1) <u>Hopf alg.</u>

k : field

<u>Def</u> Hopf alg H is

- unital assoc. k-alg
- $\Delta : H \to H \otimes H$ <u>coproduct</u> (coassoc.)
- $\varepsilon : H \to k$ <u>counit</u>
- $S : H \to H$ <u>antipode</u>

} alg. hom.

$\Delta(ab) = \Delta(a)\Delta(b)$

$\Delta(a) = \sum a' \otimes a''$

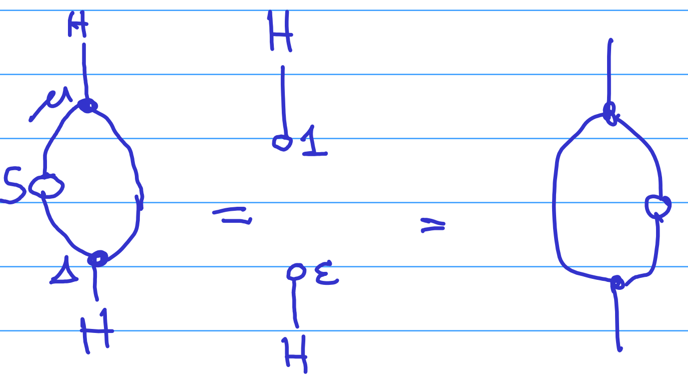
$\Delta(a)\Delta(b) = \sum a'b' \otimes a''b''$

s.th. $\forall a \in H$:

$$\left.\begin{aligned} \sum S(a')a'' &= \varepsilon(a)\cdot 1 \\ &= \sum a' S(a'') \end{aligned}\right]$$

———— , ————

string diagram (in Vec_k)



<u>Prop</u> S is an alg. antihom. $S(ab) = S(b)S(a)$

<u>Examples</u>

1) group alg. $k[G]$ G : finite group

$\uparrow$ k-v.sp. with basis G $\{e_g\}_{g \in G}$

$$e_g \cdot e_h = e_{g \cdot h}$$

$\Delta(e_g) = e_g \otimes e_g \qquad \varepsilon(e_g) = 1$

$S(e_g) = e_{g^{-1}} \qquad \rightsquigarrow$ cocomm.

2) function alg $\mathrm{Fun}(G,k) \qquad \varphi, \psi \in \mathrm{Fun}(G,k)$

$(\varphi \cdot \psi)(g) := \varphi(g)\psi(g) \qquad \rightsquigarrow$ comm.

$1(g) = 1$

$(\Delta(\varphi))(g,h) := \varphi(gh) \qquad \varepsilon(\varphi) = \varphi(e)$

$S(\varphi) := (g \longmapsto \varphi(g^{-1}))$

for G finite

$\mathrm{Fun}(G,k) \otimes \mathrm{Fun}(G,k) \xrightarrow{\sim} \mathrm{Fun}(G \times G, k)$

$\varphi \otimes \psi \longmapsto ((g,h) \mapsto \varphi(g)\psi(h))$

$\sum S(a')a'' = \varepsilon(a) \cdot 1$

$\varphi \in \mathrm{Fun}(G,k)$

$$\left(\sum S(\varphi') \cdot \varphi''\right)(g) = \sum \underbrace{(S(\varphi'))(g)}_{=\varphi'(g^{-1})} \varphi''(g)$$

$$= \Delta(\varphi)(g^{-1}, g) = \varphi(g^{-1}g) = \varphi(e) = \varepsilon(\varphi)$$

$g \longmapsto \varepsilon(\varphi) \qquad \varepsilon(\varphi) \cdot 1$

3) $H = u_q(sl_2)$

6.1. **Small quantum group.** Let us set $q = e^{\frac{2\pi i}{r}}$ for some odd integer $r \geqslant 3$. We recall that $\bar{U}_q\mathfrak{sl}_2$ is constructed as the $\mathbb{C}$-algebra with generators $\{E, F, K\}$ and relations

$$E^r = F^r = 0, \qquad K^r = 1,$$

$$KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F, \qquad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

A Hopf algebra structure on $\bar{U}_q\mathfrak{sl}_2$ is obtained by setting

$$\begin{aligned} \Delta(E) &= E \otimes K + 1 \otimes E, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1}, \\ \Delta(F) &= K^{-1} \otimes F + F \otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF, \\ \Delta(K) &= K \otimes K, & \varepsilon(K) &= 1, & S(K) &= K^{-1}. \end{aligned}$$

A basis of $\bar{U}_q\mathfrak{sl}_2$ is given by

$$\{E^a F^b K^c \mid 0 \leqslant a, b, c \leqslant r - 1\}.$$

H-mod

obj: H-modules k-v.sp. M

action $H \otimes M \xrightarrow{\rho} M$

$(h, m) \longmapsto h.m$

fin. dim.

morph: $M \xrightarrow{f} N$ interfw.

