MSc Seminar on Hopf algebras, tensor categories and three-manifold invariants

Ingo Runkel Stand: 15. September 2020

The seminar talks are 90 minutes each, so it is recommended to plan them for **70-80 minutes** to allow for questions and comments. You are asked to prepare a **handout** of 2-3 pages on your topic and present this to me **2 weeks before** the seminar. We will meet a few days after I receive the handout to discuss the content of your talk.

The outlines below of the individual seminars state the material that should be covered. Since you only have around 80 minutes it is usually not possible to include all (or even most of the) proofs – please choose what you present with the aim to make the seminar **comprehensible and instructive for the audience**. The references given should cover most aspects of the seminar, possibly multiple times.

1 Ribbon Hopf algebras and ribbon categories

H1) Review of Hopf algebras

Definition of a Hopf algebra H. Examples: Functions on a group, group-algebra, $u_q(sl(2))$ (the finite-dimensional root-of-unity version, "restricted quantum group"). Category of representations H-mod, tensor product functor on H-mod from coproduct (just that it is a well-defined functor, coherence comes in the next talk).

[Ka, Sec. III.3] (Definition of a Hopf algebra, antipode is a property not extra structure – do not try to cover everthing in that section, it is too much), see also [Ma, Sec. 1.3];
[Ka, Sec. III.5] (Hopf algebra modules, module structure on $M \otimes N$ – this is just the first half page of that section);
[DNR, Sec. 4.3] (One place where one can find the group algebra example and its dual, functions on the group – one is always cocommutative, the other always commutative), see also [Ma, Sec. 1.5];
[Ka, Sec. VI.5] (The definition of u_q is Def. VI.5.6, a basis is given in Prop. VI.5.8, that it is a Hopf algebra is stated in Prop. IX.6.1.)

H2) Quasi-triangular Hopf algebras and braided monoidal categories

Definition of a monoidal category (unit object, tensor product functor, associator and unitor natural isomorphism, coherence pentagon and triangles). H-mod is a monoidal category. Definition of a braiding (natural isomorphism $U \otimes V \to V \otimes U$, hexagon diagrams). Quasi-triangular Hopf algebras, R-matrix, H-mod is braided. Examples: group algebra, function algebra on a group, $u_q(sl(2))$.

[Ka, Sec. XI.2, XIII.1] (Definition of a monoidal category and a braided monoidal category)
[Ka, Sec. XI.3, XIII.1.3] (H-mod is monoidal and braided monoidal)
[reference for group example to be added]
[Ka, Sec. XI.7] (Prop. IX.7.1 gives a formula for the R-matrix of u_q. This could be stated but should not be proved.)

H3) Ribbon Hopf algebras and ribbon categories

Dualities in monoidal categories (left and right duals). H-mod has left and right duals. Ribbon twist in a braided category. Ribbon Hopf algebra, H-mod is ribbon. Examples: group algebra, function algebra on a group, $u_q(sl(2))$. Quantum dimension in a ribbon category, quantum dimension in a Hopf algebra, the quantum dimension can be zero.

[Ka, Sec. XIV]
[more references to be added]

2 Link invariants and ribbon tangles

L1) Jones polynomial and Kauffman bracket

Knots and links in $\mathbb{R}^3$. Linking number of a knot. Kauffman bracket, writhe of a knot and the Jones polynomial. Invariants of framed links.

[Oh, Sec. 1.1, 1.2]

L2) Invariants of ribbon graphs

Monoidal functors and strict monoidal functors. Strict monoidal categories and MacLane coherence (just quote, not prove). Ribbon graphs over V, category Rib_V. Theorem I.2.5 in [Tu] on functors from Rib_V to a ribbon category. Sketch of proof.

[Ka, Sec. XI.5] (For stritification of monoidal categories)
[Tu, Sec. I.2.1, I.2.2]
[More references to be added]

L3) Examples of invariants of ribbon graphs

Invariant of the Hopf link in H-mod for a ribbon Hopf algebra H. Examples of group algebra, function algebra, $u_q(sl(2))$. Relation of $u_q(sl(2))$ example to Jones polynomial. Definition of factorisable Hopf algebra. Definition of a modular tensor category.

