_n} (\mathcal{L}_{(X_2,Y_2)}^k(A, B; C))_n \cong (\mathcal{L}_{(X_1 \cup_{X_0} X_2, Y_1 \cup_{Y_0} Y_2)}^k(A, B; C))_n.$$

Here we use that tensoring a commutative S-algebra with a pointed simplicial set is compatible with pushouts of simplicial sets, hence we get

$$
\left(\bigwedge_{(X_1)_n \backslash (Y_1)_n} A \wedge \bigwedge_{(Y_1)_n \backslash *} B \wedge C \right) \bigwedge_{\left(\bigwedge_{(X_0)_n \backslash (Y_0)_n} A \wedge \bigwedge_{(Y_0)_n \backslash *} B \wedge C \right)} \left(\bigwedge_{(X_2)_n \backslash (Y_2)_n} A \wedge \bigwedge_{(Y_2)_n \backslash *} B \wedge C \right)
$$

$$
\cong \bigwedge_{(X_1 \cup_{X_0} X_2)_n \backslash (Y_1 \cup_{Y_0} Y_2)_n} A \wedge \bigwedge_{y \in (Y_1 \cup_{Y_0} Y_2)_n \backslash *} B \wedge C.
$$

$\square$

Remark 2.11. An immediate consequence of Proposition 2.10 is a suitable form of homotopy invariance for the relative Loday construction. If you replace the pair (X, Y) of pointed simplicial sets with a homotopy equivalent one (X', Y') and if the equivalence is a homotopy equivalence of pairs, then the relative Loday constructions $\mathcal{L}^k_{(X,Y)}(A, B; C)_\bullet$ and $\mathcal{L}^k_{(X',Y')}(A, B; C)_\bullet$ are homotopy equivalent as augmented simplicial commutative C-algebras.

Let A be an augmented commutative C-algebra, *i.e.*, in addition to the map $g \circ f \colon A \to C$ we have a map $\eta \colon C \to A$, such that $g \circ f \circ \eta = \mathrm{id}_C$. In that case, we can identify the relative Loday construction $\mathcal{L}^C_{(X,Y)}(A, C; C)_\bullet$ with the Loday construction of the quotient:

Proposition 2.12. *Let A be an augmented commutative C-algebra. Then there is an isomorphism of simplicial augmented commutative C-algebras*

$$
\text{(2.12.1)} \qquad \mathcal{L}^C_{(X,Y)}(A, C; C)_\bullet \cong \mathcal{L}^C_{X/Y}(A; C)_\bullet
$$

where X/Y has the equivalence class of Y as a basepoint.

Proof. We use Proposition 2.10 above and obtain that

$$
\mathcal{L}^C_{(X,Y)}(A, C; C)_\bullet \cong \mathcal{L}^C_X(A; C)_\bullet \wedge_{\mathcal{L}^C_Y(A;C)_\bullet} \mathcal{L}^C_Y(C; C)_\bullet
$$

but $\mathcal{L}^C_Y(C; C)_\bullet$ is isomorphic to the constant simplicial object $C_\bullet$ with C in every simplicial degree. Thus

$$
\mathcal{L}^C_{(X,Y)}(A, C; C)_\bullet \cong \mathcal{L}^C_X(A; C)_\bullet \wedge_{\mathcal{L}^C_Y(A;C)_\bullet} C_\bullet \cong \mathcal{L}^C_{X/Y}(A; C)_\bullet \wedge_C C \cong \mathcal{L}^C_{X/Y}(A; C)_\bullet
$$

as claimed. $\square$

Proposition 2.12 immediately gives rise to the following spectral sequence.

Proposition 2.13. *If C is a cofibrant commutative S-algebra and A is a cofibrant commutative augmented C-algebra and if Y is a pointed simplicial subset of X, then there is a spectral sequence*

$$
E^2_{s,t} = \mathsf{Tor}^{\pi_*(\mathcal{L}^C_Y(A;C)_\bullet)}_{s,t}(\pi_*(\mathcal{L}^C_X(A; C)_\bullet), \pi_* C) \Rightarrow \pi_*(\mathcal{L}^C_{X/Y}(A; C)_\bullet).
$$

Proof. The isomorphism from Proposition 2.12

$$
\mathcal{L}^C_X(A; C)_\bullet \wedge_{\mathcal{L}^C_Y(A;C)_\bullet} C \cong \mathcal{L}^C_{X/Y}(A; C)_\bullet
$$

induces a weak equivalence

$$
\mathcal{L}^C_X(A; C)_\bullet \wedge^L_{\mathcal{L}^C_Y(A;C)_\bullet} C \simeq \mathcal{L}^C_{X/Y}(A; C)_\bullet
$$

and the cofibrancy assumptions ensure that we get the associated Künneth spectral sequence. $\square$

3. Spectral sequences with the relative Loday construction

In this section we set up some spectral sequences. Let S be the sphere spectrum and let R be a commutative S-algebra. Unadorned smash products will be over S. We first recall some properties of the category of commutative R-algebras: The category of commutative R-algebras is a topological model category ([EKMM, VII.4.10]). This implies that it is tensored over the category of unbased spaces and that for every sequence of cofibrations $R \to A \to B$ of commutative S-algebras and every relative CW-complex (L, K) the map

$$(A \otimes L) \wedge_{(A \otimes K)} (B \otimes K) \to B \otimes L$$

is a cofibration. For a simplicial finite set X and commutative R-algebra A there is a natural isomorphism (see [EKMM, VII.3.2]):

$$|\mathcal{L}_X^R(A)_\bullet| \cong A \otimes |X|.$$

We define the Loday construction $\mathcal{L}_{|X|}^R(A)$ as $A \otimes |X|$. In the pointed setting, we can use the inclusion of a one-point space into the basepoint to make $\mathcal{L}_{|X|}^R(A)$ into a commutative augmented A-algebra. We can then also define $\mathcal{L}_{|X|}^R(A; C) = \mathcal{L}_{|X|}^R(A) \otimes_A C$, and we can define the relative Loday construction for a pair of pointed CW complexes $Y \subset X$ and a sequence of maps of commutative S-algebras $R \to A \to B \to C$ using Proposition 2.10 (a) as

$$(3.0.1) \qquad \mathcal{L}_{(X,Y)}^R(A, B; C) := \mathcal{L}_X^R(A; C) \wedge_{\mathcal{L}_Y^R(A;C)}^L \mathcal{L}_Y^R(B; C).$$

Theorem 3.1. *Let A be a cofibrant commutative S-algebra, and let B be a cofibrant commutative A-algebra. There is an equivalence of augmented commutative B-algebras*

$$\mathcal{L}_{(D^n, S^{n-1})}(A, B) \simeq \mathsf{THH}^{[n-1], A}(B)$$

for all n.

Proof. We proceed by induction on n. For $n = 1$, $\mathcal{L}_{(D^1, S^0)}(A, B)$ is the two-sided bar construction $B(B, A, B)$ which is a model for $B \wedge_A^L B$. Here D^1 has the standard simplicial structure with a single 1-cell.

Since we assumed B to be a cofibrant commutative A-algebra, $B \wedge_A^L B$ is weakly equivalent to $B \wedge_A B$ which is $\mathsf{THH}^{[0], A}(B)$. For simplicity we will use the tensor over spaces for the rest of this proof. For the inductive step, we assume that $\mathcal{L}_{(D^n, S^{n-1})}(A, B) \simeq \mathsf{THH}^{[n-1], A}(B)$. By decomposing the n-sphere into two hemispheres glued along an $(n-1)$-sphere as in [V, Proposition 2.1.3 and Section 7], we know that $\mathsf{THH}^{[n], A}(B)$ is weakly equivalent to the bar construction $B^A(B, \mathsf{THH}^{[n-1], A}(B), B)$.

