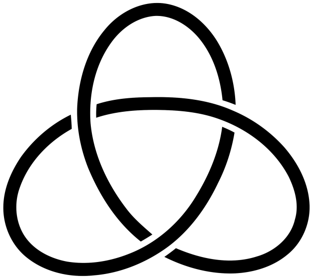
L1) Jones polynomial and Kaufman bracket

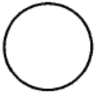
Introduction to knots and links, the Kaufman bracket, writhe of a knot, the Jones polynomial and framed links.



1. Introduction to knots and links

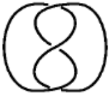
Definition 1.1 (Knots and links)

A knot is the image of a smooth embedding of the one sphere S^1 onto $\mathbb{R}^3$.
A link of l components is the image of a smooth embedding of the disjoint union of l circles into $\mathbb{R}^3$.



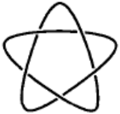
trivial knot K_0

trefoil knot $K_{\bar{3}_1}$



$K'_{\bar{3}_1}$

figure-eight knot K_{4_1}



$(5, 2)$ torus knot $K_{\bar{5}_1}$

$(7, 3)$ two-bridge knot K_{5_2}

1. Introduction to knots and links

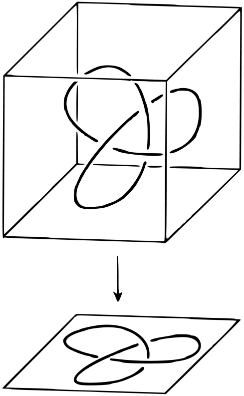
Definition 1.2 (Isotopoic knots and links)

Two knots (or two links) K and K' are called *isotopic* if K is obtained from K' by a continuous deformation such that there is no self-intersection at any time during the deformation.

1. Introduction to knots and links

Definition 1.3 (Diagrams of knots and links)

A diagram of a link is the image of a projektion $\mathbb{R}^3 \to \mathbb{R}^2$ with at most finitely many transversal double points such that the two paths at each double point are assigned to be the over path and the under path respectively.



1. Introduction to knots and links

Definition 1.4 (Isotopoic diagrams)

Two link diagrams D and D' are called *isotopic* if D is obtained from D' by a continuous deformation such that there is no self-intersection at any time during the deformation.

Definition 1.5 (The Reidemeister moves)

A Reidemeister move operates on a small region of a diagram and is one of this types:

- R1: Twist and untwist in either direction.
- R2: Move one loop completely over another.
- R3: Move a loop completely over or under a crossing.

1. Introduction to knots and links

Theorem 1.6
Let K and K' be two links and D and D' diagrams of them. Then K is isotopic to K' if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the Reidemeister moves R1, R2 and R3.

1. Introduction to knots and links

The theorem has the following symbolic representation,

$\{\text{links}\} \,/\, \mathbb{R}^3\text{-isotopy} = \{\text{link diagrams}\} \,/\, \text{R1, R2, R3 and } \mathbb{R}^2\text{-isotopy}$

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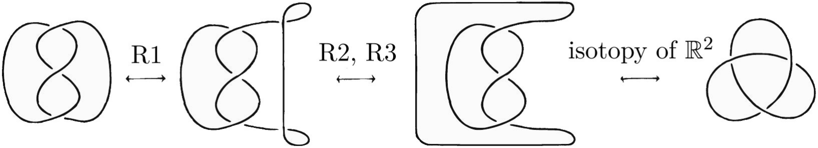
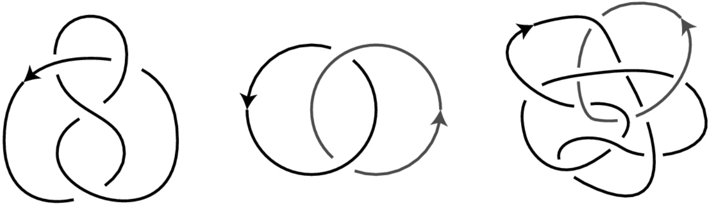
Example 1.7 (Isotopie of the Treefoilknot)



Definition 1.8

An *oriented link* is the image of an embedding of the disjoint union of oriented circles into $\mathbb{R}^3$. An *oriented diagram* is defined similarly as an immersion of oriented circles in $\mathbb{R}^2$.



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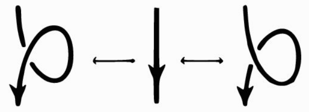
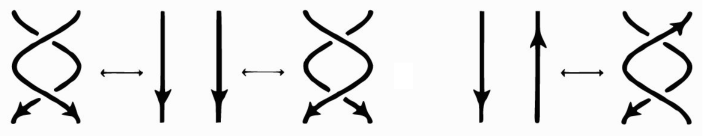
Definition 1.9 (The oriented Reidemeister moves)

The oriented Reidemeister moves on a oriented link diagram are defined als follows.

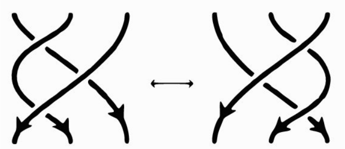
$\overrightarrow{\mathrm{R1}}$:



$\overrightarrow{\mathrm{R2}}$:



$\overrightarrow{\mathrm{R3}}$:



1. Introduction to knots and links

Theorem 1.10

Let K and K' be two oriented links and D and D' oriented diagrams of them. Then K is isotopic to K' if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the modified Reidemeister moves $\overrightarrow{\text{R1}}$, $\overrightarrow{\text{R2}}$ and $\overrightarrow{\text{R3}}$.