$\forall m \in M, h \in H$: $f(h.m) = h.f(m)$

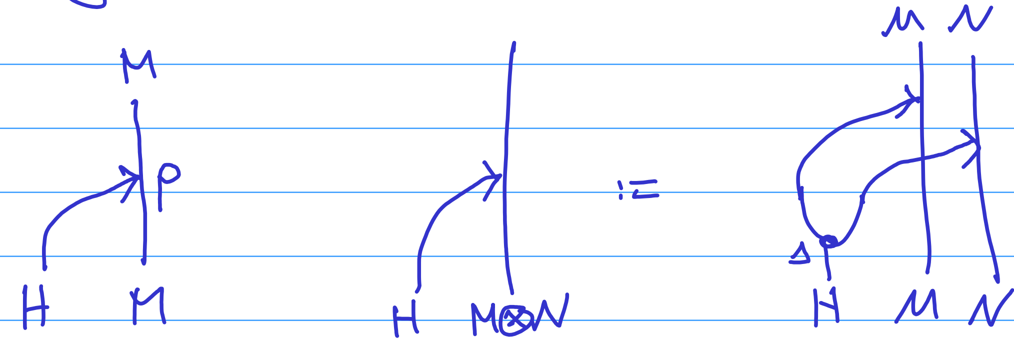
Functor $H\text{-mod} \times H\text{-mod} \longrightarrow H\text{-mod}$

$(M, N) \longmapsto M \otimes_k N$

with $(m \in M, n \in N)$

$h.(m \otimes n) := \sum h'.m \otimes h''.n$

Str. diag.



$$(f, g) \longmapsto f \otimes_k g$$

Is an intertw. :

$\forall h \in H, m \in M, n \in N,$

$$(f\otimes g)\big(h.(m\otimes n)\big) = \sum (f\otimes g)(h'm \otimes h''n)$$

$$= \sum f(h'm) \otimes g(h''n) = h.\big(f(m) \otimes g(n)\big)$$

$$= h.\big((f\otimes g)(m\otimes n)\big) \quad \checkmark$$

2) Monoidal cat.

Def: A <u>monoidal cat.</u> is $(C, \otimes, 1, \alpha, \lambda, \rho)$

C : cat

$\otimes : C \times C \to C$ functor, $1 \in C$

E.g. H-mod, $M \otimes N$ as before

$1 = k$ with action $x \in k, h \in H$

$h.x := \varepsilon(h) x$

$$\alpha : \begin{tikzcd} & C \times C \arrow[dr, "\otimes"] \arrow[dd, phantom, "\Downarrow\alpha"] & \\ C \times C \times C \arrow[ur, "id \times \otimes"] \arrow[dr, "\otimes \times id"'] & & C \\ & C \times C \arrow[ur, "\otimes"'] & \end{tikzcd}$$

$\alpha_{U,V,W} : U \otimes (V \otimes W) \xrightarrow{\sim} (U \otimes V) \otimes W$

$$\lambda, \rho : \begin{tikzcd} C \arrow[r, bend left=60, "1 \otimes (-)"] \arrow[r, "id"] \arrow[r, bend right=60, "(\ ) \otimes 1"'] & C \end{tikzcd} \quad \Downarrow\lambda, \ \Uparrow\rho$$

$\lambda_U : 1 \otimes U \xrightarrow{\sim} U$

$\rho_U : U \otimes 1 \xrightarrow{\sim} U$

E.g. H-mod α is <u>from vec_k</u>

λ, ρ 〃

$\lambda_M : k \otimes_k M \longrightarrow M$

$x \otimes m \longmapsto xm$

intertw. $\lambda_M(\underbrace{h.(x \otimes m)}_{= h'x \otimes h''m})$

$$\overset{!}{=} \varepsilon(h')x \otimes h''m$$
$$= x \otimes h.m$$
$$\lambda_M(x \otimes h.m) = x\,h.m = h.(xm)$$
$$= h.\,\lambda_M(x \otimes m) \qquad \checkmark$$

s.th.