We also know that $\mathcal{L}_{(D^{n+1}, S^n)}(A, B)$ can be built from two half-disks of dimension $n+1$, part of whose boundary (the outside edge) has B's over it, and the other part (the n-disk we glue along) has A's over it. So by (2.10.2),

$$\mathcal{L}_{(D^{n+1}, S^n)}(A, B) = \mathcal{L}_{(D^{n+1} \bigcup_{D^n} D^{n+1}, D^n \bigcup_{S^{n-1}} D^n)}(A, B)$$
$$\cong \mathcal{L}_{(D^{n+1}, D^n)}(A, B) \wedge_{\mathcal{L}_{(D^n, S^{n-1})}(A,B)} \mathcal{L}_{(D^{n+1}, D^n)}(A, B).$$

For example, when $n = 1$ we have

$$\mathcal{L}_{(D^2, S^1)}(A, B) \simeq \mathcal{L}_{\left(\!\text{◖},\text{◯}\right)}(A, B) \wedge_{\mathcal{L}_{\left(|,.\right)}(A,B)} \mathcal{L}_{\left(\text{◗},.\text{◗}\right)}(A, B)$$

So we have

$$\mathcal{L}_{(D^{n+1},S^n)}(A,B) \cong \mathcal{L}_{(D^{n+1},D^n)}(A,B) \wedge_{\mathcal{L}_{(D^n,S^{n-1})}(A,B)} \mathcal{L}_{(D^{n+1},D^n)}(A,B)$$

$$\simeq \mathcal{L}_{(*,*)}(A,B) \wedge^L_{\mathcal{L}_{(D^n,S^{n-1})}(A,B)} \mathcal{L}_{(*,*)}(A,B) \qquad \text{(by homotopy invariance)}$$

$$\simeq B \wedge^L_{\mathcal{L}_{(D^n,S^{n-1})}(A,B)} B$$

$$\simeq B^A(B, \mathcal{L}_{(D^n,S^{n-1})}(A,B), B)$$

$$\simeq B^A(B, \mathsf{THH}^{[n-1],A}(B), B) \qquad \text{(by assumption)}$$

$$\simeq \mathsf{THH}^{[n],A}(B). \qquad \text{(by [V])}$$

$\square$

Let C be a commutative R-algebra. Let $\mathcal{C}_{R/C}$ and $\mathcal{C}_{C/C}$ denote the categories of commutative R-algebras over C and of commutative C-algebras over C. We denote by $\mathcal{T}$ the category of based spaces. We have a functor

$$\bar{\otimes}_C \colon \mathcal{C}_{R/C} \times \mathcal{T} \to \mathcal{C}_{C/C}$$

defined by $(A, X) \mapsto A \bar{\otimes}_C X := (A \otimes X) \wedge_A C$. Here, the map $A \to A \otimes X$ is given by the composition of the isomorphism $A \cong A \otimes *$ with the map induced by the inclusion of the basepoint. The augmentation $A \bar{\otimes}_C X \to C$ is given by

$$(A \otimes X) \wedge_A C \to (A \otimes *) \otimes_A C \cong C.$$

We have a natural homeomorphism

$$\mathcal{C}_{C/C}(A \bar{\otimes}_C X, B) \cong \mathcal{T}(X, \mathcal{C}_{R/C}(A,B)).$$

Let $D \to E$ be a map in $\mathcal{C}_{R/C}$ such that the underlying map of commutative R-algebras is a cofibration. Let $K \to L$ be an inclusion of based spaces such that (L, K) is a relative CW-complex. Then the natural map

$$(D \bar{\otimes}_C L) \wedge_{(D \bar{\otimes}_C K)} (E \bar{\otimes}_C K) \to E \bar{\otimes}_C L$$

is a cofibration of commutative R-algebras. For $A \in \mathcal{C}_{R/C}$ and a simplicial finite pointed set X, we have a natural isomorphism of C-algebras over C:

$$|\mathcal{L}^R_{(X,*)}(A;C)_\bullet| \cong A \bar{\otimes}_C |X|.$$

Theorem 3.2. *Let $S \to A \to B \to C$ be a sequence of cofibrations of commutative S-algebras. Then*

(a) $\mathsf{THH}^{[n],A}(B) \cong A \wedge_{\mathsf{THH}^{[n]}(A)} \mathsf{THH}^{[n]}(B)$ *and*

(b) $\mathsf{THH}^{[n],A}(B;C) \cong C \wedge_{\mathsf{THH}^{[n]}(A;C)} \mathsf{THH}^{[n]}(B;C)$.

In both cases, the smash product models the derived smash product.

Proof. In order to show (a) we first prove that

$$A \wedge_{\mathsf{THH}^{[n]}(A)} \mathsf{THH}^{[n]}(B)$$

models the derived smash product of A and $\mathsf{THH}^{[n]}(B)$ over $\mathsf{THH}^{[n]}(A)$. For this we first show that $\mathsf{THH}^{[n]}(A)$ is a cofibrant commutative S-algebra: Since A is a cofibrant commutative S-algebra, it suffices to show that the unit $A \to \mathsf{THH}^{[n]}(A)$ or equivalently that the map $A \otimes * \to A \otimes S^n$ is a cofibration of commutative S-algebras. By the properties listed above the map

$$(S \otimes S^n) \wedge_{(S \otimes *)} (A \otimes *) \to (A \otimes S^n)$$

is a cofibration. As $S \otimes * \cong S \otimes S^n$, we get that

$$(A \otimes *) \cong (S \otimes S^n) \wedge_{(S \otimes *)} (A \otimes *).$$

Thus, $\mathsf{THH}^{[n]}(A)$ is cofibrant. A standard argument then shows that $\mathsf{THH}^{[n]}(A) \to \mathsf{THH}^{[n]}(B)$ is a cofibration of commutative S-algebras: Since $A \to B$ is a cofibration, the map

$$(A \otimes S^n) \otimes_{(A \otimes *)} (B \otimes *) \to (B \otimes S^n)$$

is a cofibration. Because cofibrations are stable under cobase change the map

$$(A \otimes S^n) \to (A \otimes S^n) \otimes_{(A \otimes *)} (B \otimes *)$$

is a cofibration. Thus $A \otimes S^n \to B \otimes S^n$ is a cofibration. We therefore get by [EKMM, VII.7.4] that the functor

$$- \wedge_{\mathsf{THH}^{[n]}(A)} \mathsf{THH}^{[n]}(B)$$

preserves weak equivalences between cofibrant commutative S-algebras. We factor the map $\mathsf{THH}^{[n]}(A) \to A$ as a cofibration followed by an acyclic fibration

$$\mathsf{THH}^{[n]}(A) \rightarrowtail \tilde{A} \overset{\simeq}{\twoheadrightarrow} A$$

and obtain a weak equivalence

$$\tilde{A} \wedge_{\mathsf{THH}^{[n]}(A)} \mathsf{THH}^{[n]}(B) \overset{\simeq}{\to} A \wedge_{\mathsf{THH}^{[n]}(A)} \mathsf{THH}^{[n]}(B).$$

By [EKMM, VII.6.7] the S-algebra

$$\tilde{A} \wedge_{\mathsf{THH}^{[n]}(A)} \mathsf{THH}^{[n]}(B)$$

models the derived smash product of A and $\mathsf{THH}^{[n]}(B)$ over $\mathsf{THH}^{[n]}(A)$.

We now show that there is an isomorphism of commutative S-algebras

$$\mathsf{THH}^{[n],A}(B) \cong A \wedge_{\mathsf{THH}^{[n]}(A)} \mathsf{THH}^{[n]}(B).$$

We start with the isomorphism of commutative S-algebras

$$\begin{aligned} A \wedge_{\mathsf{THH}^{[n]}(A)} \mathsf{THH}^{[n]}(B) &\cong |\mathcal{L}_*(A)_\bullet| \wedge_{|\mathcal{L}_{S^n}(A)_\bullet|} |\mathcal{L}_{S^n}(B)_\bullet| \\ &\cong |\mathcal{L}_*(A)_\bullet \wedge_{\mathcal{L}_{S^n}(A)_\bullet} \mathcal{L}_{S^n}(B)_\bullet|. \end{aligned}$$

By a comparison of coequalizer diagrams we have, for all n, isomorphisms of commutative S-algebras:

$$A \wedge_{A^{\wedge n}} (B^{\wedge n}) \cong \underbrace{B \wedge_A \ldots \wedge_A B}_{n}$$

and these induce an isomorphism of simplicial commutative S-algebras

$$\mathcal{L}_*(A)_\bullet \wedge_{\mathcal{L}_{S^n}(A)_\bullet} \mathcal{L}_{S^n}(B)_\bullet \cong \mathcal{L}_{S^n}^A(B)_\bullet.$$

This proves part (a) of the theorem.