[More references to be added]

L4) Hennings invariant for links

Definition of a projective representation of an algebra. H in H-mod is projective. Observation: For H non-semisimple, the invariant from L2 vanishes if one of the ribbons is coloured by a projective object. Define the Hennings invariants for links, show that it can be non-zero. [More details to be added later.]

[He]

3 Surgery and invariants of closed three-manifolds

S1) Surgery presentation of three-manifolds

Glueing of solid tori, mapping class group of a torus. Glue two solid tori to obtain $S^2 \times S^1$ and S^3. Theorems 4.1.9, 4.1.11 and Examples 4.1.10 in [BK]. Fenn-Rourke moves. Kirby calculus with embedded ribbon graph.

[BK, Sec. 4.1] (for surgery with and without embedded ribbon graph);
[Tu, Sec. II.3.1] (for surgery with embedded ribbon graphs)

S2) Surgery invariants

Recall definition of a modular tensor category and of a factorisable Hopf algebra. Note that only the group examples are semisimple. Theorem 4.1.12 and 4.1.16, Examples 4.1.13 in [BK] (see also Theorem II.2.2.2 and II.2.3.2 in [Tu]). Sketch of proof of invariance under Fenn-Rourke moves.

[BK, Sec. 4.1], [Tu, Sec. II.2], [Oh] (just before Thm. 8.11 there is an alternative definition of the linking matrix and $\sigma_\pm$ that does not need a bounding 4-manifold)

S3) Surgery invariants from Hennings invariant

[More details to be added later.]

[He]

4 Topological field theory as a functor

T1) Bordism category and definition of a TFT

Topological field theory as a symmetric monoidal functor from a category of d-dimensional bordisms to vector spaces. Basic properties (in particular: state spaces are finite-dimensional). Examples in 1d and 2d.

[BK, Sec. 4.2, 4.3] (Definition of d-dim TQFT, 2d example)
[CR, Sec. 2.2, 2.4, 3.1, 3.3] (Definition of d-dim TQFT, basic properties, 1d and 2d examples)

T2) Reshetikhin-Turaev construction

Bordisms with embedded ribbon graphs, category of such bordisms. Definition of state spaces and assignment of linear maps to bordisms by reduction to closed three-manifolds and surgery invariants. Maybe idea of proof.

[BK, Sec. 4.4] (Construction of 3d TQFT from a modular category in several steps – this (while already long) is a summary of the more extensive treatment in [Tu])
[Tu, Sec. IV.1, IV.2] (but skip everything to do with modular functors)

T3) The universal construction

Construct a functor from bordisms to vector spaces from an invariant of closed manifolds (possibly with embedded ribbon graphs). The resulting functor may or may not be monoidal. Applied to the Reshitikhin-Turaev invariant of a three-manifold for a modular tensor category, it produces the Reshetikhin-Turaev TQFT.

[Co, Prop. 3.5], [BHMV, Prop. 1.1].

Literature

[BHMV] Blanchet, Habegger, Masbaum, Vogel, Topological quantum field theories derived from the Kauffman bracket, Topology 34 (1995), 883–927.

[BK] Bakalov, Kirillov, Lectures on Tensor Categories and Modular Functors (AMS 2001)

[Co] Costantino, Notes on Topological Quantum Field Theories, https://doi.org/10.5802/wbln.7

[CR] Carqueville, Runkel, Introductory lectures on topological quantum field theory, arXiv:1705.05734

[DNR] Dascalescu, Nastasescu, Raianu, Hopf algebras – an introduction

[He] Hennings, Invariants of lins and 3-manifolds obtained from Hopf algebras, J. London Math. Soc. 54 (1996)

[Ka] Kassel, Quantum groups

[Ma] Majid, Foundations of Quantum Group Theory

[Oh] Ohtsuki, Quantum Invariants, A Study of Knot, 3-Manifolds, and Their Sets (World Scientific 2001)

[Tu] Turaev, Quantum Invariants of Knots and 3-Manifolds (de Gruyter, 2010, 2nd edition)