Pentagon

$$\begin{tikzcd}[column sep=small]
 & u(v(w\,x)) \arrow[dl] \arrow[dr] & \\
(uv)(wx) \arrow[dd] & & u((v\,w)\,x) \arrow[d] \\
 & & (u\,(vw))\,x \arrow[dl] \\
 & ((u\,v)\,w)\,x &
\end{tikzcd}$$

Triangle

$$\begin{tikzcd}
v(I\,w) \arrow[rr] \arrow[dr] & & (vI)w \arrow[dl] \\
 & vw &
\end{tikzcd}$$

<u>Prop</u> H-mod is a mon. cat.

$$
\begin{array}{c}
((((X_1 \, 1) \, X_2)((X_3 \, (1 \, (1 \, X_4))) \, X_5)) \, X_6) \, 1 \\
\Big\downarrow{\scriptstyle \alpha, \lambda, \rho} \\
((1 \, 1)((X_1 \, 1) \, X_2))(((X_3 \, X_4)(X_5 \, 1)) \, X_6)
\end{array}
$$

MacLane coherence thm.

3) Braided mon. cat.

Def: A braiding on a mon. cat. is a nat iso.

$$\begin{tikzcd} C \times C \arrow[r, "\otimes"] \arrow[d, "\text{swap}"'] \arrow[dr, phantom, "\Downarrow c"] & C \\ C \times C \arrow[ur, "\otimes"'] & \end{tikzcd}$$

$$c_{U,V} : U \otimes V \xrightarrow{\sim} V \otimes U$$

s.th. two hexagons hold

$$\begin{tikzcd} & (UV)W \arrow[r, "c_{UV,W}"] & W(UV) \arrow[dr] & \\ U(VW) \arrow[ur] \arrow[dr, "id\, c_{V,W}"'] & & & (WU)V \\ & U(WV) \arrow[r] & (UW)V \arrow[ur, "c_{U,W}\, id"'] & \end{tikzcd}$$

$$\begin{tikzcd} & U(VW) \arrow[r, "c_{U,VW}"] & (VW)U \arrow[dr] & \\ (UV)W \arrow[ur, "\alpha^{-1}"] \arrow[dr, "c_{U,V}\, id"'] & & & V(WU) \\ & (VU)W \arrow[r] & V(UW) \arrow[ur, "id\, c_{U,W}"'] & \end{tikzcd}$$

E.g. H-mod:

Want $M \otimes_k N \longrightarrow N \otimes_k M$

Try 1 $\quad \tau_{M,N} : M \otimes_k N \longrightarrow N \otimes_k M$

$m \otimes n \longmapsto n \otimes m$

Intertw.? $\quad \tau_{M,N}(\underbrace{h.(m \otimes n)}_{= \sum h'm \otimes h''n})$

$= \sum h''.n \otimes h'm \qquad (*)$

$h.\tau(m \otimes n) = \sum h'.n \otimes h''.m \qquad (**)$

Choose $M = N = H \leftarrow h, a \in H \quad h.a = h \cdot a$ $\quad {}_H H$

$m = 1^{\in H}, \; n = 1$

$(*) = \sum h'' \otimes h' = \Delta^{op}(h)$

$(**) = \Delta(h)$

$\rightsquigarrow$ would need H cocomm.

Try 2 Suppose we are given braiding c on H-mod.

Write

nat. xfer in H-mod

$$\begin{tikzcd}
M\otimes N \arrow[rr, "c_{M,N}"] \arrow[dr, "r_{M,N}"'] & & N\otimes M \arrow[dl, bend left] \\
& M\otimes N \arrow[ur, "\tau_{M,N}"'] &
\end{tikzcd}$$

only lin., not intertw.

nat xfer in vect

$\underline{r_{M,N}\,?}$

$$r_{H,H} : H\otimes H \longrightarrow H\otimes H$$
$$1\otimes 1 \longmapsto R$$

$$c_{H,H}(1\otimes 1) = \tau(R) = R_{21}$$

$$R = \sum_i R_i^1 \otimes R_i^2$$

$$R_{21} = \sum_i R_i^2 \otimes R_i^1$$

Let M, N be in H-mod.