We now prove part (b). We again first show that

$$C \wedge_{\mathsf{THH}^{[n]}(A;C)} \mathsf{THH}^{[n]}(B;C)$$

models the derived smash product of C and $\mathsf{THH}^{[n]}(B;C)$ over $\mathsf{THH}^{[n]}(A;C)$. For this it suffices to show that $\mathsf{THH}^{[n]}(A;C)$ is a cofibrant commutative S-algebra and that the map $\mathsf{THH}^{[n]}(A;C) \to \mathsf{THH}^{[n]}(B;C)$ is a cofibration of commutative S-algebras. The morphism $C \to \mathsf{THH}^{[n]}(A;C)$ is a cofibration because

$$C \to A \,\bar{\otimes}_C\, S^n = (A \otimes S^n) \wedge_A C$$

is a cofibration. Thus $\mathsf{THH}^{[n]}(A;C)$ is cofibrant. The map $\mathsf{THH}^{[n]}(A;C) \to \mathsf{THH}^{[n]}(B;C)$ is a cofibration because $A \,\bar{\otimes}_C\, S^n \to B \,\bar{\otimes}_C\, S^n$ can be written as

$$A \,\bar{\otimes}_C\, S^n \to (A \,\bar{\otimes}_C\, S^n) \wedge_{(A \bar{\otimes}_C *)} (B \,\bar{\otimes}_C\, *) \rightarrowtail B \,\bar{\otimes}_C\, S^n.$$

The first map of the composition is an isomorphism, because the map $A \bar{\otimes}_C * \to B \bar{\otimes}_C *$ identifies with the identity of C.

It remains to prove that there is an isomorphism of commutative S-algebras

$$\mathsf{THH}^{[n],A}(B;C) \cong C \wedge_{\mathsf{THH}^{[n]}(A;C)} \mathsf{THH}^{[n]}(B;C).$$

This follows as above by using that we have an isomorphism of commutative S-algebras

$$C \wedge_{(A^{\wedge n} \wedge C)} (B^{\wedge n} \wedge C) \cong B^{\wedge_A n} \wedge_A C$$

for all $n \geqslant 0$. $\qquad\square$

Remark 3.3. The proof shows that Theorem 3.2 also holds for general finite pointed simplicial sets X and a sequence of cofibrations of commutative S-algebras $S \to A \to B \to C$, giving us isomorphisms

(a) $\mathcal{L}_{|X|}^A(B) \cong A \wedge_{\mathcal{L}_{|X|}(A)} \mathcal{L}_{|X|}(B)$ and
(b) $\mathcal{L}_{|X|}^A(B;C) \cong C \wedge_{\mathcal{L}_{|X|}(A;C)} \mathcal{L}_{|X|}(B;C)$.

Remark 3.4. It is known that for topological Hochschild homology, there is a difference between Galois descent and étale descent: John Rognes [R] developed the notion of Galois extensions for commutative S-algebras and showed that for a Galois extension $A \to B$ with finite Galois group G the canonical map $B \to \mathsf{THH}^A(B)$ is a weak equivalence [R, Lemma 9.2.6]. Akhil Mathew [M] provided an example of such a Galois extension that does *not* satisfy étale descent, *i.e.*, the pushout map

$$B \wedge_A \mathsf{THH}(A) \to \mathsf{THH}(B)$$

is not a weak equivalence. Theorem 3.2 doesn't contradict this. We take a finite Galois extension $A \to B$. Then we obtain a weak equivalences

$$B \to \mathsf{THH}^A(B) \cong A \wedge_{\mathsf{THH}(A)} \mathsf{THH}(B).$$

But if we then smash this equivalence with $\mathsf{THH}(A)$ over A the resulting equivalence

(3.4.1) $$B \wedge_A \mathsf{THH}(A) \simeq (A \wedge_{\mathsf{THH}(A)} \mathsf{THH}(B)) \wedge_A \mathsf{THH}(A)$$

cannot be reduced to the statement that $B \wedge_A \mathsf{THH}(A)$ is equivalent to $\mathsf{THH}(B)$: On the right hand side of (3.4.1) we cannot reduce the $\mathsf{THH}(A)$-term because in the smash product we use the augmentation map $\mathsf{THH}(A) \to A$ and its composite with the unit is not equivalent to the identity map.

Let R be a commutative S-algebra, and $\mathcal{C}_R$ the category of commutative R-algebras. Let D be the category $\{b \longleftarrow a \longrightarrow c\}$. Then the category $^D\mathcal{C}_R$ of functors from D to $\mathcal{C}_R$ admits a model category structure, where the weak equivalences (resp. fibrations) are the maps that are objectwise weak equivalences (resp. fibrations). We have a cofibrant replacement functor $^D\mathcal{C}_R \to {}^D\mathcal{C}_R$. The homotopy pushout $B \tilde{\wedge}_A C$ of a diagram $B \longleftarrow A \longrightarrow C$ in $\mathcal{C}_R$ is constructed by taking the chosen cofibrant replacement $B' \longleftarrow A' \longrightarrow C'$ of the diagram and then taking the usual pushout $B' \wedge_{A'} C'$. One gets a functor

$$(-) \tilde{\wedge}_{(-)} (-) : {}^D\mathcal{C}_R \to \mathcal{C}_R.$$

This functor sends weak equivalences to weak equivalences. There is natural map

$$B \tilde{\wedge}_A C \to B \wedge_A C$$

which is a weak equivalence when A is cofibrant and $A \to B$ and $A \to C$ are cofibrations. If A is cofibrant, then the homotopy pushout $B \tilde{\wedge}_A C$ is equivalent to the derived smash product $B \wedge_A^L C$ of B and C over A. One can show:

Lemma 3.5. *For a commutative diagram*

$$
\begin{array}{ccc}
E \longleftarrow D \longrightarrow F \\
\uparrow \qquad \uparrow \qquad \uparrow \\
B \longleftarrow A \longrightarrow C \\
\downarrow \qquad \downarrow \qquad \downarrow \\
H \longleftarrow G \longrightarrow I
\end{array}
$$

in $\mathcal{C}_R$ there is a zig-zag of weak equivalences

$$(E \,\tilde{\wedge}_D\, F) \,\tilde{\wedge}_{(B\tilde{\wedge}_A C)}\, (H \,\tilde{\wedge}_G\, I) \simeq (E \,\tilde{\wedge}_B\, H) \,\tilde{\wedge}_{(D\tilde{\wedge}_A G)}\, (F \,\tilde{\wedge}_C\, I)$$

over $(E \wedge_B H) \wedge_{(D\wedge_A G)} (F \wedge_C I)$ where

$$(E \,\tilde{\wedge}_D\, F) \,\tilde{\wedge}_{(B\tilde{\wedge}_A C)}\, (H \,\tilde{\wedge}_G\, I) \to (E \wedge_B H) \wedge_{(D\wedge_A G)} (F \wedge_C I)$$

is given by the compositions of the morphisms

$$(E \,\tilde{\wedge}_D\, F) \,\tilde{\wedge}_{(B\tilde{\wedge}_A C)}\, (H \,\tilde{\wedge}_G\, I) \to (E \wedge_D F) \,\tilde{\wedge}_{(B\wedge_A C)}\, (H \wedge_G I) \to (E \wedge_D F) \wedge_{(B\wedge_A C)} (H \wedge_G I)$$

with the standard isomorphism

$$(E \wedge_D F)\wedge_{(B\wedge_A C)}(H \wedge_G I) \cong (E \wedge_B H) \wedge_{(D\wedge_A G)} (F \wedge_C I)$$

and the morphism

$$(E \,\tilde{\wedge}_B\, H) \,\tilde{\wedge}_{(D\tilde{\wedge}_A G)}\, (F \,\tilde{\wedge}_C\, I) \to (E \wedge_B H) \wedge_{(D\wedge_A G)} (F \wedge_C I)$$

is given by

$$(E \,\tilde{\wedge}_B\, H) \,\tilde{\wedge}_{(D\tilde{\wedge}_A G)}\, (F \,\tilde{\wedge}_C\, I) \to (E \wedge_B H) \,\tilde{\wedge}_{(D\wedge_A G)}\, (F \wedge_C I) \to (E \wedge_B H) \wedge_{(D\wedge_A G)} (F \wedge_C I).$$

Theorem 3.6. *Let $S \to A \to B \to C$ be a sequence of cofibrations of commutative S-algebras. Then*

$$\mathsf{THH}^{[n]}(B;C) \simeq \mathsf{THH}^{[n]}(A;C) \wedge^L_{\mathsf{THH}^{[n-1],A}(C)} \mathsf{THH}^{[n-1],B}(C),$$

where we regard $\mathsf{THH}^{[1]}(A;C) = C \wedge^L_{C\wedge_S A} C$ as a $\mathsf{THH}^{[0],A}(C) = C \wedge^L_A C$-algebra using the map $A \to C \wedge_S A$, and then once we know how to give $\mathsf{THH}^{[n]}(A;C)$ a $\mathsf{THH}^{[n-1],A}(C)$-algebra structure, we use that to get a $\mathsf{THH}^{[n],A}(C) \simeq C\wedge^L_{\mathsf{THH}^{[n-1],A}(C)}C$-algebra structure on $\mathsf{THH}^{[n+1]}(A;C) \simeq C \wedge^L_{\mathsf{THH}^{[n]}(A;C)} C$.

Proof. We work in the model category of commutative S-algebras. For a map of commutative S-algebras $D \to E$ we define commutative S-algebras $T^{[n],D}(E)$ augmented over E inductively as follows: Let $T^{[0],D}(E)$ be $E \,\tilde{\wedge}_D\, E$ and let $T^{[0],D}(E) \to E$ be defined by

$$E \,\tilde{\wedge}_D\, E \to E \wedge_D E \to E.$$

Set $T^{[n+1],D}(E) := E \,\tilde{\wedge}_{T^{[n],D}(E)}\, E$ and define $T^{[n+1],D}(E) \to E$ by

$$E \,\tilde{\wedge}_{T^{[n],D}(E)}\, E \to E \wedge_{T^{[n],D}(E)} E \to E.$$

The $T^{[n],(-)}(E)$ are then endofunctors on the category of commutative S-algebras over E. Using the decomposition $S^n = D^n \cup_{S^{n-1}} D^n$, one can show that there are zig-zags of weak equivalences over C (compare with [V, Section 7])

$$
\begin{aligned}
\mathsf{THH}^{[n],A}(C) &\simeq C \,\tilde{\wedge}_{\mathsf{THH}^{[n-1],A}(C)}\, C \\
\mathsf{THH}^{[n]}(A;C) &\simeq C \,\tilde{\wedge}_{\mathsf{THH}^{[n-1]}(A;C)}\, C.
\end{aligned}
$$

With that it follows that there are equivalences over C

$$\mathsf{THH}^{[n],A}(C) \quad \simeq \quad T^{[n],A}(C)$$
$$\mathsf{THH}^{[n]}(A;C) \quad \simeq \quad T^{[n-1],C\wedge_S A}(C).$$

The same is true for B instead of A.

It thus suffices to show:

$$T^{[n],C\wedge_S B}(C) \simeq T^{[n],C\wedge_S A}(C) \, \tilde{\wedge}_{T^{[n],A}(C)} \, T^{[n],B}(C).$$

We prove by induction on n that these S-algebras are equivalent via a zig-zag of weak equivalences over C where the augmentation of the right-hand side is given by

$$T^{[n],C\wedge_S A}(C) \, \tilde{\wedge}_{T^{[n],A}(C)} \, T^{[n],B}(C) \to T^{[n],C\wedge_S A}(C)\wedge_{T^{[n],A}(C)}T^{[n],B}(C) \to C.$$

We have an isomorphism $T^{[0],C\wedge_S B}(C) \cong T^{[0],(C\wedge_S A)\wedge_A B}(C)$ over C. Because of the cofibrancy assumptions the map

$$(C \wedge_S A) \, \tilde{\wedge}_A \, B \to (C \wedge_S A) \wedge_A B$$

is a weak equivalence. It induces a weak equivalence

$$(C \, \tilde{\wedge}_C \, C) \, \tilde{\wedge}_{((C\wedge_S A)\tilde{\wedge}_A B)} \, (C \, \tilde{\wedge}_C \, C) \xrightarrow{\simeq} C \, \tilde{\wedge}_{((C\wedge_S A)\wedge_A B)} \, C = T^{[0],(C\wedge_S A)\wedge_A B}(C).$$

This is a map over C if we endow the left-hand side with the augmentation

$$(C \, \tilde{\wedge}_C \, C) \, \tilde{\wedge}_{((C\wedge_S A)\tilde{\wedge}_A B)} \, (C \, \tilde{\wedge}_C \, C) \to (C \, \tilde{\wedge}_C \, C) \wedge_{((C\wedge_S A)\tilde{\wedge}_A B)} \, (C \, \tilde{\wedge}_C \, C) \to C \, \tilde{\wedge}_C \, C \to C.$$

By Lemma 3.5, we have an equivalence

$$(C \, \tilde{\wedge}_C \, C) \, \tilde{\wedge}_{((C\wedge_S A)\tilde{\wedge}_A B)} \, (C \, \tilde{\wedge}_C \, C) \simeq (C \, \tilde{\wedge}_{(C\wedge_S A)} \, C) \, \tilde{\wedge}_{(C\tilde{\wedge}_A C)} \, (C \, \tilde{\wedge}_B \, C)$$

and the right-hand side is equal to $T^{[0],C\wedge_S A}(C) \, \tilde{\wedge}_{T^{[0],A}(C)} \, T^{[0],B}(C)$. The compatibility of the equivalence with the isomorphism

$$(C\wedge_C C)\wedge_{((C\wedge_S A)\wedge_A B)}(C\wedge_C C) \cong (C\wedge_{(C\wedge_S A)}C)\wedge_{(C\wedge_A C)}(C\wedge_B C)$$

implies that it is an equivalence over C.

We now assume that the claim is true for n. Set $T' = T^{[n],C\wedge_S A}(C) \, \tilde{\wedge}_{T^{[n],A}(C)} \, T^{[n],B}(C)$. By the induction hypothesis we have

$$T^{[n+1],C\wedge_S B}(C) = C \, \tilde{\wedge}_{T^{[n],C\wedge_S B}(C)} \, C \simeq C \, \tilde{\wedge}_{T'} \, C.$$

via a zig-zag of weak equivalences over C. The weak equivalence

$$(C \, \tilde{\wedge}_C \, C) \, \tilde{\wedge}_{T'} \, (C \, \tilde{\wedge}_C \, C) \xrightarrow{\simeq} C \, \tilde{\wedge}_{T'} \, C$$

is a map over C if we endow the right-hand side with the augmentation

$$(C \, \tilde{\wedge}_C \, C) \, \tilde{\wedge}_{T'} \, (C \, \tilde{\wedge}_C \, C) \to (C \, \tilde{\wedge}_C \, C)\wedge_{T'}(C \, \tilde{\wedge}_C \, C) \to C \, \tilde{\wedge}_C \, C \to C.$$

By Lemma 3.5 we have an equivalence

$$(C \, \tilde{\wedge}_C \, C) \, \tilde{\wedge}_{T'} \, (C \, \tilde{\wedge}_C \, C) \simeq (C \, \tilde{\wedge}_{T^{[n],C\wedge_S A}(C)} \, C) \, \tilde{\wedge}_{(C\tilde{\wedge}_{T^{[n],A}(C)}C)} \, (C \, \tilde{\wedge}_{T^{[n],B}(C)} \, C)$$

and the right-hand side is equal to $T^{[n+1],C\wedge_S A}(C) \, \tilde{\wedge}_{T^{[n+1],A}(C)} \, T^{[n+1],B}(C)$. Because of the compatibility with the isomorphism

$$(C \wedge_C C)\wedge_{(T^{[n],C\wedge_S A}(C)\wedge_{T^{[n],A}(C)}T^{[n],B}(C))}(C\wedge_C C)$$

$$\cong \quad (C\wedge_{T^{[n],C\wedge_S A}(C)}C)\wedge_{(C\wedge_{T^{[n],A}(C)}C)}(C\wedge_{T^{[n],B}(C)}C)$$

it is an equivalence over C. This shows the induction step. $\qquad\square$

4. Applications

4.1. Thom spectra.

Example 4.1. Schlichtkrull [Sch] gives a general formula for the Loday construction on Thom spectra $\mathcal{L}_X(T(f); M)$ where $f \colon A \to BF_{h\mathcal{I}}$ is an E_∞-map, A is a grouplike E_∞-space, and $BF_{h\mathcal{I}}$ is a model for $BF = BGL_1(S)$, the classifying space for stable spherical fibrations. The commutative S-algebra $T(f)$ is the associated Thom spectrum for f and M is any $T(f)$-module.

If we set $B = C$ in Theorem 3.2, then we obtain

$$(4.1.1) \qquad \mathsf{THH}^{[n],A}(B) \simeq B \wedge^L_{\mathsf{THH}^{[n]}(A;B)} \mathsf{THH}^{[n]}(B)$$

so if there is a factorization

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & BF_{h\mathcal{I}} \\
{\scriptstyle h}\downarrow & \nearrow{\scriptstyle g} & \\
B & &
\end{array}
$$

such that h is a map of grouplike E_∞-spaces, then we get an induced map of commutative S-algebras $T(f) \to T(g)$.

For X a sphere and $M = T(g)$, we obtain [Sch, Theorem 1]

$$\mathsf{THH}^{[n]}(T(f); T(g)) \simeq T(g) \wedge \Omega^\infty(S^n \wedge \mathbb{A})_+$$

where $\mathbb{A}$ denotes the spectrum associated to A such that the map from A to the underlying infinite loop space of $\mathbb{A}$, $\Omega^\infty \mathbb{A}$, is a weak equivalence.