Let $m \in M$, $n \in N$

$$f : H \longrightarrow M \qquad\qquad g : H \longrightarrow N$$
$$h \longmapsto h.m \qquad\qquad h \longmapsto h.n$$

are interdw.

$$\begin{tikzcd}
H\otimes H \arrow[rr, "c_{HH}"] \arrow[dr, "r_{HH}"] \arrow[dd, "f\otimes g"'] \arrow[rr, phantom, shift right=3, "\#"] & & H\otimes H \arrow[dd, "g\otimes f"] \\
& H\otimes H \arrow[ur, "\tau_{HH}"] \arrow[d, "f\otimes g"] \arrow[r, phantom, "\#"] & \\
M\otimes N \arrow[rr, "c_{M,N}"'] \arrow[r, "r_{MN}"] \arrow[rr, phantom, shift left=3, "\#"] & M\otimes N \arrow[r, "\tau_{MN}"] & N\otimes M
\end{tikzcd}$$

hence this commutes!

$$\begin{tikzcd}
1\otimes 1 \arrow[r, mapsto] \arrow[d, mapsto] & R = \sum R^1\otimes R^2 \arrow[d, mapsto] \\
& \sum f(R^1)\otimes g(R^2) \\
\underset{= m\otimes n}{f(1)\otimes g(1)} \arrow[r, mapsto] & r_{M,N}(m\otimes n) \overset{(*)}{=} R^1.m\otimes R^2.n
\end{tikzcd}$$

$$(*) \quad f(R^1) = f(R^1\cdot 1) = R^1.f(1) = R^1.m$$

Thus: $\quad r_{M,N}(m\otimes n) = R.(m\otimes n)$

braiding is: $\quad c_{M,N}(m\otimes n) = \sum R_2.n\otimes R_1.m$

Def: A <u>quasi-triangular Hopf alg</u> is a Hopf alg. together with an <u>R-matrix</u> $R \in H \otimes H$ s.th.

c_{MN} is iso & intertw.

- R is invertible, $\forall h$: $\Delta^{op}(h) = R\,\Delta(h)\,R^{-1}$
- $(\Delta \otimes id)(R) = R_{13} R_{23} = \sum_{i,j} R_i^1 \otimes R_j^1 \otimes R_i^2 R_j^2$
- $(id \otimes \Delta)(R) = R_{13} R_{12}$

Prop H q. tr. H. alg $\Rightarrow$ H-mod br. mon. with braiding $(*)$.

Prop If c is a braiding on H-mod, then $\exists$ R-matrix $R \in H \otimes H$ s.th.

$$c_{MN}(m \otimes n) = \sum R^2 n \otimes R^1 m \qquad (*)$$

E.g.

- group alg $k[G]$ $\qquad \Delta(e_g) = e_g \otimes e_g = \Delta^{op}(e_g)$

Can choose $R = 1 \otimes 1$

Criterion for non-existence of R-matrix: Find two H-mod. M, N s.th. $M \otimes N \not\cong N \otimes M$ as H-mod.

⚠ $\exists H$ s.th. $M \otimes N \simeq N \otimes M$ for all H-mod M, N
but H-mod not braidable

- function alg $\mathrm{Fun}(G, k)$

Ex braiding on H-mod exists iff G abelian.

so: let G be abelian

pick $\gamma : G \times G \longrightarrow k$ bihom.

$$R = \sum_{g,h \in G} \gamma(g,h)\, \delta_g \otimes \delta_h$$

$$\delta_g : G \longrightarrow k$$
$$h \longmapsto \begin{cases} 1 & ; g = h \\ 0 & : \text{else} \end{cases}$$

- $u_q sl_2$

A *pivotal element* $g \in \bar{U}_q\mathfrak{sl}_2$ is given by $g := K$, and an *R-matrix* $R \in \bar{U}_q\mathfrak{sl}_2 \otimes \bar{U}_q\mathfrak{sl}_2$ is given by

$$R := \frac{1}{r} \sum_{a,b,c=0}^{r-1} \frac{\{1\}^a}{[a]!} q^{\frac{a(a-1)}{2} - 2bc} K^b E^a \otimes K^c F^a.$$

A *ribbon element* $v \in \bar{U}_q\mathfrak{sl}_2$ is given by

$$v := \frac{i^{\frac{r-1}{2}}}{\sqrt{r}} \sum_{a,b=0}^{r-1} \frac{\{-1\}^a}{[a]!} q^{-\frac{a(a-1)}{2} + \frac{(r+1)(a-b-1)^2}{2}} F^a K^b E^a.$$

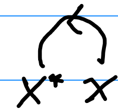
$$\{k\} := q^k - q^{-k}, \quad [k] := \frac{\{k\}}{\{1\}}, \quad [k]! := [k][k-1]\cdots[1].$$