Our juggling formula (4.1.1) gives a formula for higher THH of $T(g)$ as a commutative $T(f)$-algebra:

$$\mathsf{THH}^{[n],T(f)}(T(g)) \simeq T(g) \wedge^L_{\mathsf{THH}^{[n]}(T(f); T(g))} \mathsf{THH}^{[n]}(T(g))$$
$$\simeq T(g) \wedge^L_{T(g) \wedge \Omega^\infty(S^n \wedge \mathbb{A})_+} T(g) \wedge \Omega^\infty(S^n \wedge \mathbb{B})_+.$$

Important examples of such factorizations are listed for instance in [Be, section 3]. For example we can consider $BSU \to BU$, $BU \to BSO$ or $B\mathrm{String} \to B\mathrm{Spin}$ to get $\mathsf{THH}^{[n],MSU}(MU)$, $\mathsf{THH}^{[n],MU}(MSO)$ or $\mathsf{THH}^{[n],M\mathrm{String}}(M\mathrm{Spin})$. As these examples give rise to Hopf-Galois extensions (but not Galois extensions) of ring spectra (see [R]), the above relative THH-terms will be non-trivial.

4.2. $\mathsf{THH}^{[n],HA}(H\mathbb{F}_p)$ for commutative pointed $\mathbb{F}_p$-monoid algebras A.

Hesselholt and Madsen [HM, Theorem 7.1] showed a splitting result for topological Hochschild homology of pointed monoid rings. There is a straightforward generalization of this splitting result to higher order topological Hochschild homology in the commutative case. Let Π be a discrete pointed commutative monoid, *i.e.*, a commutative monoid in the category of based spaces with smash product. Assume moreover that Π is augmented, that is, admits a map of pointed monoids to the pointed monoid $\{1, *\}$, where 1 is the unit and $*$ the base point. As long as $1 \neq *$ in Π, there always is such a map: we can send all invertible elements in the monoid to 1 and all the rest to $*$. In general, however, such an augmentation is not unique, so it needs to be part of the data. For a commutative ring A, Hesselholt-Madsen define the monoid algebra $A[\Pi]$ as the free A-module on the elements of Π modulo $A \cdot *$, with multiplication induced by Π's multiplication.

The analogue of [HM, Theorem 7.1] is a splitting of augmented commutative HA-algebras, for any commutative ring A:

$$(4.1.2) \qquad \mathsf{THH}^{[n]}(HA[\Pi]) \cong \mathsf{THH}^{[n]}(HA) \wedge_{HA} \mathsf{THH}^{[n],HA}(HA[\Pi]).$$

Note that $\mathsf{THH}^{[n],HA}(HA[\Pi])$ is equivalent to the smash product over the sphere spectrum of HA with the cyclic nerve of Π, and combining this with Remark 2.9, $\pi_*(\mathsf{THH}^{[n],HA}(HA[\Pi])) \cong \mathsf{HH}^{[n],A}_*(A[\Pi])$.

Theorem 4.2. *For any commutative algebra A and any augmented commutative pointed monoid Π with $* \neq 1$, for the monoid algebra $A[\Pi]$ there is a weak equivalence*

$$\mathsf{THH}^{[n],HA[\Pi]}(HA) \simeq HA \wedge^L_{\mathsf{THH}^{[n],HA}(HA[\Pi])} HA[\Pi]$$

of commutative augmented HA-algebras.

Proof. Theorem 3.2 applied to a model of the augmentation map $HA[\Pi] \to HA$ that is a cofibration yields that

$$(4.2.1) \qquad \mathsf{THH}^{[n],HA[\Pi]}(HA) \simeq \mathsf{THH}^{[n]}(HA) \wedge^L_{\mathsf{THH}^{[n]}(HA[\Pi])} HA[\Pi].$$

We use the two-sided bar construction as model for the above derived smash product and use the splittings in (4.1.2) and (4.2.1) to obtain

$$B(\mathsf{THH}^{[n]}(HA), \mathsf{THH}^{[n]}(HA[\Pi]), HA[\Pi])$$

$$\simeq B(\mathsf{THH}^{[n]}(HA), \mathsf{THH}^{[n]}(HA) \wedge_{HA} \mathsf{THH}^{[n],HA}(HA[\Pi]), HA[\Pi])$$

$$\simeq B(\mathsf{THH}^{[n]}(HA), \mathsf{THH}^{[n]}(HA), HA) \wedge_{HA} B(HA, \mathsf{THH}^{[n],HA}(HA[\Pi]), HA[\Pi])$$

$$\simeq HA \wedge_{HA} B(HA, \mathsf{THH}^{[n],HA}(HA[\Pi]), HA[\Pi])$$

which is a model of $HA \wedge^L_{\mathsf{THH}^{[n],HA}(HA[\Pi])} HA[\Pi]$. $\qquad \square$

We apply the above result to special cases of pointed commutative monoids, where we can identify the necessary ingredients for the above result.

Proposition 4.3.

(a) *Consider the polynomial algebra $\mathbb{F}_p[x]$ over $\mathbb{F}_p$ (with $|x| = 0$, augmented by sending $x \mapsto 0$), and let $B'_1(x) = \mathbb{F}_p[x]$ and $B'_{n+1}(x) = \mathsf{Tor}^{B'_n(x)}_{*,*}(\mathbb{F}_p, \mathbb{F}_p)$ with total grading. Then*

$$\mathsf{THH}^{[n],H\mathbb{F}_p[x]}_*(H\mathbb{F}_p) \cong B'_{n+2}(x).$$

(b) *Let m be a natural number such that p divides m, and let $B''_1(m) = \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$ with $|\varepsilon x| = 1$ and $|\varphi^0 x| = 2$ and $B''_{n+1}(m) = \mathsf{Tor}^{B''_n(m)}(\mathbb{F}_p, \mathbb{F}_p)$. Then*

$$\mathsf{THH}^{[n],H\mathbb{F}_p[x]/x^m}_*(H\mathbb{F}_p) \cong B''_{n+1}(m).$$

(c) *Let G be a finitely generated abelian group, so, $G = \mathbb{Z}^m \oplus \bigoplus_{i=1}^N \mathbb{Z}/q_i^{\ell_i}$ for some primes q_i. Then $\mathsf{THH}^{[n],H\mathbb{F}_p[G]}_*(H\mathbb{F}_p)$ can be expressed in terms of a tensor product of factors that are isomorphic to $\mathsf{THH}^{[n],H\mathbb{F}_p[x]}_*(H\mathbb{F}_p)$ or $\mathsf{THH}^{[n],H\mathbb{F}_p[x]/(x^{p^\ell})}_*(H\mathbb{F}_p)$ for some ℓ.*

Proof. We can rewrite $H\mathbb{F}_p \wedge^L_{\mathsf{THH}^{[n],H\mathbb{F}_p}(H\mathbb{F}_p[\Pi])} H\mathbb{F}_p[\Pi]$ as

$$H\mathbb{F}_p \wedge_{H\mathbb{F}_p[\Pi]} H\mathbb{F}_p[\Pi] \wedge^L_{\mathsf{THH}^{[n],H\mathbb{F}_p}(H\mathbb{F}_p[\Pi])} H\mathbb{F}_p[\Pi]$$

which is equivalent to

$$H\mathbb{F}_p \wedge_{H\mathbb{F}_p[\Pi]} \mathsf{THH}^{[n+1],H\mathbb{F}_p}(H\mathbb{F}_p[\Pi]).$$

In [BLPRZ] $\pi_*(\mathsf{THH}^{[n+1],H\mathbb{F}_p}(H\mathbb{F}_p[\Pi])) \cong \mathsf{HH}^{[n+1],\mathbb{F}_p}_*(\mathbb{F}_p[\Pi])$ is calculated in the cases of the Proposition:

For (a) we consider the pointed monoid $\Pi = \{0, 1, x, x^2, \ldots\}$ whose associated pointed monoid ring $\mathbb{F}_p[\Pi]$ is the ring of polynomials over $\mathbb{F}_p$. In [BLPRZ, Theorem 8.6], we show inductively that

$$\mathsf{HH}^{[n],\mathbb{F}_p}_*(\mathbb{F}_p[x]) \cong \mathbb{F}_p[x] \otimes B'_{n+1}(x).$$

We also get inductively that the augmentation on $\mathsf{HH}^{[n]}_*(\mathbb{F}_p[x])$ is the identity on the $\mathbb{F}_p[x]$ factor and for degree reasons the obvious augmentation on $B'_{n+1}(x)$. Therefore the claim follows.

Higher Hochschild homology of truncated polynomial algebras of the form $\mathbb{F}_p[x]/x^m$ for m divisible by p was calculated in [BLPRZ] (the case $m = p^\ell$) and [BHLPRZ] (the general case). The result in those cases is

$$\mathsf{HH}_*^{[n]}(H\mathbb{F}_p[x]/x^m) \cong \mathbb{F}_p[x]/x^m \otimes B_n''(m)$$

where $B_1''(m) = \Lambda_{\mathbb{F}_p}(\varepsilon x) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 x)$ with $|\varepsilon x| = 1$ and $|\varphi^0 x| = 2$ and where $B_{n+1}''(m) = \mathsf{Tor}^{B_n''(m)}(\mathbb{F}_p, \mathbb{F}_p)$. This implies (b).

For a finitely generated abelian group as in (c) the group ring splits as

$$\mathbb{F}_p[\mathbb{Z}]^{\otimes m} \otimes \bigotimes_{i=1}^N \mathbb{F}_p[\mathbb{Z}/q_i^{\ell_i}].$$

The torsion groups with torsion prime to p do not contribute to higher (topological) Hochschild homology because they are étale over $\mathbb{F}_p$ (see [BLPRZ, Theorem 9.1] and [H, Theorem 7.9]). For the free factors we use the fact that $\mathbb{F}_p[\mathbb{Z}] = \mathbb{F}_p[x^{\pm 1}]$ is étale over $\mathbb{F}_p[x]$; for the factors with torsion that is a power of p, we use the fact that $\mathbb{F}_p[\mathbb{Z}/p^\ell] \cong \mathbb{F}_p[y]/(y^{p^\ell} - 1) \cong \mathbb{F}_p[x]/(x^{p^\ell})$ by taking $x = y - 1$. $\qquad\square$

Remark 4.4. Let k be a commutative ring and let A be a commutative k-algebra. In [BHLPRZ] we define higher order Shukla homology of A over k as

$$\mathsf{Shukla}_*^{[n],k}(A) := \mathsf{THH}_*^{[n],Hk}(HA).$$

Thus the calculations above determine higher order Shukla homology for commutative pointed monoid algebras over $\mathbb{F}_p$, $\mathsf{Shukla}_*^{[n],\mathbb{F}_p[\Pi]}(\mathbb{F}_p)$.

4.3. The examples ko, ku, ℓ and tmf. Angeltveit and Rognes calculate in [AR, 5.13, 6.2] $H_*(\mathsf{THH}(ko); H\mathbb{F}_2)$, $H_*(\mathsf{THH}(tmf); H\mathbb{F}_2)$, $H_*(\mathsf{THH}(ku); H\mathbb{F}_2)$. In addition, for any odd prime p they determine $H_*(\mathsf{THH}(\ell); H\mathbb{F}_p)$ where $\ell \to ku_{(p)}$ is the Adams summand of p-local connective topological complex K-theory.

The following lemma collects some immediate consequences of their work, which were already noticed in [G]. These will be the basis of the calculations in the results that follow the lemma. The index of a generator denotes its degree.

Lemma 4.5.

(a)
$$\mathsf{THH}_*(ko; H\mathbb{F}_2) \cong \Lambda(x_5, x_7) \otimes \mathbb{F}_2[\mu_8].$$

(b)
$$\mathsf{THH}_*(tmf; H\mathbb{F}_2) \cong \Lambda(x_9, x_{13}, x_{15}) \otimes \mathbb{F}_2[\mu_{16}].$$

(c)
$$\mathsf{THH}_*(ku; H\mathbb{F}_2) \cong \Lambda(x_3, x_7) \otimes \mathbb{F}_2[\mu_8].$$

(d) *At any odd prime:*
$$\mathsf{THH}_*(\ell; H\mathbb{F}_p) \cong \Lambda(x_{2p-1}, x_{2p^2-1}) \otimes \mathbb{F}_p[y_{2p^2}].$$

Proof. In all four cases Angeltveit and Rognes show that $H_*(\mathsf{THH}(E); H\mathbb{F}_p)$ is of the form $H_*(E; \mathbb{F}_p) \otimes A_E$ with A_E as follows:

(4.5.1)
$$A_E = \begin{cases} \Lambda_{\mathbb{F}_2}(\sigma\bar\xi_1^4, \sigma\bar\xi_2^2) \otimes \mathbb{F}_2[\sigma\bar\xi_3], & E = ko, \\ \Lambda_{\mathbb{F}_2}(\sigma\bar\xi_1^8, \sigma\bar\xi_2^4, \sigma\bar\xi_3^2) \otimes \mathbb{F}_2[\sigma\bar\xi_4], & E = tmf, \\ \Lambda_{\mathbb{F}_2}(\sigma\bar\xi_1^2, \sigma\bar\xi_2^2) \otimes \mathbb{F}_2[\sigma\bar\xi_3], & E = ku, \text{ and} \\ \Lambda_{\mathbb{F}_p}(\sigma\bar\xi_1, \sigma\bar\xi_2) \otimes \mathbb{F}_p[\sigma\bar\tau_2], & E = \ell. \end{cases}$$

Here, $p = 2$ for $E = ko, tmf, ku$ and p is odd for $E = \ell$. The degrees are the usual degrees in the dual of the Steenrod algebra, hence at 2 we have $|\xi_i| = 2^i - 1$ and at odd primes $|\xi_i| = 2p^i - 2$

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and $|\tau_i| = 2p^i - 1$. We also have $|\sigma y| = |y| + 1$. The $\bar{(.)}$ denotes conjugation in the dual of the Steenrod algebra.

We rewrite $\pi_*(\mathsf{THH}(E; H\mathbb{F}_p))$ as

$$\pi_*(\mathsf{THH}(E; H\mathbb{F}_p)) \cong \pi_*(\mathsf{THH}(E) \wedge_E^L H\mathbb{F}_p)$$

$$\cong \pi_*((\mathsf{THH}(E) \wedge H\mathbb{F}_p) \wedge_{E \wedge H\mathbb{F}_p}^L H\mathbb{F}_p)$$

and thus we get a spectral sequence

$$E_{s,t}^2 = \mathsf{Tor}_{s,t}^{H_*(E;\mathbb{F}_p)}(H_*(\mathsf{THH}(E); \mathbb{F}_p), \mathbb{F}_p)$$

converging to the homotopy groups of $\mathsf{THH}(E; H\mathbb{F}_p)$. As $H_*(\mathsf{THH}(E); \mathbb{F}_p) \cong H_*(E; \mathbb{F}_p) \otimes A_E$ in all four cases, the E^2-term above is concentrated in the $s = 0$ column with

$$E_{0,*}^2 \cong A_E.$$

Counting degrees gives the claim. $\qquad\square$

We can use the equivalence $\mathsf{THH}^A(B) \simeq B \wedge_{\mathsf{THH}(A;B)}^L \mathsf{THH}(B)$ from Theorem 3.2 to deduce the following result.

Theorem 4.6. *There are additive isomorphisms*

(a) $\mathsf{THH}_*^{ko}(H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_5) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \mathbb{F}_2[\mu_2]/\mu_2^4$,
(b) $\mathsf{THH}_*^{tmf}(H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_9) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_{13}) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_{15}) \otimes \mathbb{F}_2[\mu_2]/\mu_2^8$,
(c) $\mathsf{THH}_*^{ku}(H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_3) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \mathbb{F}_2[\mu_2]/\mu_2^4$, *and*
(d) $\mathsf{THH}_*^{\ell}(H\mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(\rho^0 x_{2p-1}) \otimes \Gamma_{\mathbb{F}_p}(\rho^0 x_{2p^2-1}) \otimes \mathbb{F}_p[\mu_2]/\mu_2^{p^2}$ *when p is odd.*

Here ρ^0 raises degree by one.

Proof. We use Theorem 3.2 in the case where $B = H\mathbb{F}_p$. In [B], Bökstedt shows that $\mathsf{THH}_*(H\mathbb{F}_p) \cong \mathbb{F}_p[\mu_2]$ for all primes p. We give the details for the case ko; the arguments for the other examples are completely analogous.