4) Ribbon categories

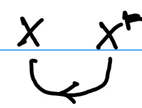
Duals $\mathcal{C}$: mon cat. , $X \in \mathcal{C}$

A (left) dual of X is $X^* \in \mathcal{C}$

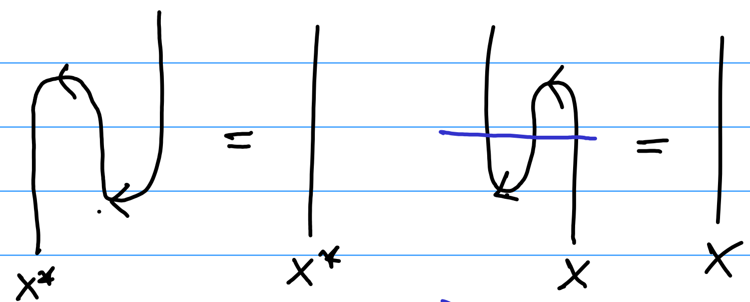
$ev_X : X^* \otimes X \longrightarrow 1$



$coev_X : 1 \longrightarrow X \otimes X^*$



s.th.



$$id_X = \left[X \xrightarrow{\lambda^{-1}} 1X \xrightarrow{coev_X\, id} (XX^*)X \xrightarrow{\alpha} X(X^*X) \xrightarrow{id\, ev_X} X1 \xrightarrow{\rho} X \right]$$

E.g. H-mod.

M $\qquad$ $M^* \ni \varphi$ dual v.sp. $\qquad$ $(h.\varphi)(m) := \varphi(S(h).m)$

Def A <u>ribbon cat.</u> is a br. mon. cat $\mathcal{C}$ s.th. every X has a left dual, together with

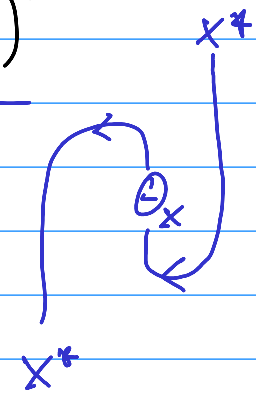
nat. iso. $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "id"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C} \end{tikzcd}$ (i.e. $\underbrace{\{\theta_X : X \to X\}}_{\text{ribbon twist}}{}_{X \in \mathcal{C}}$)

s.th.

1) $\forall X, Y$: $c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y) = \theta_{X \otimes Y}$

2) $\forall X$: $\theta_{(X^*)} = \underbrace{(\theta_X)^*}$

=



E.g. <u>H-mod</u>

$$\theta_M(m) = v^{-1}.m \quad \text{for some invertible } v \in Z(H)$$

s.th. 1) $\Delta(v) = (R_{21} R)^{-1} (v \otimes v)$

2) $S(v) = v$

Def A <u>ribbon Hopf alg.</u> is a qtr. Hopf alg with v inv., $v \in Z(H)$ s.th. 1) & 2) hold.

Prop H rib. Hopf $\Rightarrow$ H-mod ribbon cat.

- group alg $k[G]$: $R = 1 \otimes 1$, $v = 1$
- funct. alg. $\mathrm{Fun}(G, k)$, G abelian

$$R = \sum_{g,h} \gamma(g,h)\, \delta_g \otimes \delta_h$$

$$v = \sum_{g} \gamma(g,g)^{-1}\, \delta_g$$

- $u_q\mathfrak{sl}_2$:

A *pivotal element* $g \in \bar{U}_q\mathfrak{sl}_2$ is given by $g := K$, and an *R-matrix* $R \in \bar{U}_q\mathfrak{sl}_2 \otimes \bar{U}_q\mathfrak{sl}_2$ is given by

$$R := \frac{1}{r} \sum_{a,b,c=0}^{r-1} \frac{\{1\}^a}{[a]!} q^{\frac{a(a-1)}{2} - 2bc} K^b E^a \otimes K^c F^a.$$

A *ribbon element* $v \in \bar{U}_q\mathfrak{sl}_2$ is given by

$$v := \frac{i^{\frac{r-1}{2}}}{\sqrt{r}} \sum_{a,b=0}^{r-1} \frac{\{-1\}^a}{[a]!} q^{-\frac{a(a-1)}{2} + \frac{(r+1)(a-b-1)^2}{2}} F^a K^b E^a.$$