The E^2-term of the spectral sequence is

$$\mathsf{Tor}_*^{\Lambda(x_5, x_7) \otimes \mathbb{F}_2[\mu_8]}(\mathbb{F}_2, \mathbb{F}_2[\mu_2]) \implies \mathsf{THH}_*^{ko}(H\mathbb{F}_2).$$

Since both x_5 and x_7 have odd degrees, they cannot act on μ_2 other than trivially. Thus we can rewrite the left-hand side as

$$\mathsf{Tor}_*^{\Lambda(x_5, x_7)}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathsf{Tor}_*^{\mathbb{F}_2[\mu_8]}(\mathbb{F}_2, \mathbb{F}_2[\mu_2]).$$

The explicit description of the generators in [AR, Theorem 6.2] implies that the map $\mathbb{F}_2[\mu_8] \to \mathbb{F}_2[\mu_2]$ takes μ_8 to μ_2^4, because μ_8 corresponds to $\sigma\bar{\xi}_3$ and Angeltveit and Rognes show [AR, proof of 5.12] that the $\sigma\bar{\xi}_k$ satisfy

$$(\sigma\bar{\xi}_k)^2 = \sigma\bar{\xi}_{k+1}$$

for $p = 2$ and μ_2 in Bökstedt's calculation corresponds to $\sigma\bar{\xi}_1$.

Therefore the right-hand Tor is isomorphic to $\mathbb{F}_2[\mu_2]/\mu_2^4$. Hence the E^2-term is isomorphic to

$$\Gamma_{\mathbb{F}_2}(\rho^0 x_5) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \mathbb{F}_2[\mu_2]/\mu_2^4.$$

Since all the nonzero classes in this E^2-term have even total degree, the spectral sequence must collapse at E^2.

In the case of the Adams summand, ℓ, we work at odd primes and here in [AR, proof of 5.12] the relation

$$(\sigma\bar{\tau}_k)^p = \sigma\bar{\tau}_{k+1}$$

is shown. Hence $\sigma\bar{\tau}_2$ in $\mathsf{THH}_*(\ell; H\mathbb{F}_p)$ corresponds to $(\sigma\bar{\tau}_0)^{p^2}$ and $\sigma\bar{\tau}_0$ is the element that represents μ_2 at odd primes. $\qquad\square$

Remark 4.7. In order to determine for instance $\mathsf{THH}^{ko}(H\mathbb{F}_2)$ multiplicatively we would also need to control possible multiplicative extensions. Using the notation $\rho^k a$ to denote the generator $(\rho^0 a)^{(2^k)} \in \Gamma_{\mathbb{F}_2}(\rho^0 a)$, we can show by degree considerations that $(\rho^k x_7)^2 = 0$, but not whether $(\rho^k x_5)^2$ must vanish.

As a general warning we discuss the case of the bar spectral sequence in the case $\mathsf{THH}_*(ku; H\mathbb{F}_2)$. Here, we know the answer from Lemma 4.5:

$$\mathsf{THH}_*(ku; H\mathbb{F}_2) \cong \Lambda(x_3, x_7) \otimes \mathbb{F}_2[\mu_8].$$

However, if we use the bar spectral sequence we get as the E^2-term

$$\mathsf{Tor}^{\mathsf{THH}^{[0]}_*(ku; H\mathbb{F}_2)}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathsf{Tor}^{\mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_k, k \geqslant 3]}_{*,*}(\mathbb{F}_2, \mathbb{F}_2)$$

because

$$\mathsf{THH}^{[0]}_*(ku; H\mathbb{F}_2) \cong H_*(ku; \mathbb{F}_2) \cong \mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_k, k \geqslant 3]$$

(see for instance [AR, 6.1]). Hence the spectral sequence collapses because

$$E^2_{*,*} \cong \Lambda_{\mathbb{F}_2}(\sigma \bar{\xi}_1^2, \sigma \bar{\xi}_2^2, \sigma \bar{\xi}_k, k \geqslant 3)$$

and all generators are concentrated on the 1-line. But we know that the exterior generators in $\Lambda_{\mathbb{F}_2}(\sigma \bar{\xi}_k, k \geqslant 3)$ extend to form $\mathbb{F}_2[\mu_8]$, so there are highly non-trivial multiplicative extensions in this spectral sequence.

Remark 4.8. Veen established a Hopf-structure on the bar spectral sequence [V, §7] for higher order THH of $H\mathbb{F}_p$. His argument generalizes: The pinch maps $\mathbb{S}^n \to \mathbb{S}^n \vee \mathbb{S}^n$ give rise to a comultiplication

$$\mathsf{THH}^{[n]}(A; C) \to \mathsf{THH}^{[n]}(A; C) \wedge_C \mathsf{THH}^{[n]}(A; C)$$

and as the multiplication on $\mathsf{THH}^{[n]}(A; C)$ is induced by the fold map, both structures are compatible. For $\mathsf{THH}(A)$ this structure is heavily used in [AR].

If A is connective, then we can consider $A \to H(\pi_0 A)$. For $C = H\mathbb{F}_p$ this multiplication and comultiplication turns $\mathsf{THH}^{[n]}_*(A; H\mathbb{F}_p)$ into an $\mathbb{F}_p$-Hopf algebra. Veen's arguments also transfer to yield that the bar spectral sequence

$$\mathsf{Tor}^{\mathsf{THH}^{[n]}_*(A; H\mathbb{F}_p)}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \mathsf{THH}^{[n+1]}_*(A; H\mathbb{F}_p)$$

is a spectral sequence of Hopf-algebras; in particular, the differentials satisfy a Leibniz and a co-Leibniz rule and these facts let us determine the differentials in certain cases.

Theorem 4.9. *Additively,*

$$\mathsf{THH}^{[2]}(ko; H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_5) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \Lambda_{\mathbb{F}_2}(\varepsilon \mu_8).$$

Here, the degrees are $|\rho^0 x_5| = 6$, $|\rho^0 x_7| = 8$ *and* $|\varepsilon \mu_8| = 9$.

Proof. Using the Tor spectral sequence we get

$$\mathsf{Tor}^{\Lambda(x_5) \otimes \Lambda(x_7) \otimes \mathbb{F}_2[\mu_8]}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \mathsf{THH}^{[2]}_*(ko, H\mathbb{F}_2).$$

The E^2 page of the spectral sequence is of the form $\Gamma_{\mathbb{F}_2}(\rho^0 x_5) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \Lambda_{\mathbb{F}_2}(\epsilon \mu_8)$. This, in turn, is isomorphic to

$$\bigotimes_{k=0}^{\infty} \underbrace{\mathbb{F}_2(\rho^k x_5)/(\rho^k x_5)^2}_{\Lambda_{\mathbb{F}_2}(\rho^k x_5)} \otimes \bigotimes_{\ell=0}^{\infty} \underbrace{\mathbb{F}_2(\rho^k x_7)/(\rho^k x_7)^2}_{\Lambda_{\mathbb{F}_2}(\rho^k x_7)} \otimes \Lambda_{\mathbb{F}_2}(\epsilon \mu_8),$$

with bidegrees $||\epsilon \mu_8|| = (1, 8)$, and $||\rho^k x_i|| = (2^k, 2^k i)$. The claim is that the spectral sequence collapses at E^2.

As the spectral sequence above is a spectral sequence of Hopf algebras, the smallest nonzero differential must go from an indecomposable element to a primitive element. As the only primitive element in $\Gamma_{\mathbb{F}_2}(\rho^0 x_i)$ is $\rho^0 x_i$ we just need to check that no differentials hit $\rho^0 x_i$, $i = 5, 7$, or $\epsilon \mu_8$. These have bidegrees $(1, i)$ and $(1, 8)$, respectively, and thus if they are hit by

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d^r of an indecomposable $d^r(\rho^k x_j)$, which would have bidegree $(2^k - r, 2^k j + r - 1)$, we must have $2^k - r = 1$ and $2^k j + r - 1 = 5, 7,$ or 8. Substituting $r = 2^k - 1$ into the second expression, it becomes $2^k(j + 1) - 2$, where clearly $k \geqslant 1$ and $j = 5$ or 7, making $2^k(j + 1) - 2 \geqslant 10$. So the spectral sequence collapses at E^2.

$\square$

So far, we cannot rule out non-trivial multiplicative extensions. These could be nontrivial only if there is a multiplicative generator $\rho^k x_i$ or $\epsilon \mu_8$ with whose square is not zero in $\mathsf{THH}_*^{[2]}(ko, H\mathbb{F}_2)$ (although it is zero in the E^∞ term). This is possible only if $(\rho^k x_i)^2$ has filtration degree less than 2^{k+1}, or $(\epsilon \mu_8)^2$ has filtration degree less than 2. The latter is clearly impossible, since there is nothing in total degree 18 in filtration degrees 0 or 1. If $i = 7$, by similar arguments we cannot have anything in total degree $2^{k+1} \cdot 8$ in filtration degree less than 2^{k+1}: it would have to be constructed out of elements in bidegrees $(a, 5a)$, $(b, 7b)$ and possibly also one occurrence of the element of bidegree $(1, 8)$, but then the first coordinates would have to add up to at least 2^{k+1} to have the total degree equal to $2^{k+1} \cdot 8$. However, for the cases $(\rho^k x_5)^2$ one cannot rule out extensions just for degree reasons.

In a similar manner as above, we can exclude non-trivial differentials in the other three cases:

Proposition 4.10. *There are additive isomorphisms*

(a) $\mathsf{THH}_*^{[2]}(ku; H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_3) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_7) \otimes \Lambda_{\mathbb{F}_2}(\varepsilon \mu_8),$

(b) $\mathsf{THH}_*^{[2]}(tmf; H\mathbb{F}_2) \cong \Gamma_{\mathbb{F}_2}(\rho^0 x_9) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_{13}) \otimes \Gamma_{\mathbb{F}_2}(\rho^0 x_{15}) \otimes \Lambda_{\mathbb{F}_2}(\varepsilon \mu_{16}),$

(c) *and for any odd prime p we get an additive isomorphism*

$$\mathsf{THH}_*^{[2]}(\ell; H\mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(\rho^0 x_{2p-1}) \otimes \Gamma_{\mathbb{F}_p}(\rho^0 x_{2p^2-1}) \otimes \Lambda_{\mathbb{F}_p}(\varepsilon \mu_{2p^2}).$$

Proof.

(a) In the case of ku at the even prime we get a degree constraint for a differential $d^r(\rho^k x_i)$ of the form
$$(2^k - r, 2^k i + r - 1) = (1, j)$$
where j is $3, 7$ or 8. Since $r \geqslant 2$ and $2^k - r = 1$, we get $k \geqslant 2$, but that would make the internal degree at least $4(i + 1) - 2$ and this is bigger or equal to 14, hence doesn't occur.

(b) For tmf the degree constraint is
$$(2^k - r, 2^k i + r - 1) = (1, j)$$
where j is $9, 13, 15$ or 16. Again $r \geqslant 2$ and $2^k - r = 1$ imply that $k \geqslant 2$, which makes the internal degree at least 38.

(c) For the Adams summand ℓ the degree condition is
$$(p^k - r, p^k i + r - 1) = (1, j)$$
where $j = 2p - 1, 2p^2 - 1$ or $2p^2$. As before, we get that $k \geqslant 2$ and therefore we get an internal degree of at least $2p^3 - 2$ which is too big to be the degree of a primitive element.

In all cases the differentials d^r, $r \geqslant 2$, all have to be trivial and we get the result. $\square$

4.4. **A splitting for $\mathsf{THH}^{[n], Hk}(HA)$ for commutative k-algebras A.** We apply Theorem 3.6 for a sequence of cofibrations of commutative S-algebras of the form $S \to A \to B = C$. As $\mathsf{THH}^{[n-1], B}(B) \simeq B$ we obtain a weak equivalence

(4.10.1)
$$\mathsf{THH}^{[n]}(B) \simeq \mathsf{THH}^{[n]}(A; B) \wedge_{\mathsf{THH}^{[n-1], A}(B)}^L B.$$

In the special case of a sequence $S \to Hk \to HA = HA$ where A is a commutative k-algebra the formula in (4.10.1) specializes to the following result.

Proposition 4.11. *For all commutative rings k and all commutative k-algebras A the higher topological Hochschild homology of HA splits as*

$$\mathsf{THH}^{[n]}(HA) \simeq \mathsf{THH}^{[n]}(Hk; HA) \wedge^{L}_{\mathsf{Shukla}^{[n-1],k}(A)} HA.$$

If A is flat as a k-module, then higher Shukla homology reduces to higher Hochschild homology and we obtain

$$\mathsf{THH}^{[n]}(HA) \simeq \mathsf{THH}^{[n]}(Hk; HA) \wedge^{L}_{\mathsf{HH}^{[n-1],k}(A)} HA.$$

In particular, this gives splitting results for number rings: For $k = \mathbb{Z}$ and $A = \mathcal{O}_K$ a ring of integers in a number field we get

$$\mathsf{THH}^{[n]}(H\mathcal{O}_K) \simeq \mathsf{THH}^{[n]}(H\mathbb{Z}; H\mathcal{O}_K) \wedge^{L}_{\mathsf{HH}^{[n-1],\mathbb{Z}}(\mathcal{O}_K)} H\mathcal{O}_K.$$

The (topological) Hochschild homology of $\mathcal{O}_K$ is known (see [LL, LM]). However, the additive and multiplicative structure of these is complicated enough that we cannot use the iteration methods of [BLPRZ, DLR] and so we do not know the higher order topological Hochschild homology of $\mathcal{O}_K$ with unreduced coefficients so far, nor its higher Shukla homology.

Remark 4.12. Beware that the splitting

$$\mathsf{THH}^{[n]}(HA) \simeq \mathsf{THH}^{[n]}(Hk; HA) \wedge^{L}_{\mathsf{Shukla}^{[n-1],k}(A)} HA$$

$$\simeq (\mathsf{THH}^{[n]}(Hk) \wedge_{Hk} HA) \wedge^{L}_{\mathsf{Shukla}^{[n-1],k}(A)} HA$$

cannot be rearranged to

$$\mathsf{THH}^{[n]}(Hk) \wedge_{Hk} (HA \wedge^{L}_{\mathsf{Shukla}^{[n-1],k}(A)} HA) = \mathsf{THH}^{[n]}(Hk) \wedge_{Hk} \mathsf{Shukla}^{[n],k}(A)$$

because the $\mathsf{Shukla}^{[n-1],k}(A)$-action on $\mathsf{THH}^{[n]}(Hk; HA)$ does not usually factor through an action on the coefficients HA. If we could rearrange it that way, it would imply that $\mathsf{Shukla}^{[n],k}(A)$ splits off $\mathsf{THH}^{[n]}(HA)$, which is not true even for $n = 1$: for example, if we take $k = \mathbb{Z}$ and $A = \mathbb{Z}[i]$, since $\mathsf{THH}(H\mathbb{Z})$ as the topological Hochschild homology of a ring is equivalent to a product of Eilenberg Mac Lane spectra, which Bökstedt [B] identified to be

$$\mathsf{THH}(H\mathbb{Z}) \simeq H\mathbb{Z} \times \prod_{a=2}^{\infty} \Sigma^{2a-1} H(\mathbb{Z}/a\mathbb{Z}),$$

then we get the formula

$$\pi_*(\mathsf{THH}(H\mathbb{Z}) \wedge_{H\mathbb{Z}} \mathsf{Shukla}(\mathbb{Z}[i]))$$

$$\cong \pi_*(\mathsf{Shukla}(\mathbb{Z}[i])) \oplus \bigoplus_{a=2}^{\infty} \pi_*(\Sigma^{2a-1} H(\mathbb{Z}/a\mathbb{Z}) \wedge_{H\mathbb{Z}} \mathsf{Shukla}(\mathbb{Z}[i]))$$

$$\cong \mathsf{HH}_*(\mathbb{Z}[i]) \oplus \bigoplus_{a=2}^{\infty} \Big(\mathsf{HH}_{*-2a+1}(\mathbb{Z}[i]) \otimes \mathbb{Z}/a\mathbb{Z} \oplus \mathsf{Tor}(\mathsf{HH}_{*-2a}(\mathbb{Z}[i]), \mathbb{Z}/a\mathbb{Z}) \Big)$$

where $\mathsf{HH}_*(\mathbb{Z}[i]) = 0$ when $* < 0$. We also know that $\mathsf{HH}_0(\mathbb{Z}[i]) \cong \mathbb{Z}[i]$, $\mathsf{HH}_{2a-1}(\mathbb{Z}[i]) \cong \mathbb{Z}[i]/2\mathbb{Z}[i]$, and the positive even groups vanish. Thus the number of copies of $\mathbb{Z}/2\mathbb{Z}$'s in $\pi_n(\mathsf{THH}(H\mathbb{Z}) \wedge_{H\mathbb{Z}} \mathsf{Shukla}(\mathbb{Z}[i]))$ grows linearly with n. On the other hand, by [L], $\mathsf{THH}_0(\mathbb{Z}[i]) \cong \mathbb{Z}[i]$, $\mathsf{THH}_{2a-1}(\mathbb{Z}[i]) \cong \mathbb{Z}[i]/2a\mathbb{Z}[i]$, and the positive even groups vanish.

Such a splitting of $\mathsf{Shukla}^{[n],k}(A)$ off $\mathsf{THH}^{[n]}(HA)$ *does* hold under additional assumptions, for instance in the case of commutative pointed monoid rings (see (4.1.2) above).

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