Theorem 1.10

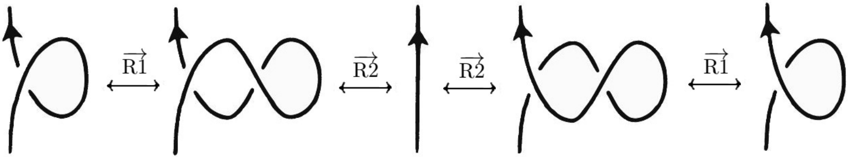
Let K and K' be two oriented links and D and D' oriented diagrams of them. Then K is isotopic to K' if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the modified Reidemeister moves $\overrightarrow{\text{R1}}$, $\overrightarrow{\text{R2}}$ and $\overrightarrow{\text{R3}}$.

Proof.

It is sufficient to verify that each of the Reidemeister moves with any orientation can be obtained as a sequence of the $\overrightarrow{\text{R1}}$, $\overrightarrow{\text{R2}}$ and $\overrightarrow{\text{R3}}$ moves. Since there are many cases we only show the R1 move.

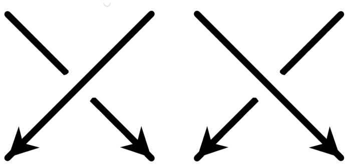
1. Introduction to knots and links

For one orientation the R1 move is $\overrightarrow{R1}$, for the other we use the following sequence.



2. Linking number

The linking number is a simple invariant on link diagrams. As we will see it is not very usefull.
In an oriented diagram we call a "left pass under right"-crossing positive and a "left pass over right"-crossing negative.



2. Linking number

Definition 2.1

The *linking number* of two components L_1 and L_2 of an oriented link is defined by

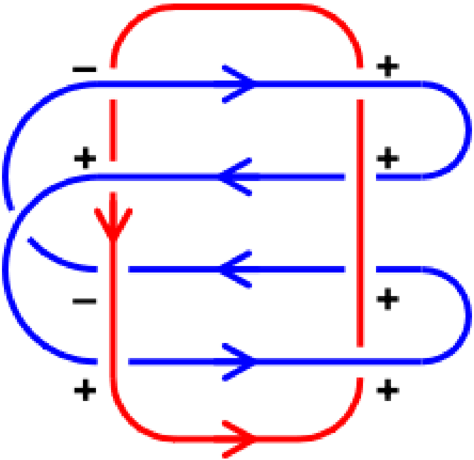
$$\mathrm{lk}(L_1, L_2) = \frac{1}{2}\Big(\#\{\text{positive crossings of two strands of } D_1 \text{ and } D_2\} \\ - \#\{\text{negative crossings of two strands of } D_1 \text{ and } D_2\}\Big),$$

where $D_1 \cup D_2$ is a diagram of $L_1 \cup L_2$.

2. Linking number

Example 2.2

This link has linkingnumber 2.



2. Linking number

Theorem 2.3

The linking number $\mathrm{lk}(L_1, L_2)$ is an isotopy invariant of an oriented link $L_1 \cup L_2$.

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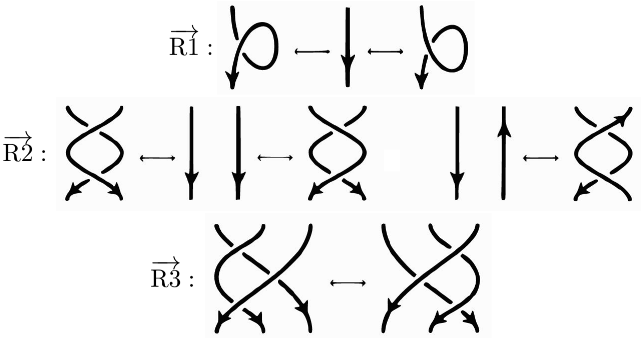
Proof.

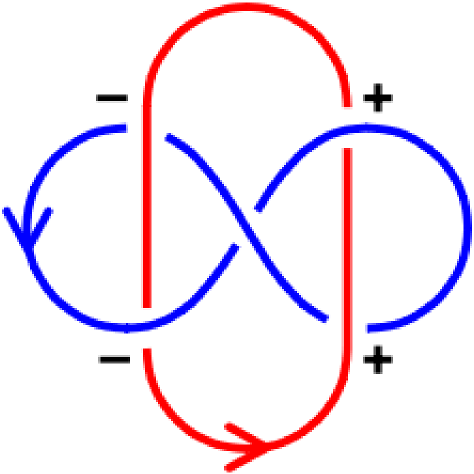
The linking number is invariant under the $\overrightarrow{\mathrm{R1}}$, $\overrightarrow{\mathrm{R2}}$ and $\overrightarrow{\mathrm{R3}}$ moves, hence the proposition is obtained from Theorem 1.10. □



2. Linking number

Example 2.4

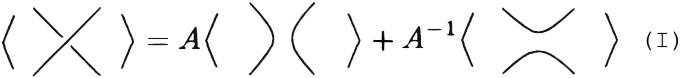
This link has linkingnumber 0 just like the disjoined union of two trivial knots.



3. The Kaufmann bracket

Definition 3.1

For a link diagram D in $\mathbb{R}^2$ the Kauffman bracket $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ of D is defined as follows. We consider the following three recursive formulae



$$\langle \bigcirc\, D \rangle = (-A^2 - A^{-2})\langle D \rangle \qquad \text{(II)}$$

$$\langle \emptyset \rangle = 1 \qquad \text{(III)}$$

where D is a diagram without crossings.

3. The Kaufmann bracket

Example 3.2

Let D be the canonical diagram of the Trefoil knot. We calculate the Kaufmann bracket of the Trefoil knot $\langle D \rangle$. At first resolve all crossings with the first formula.

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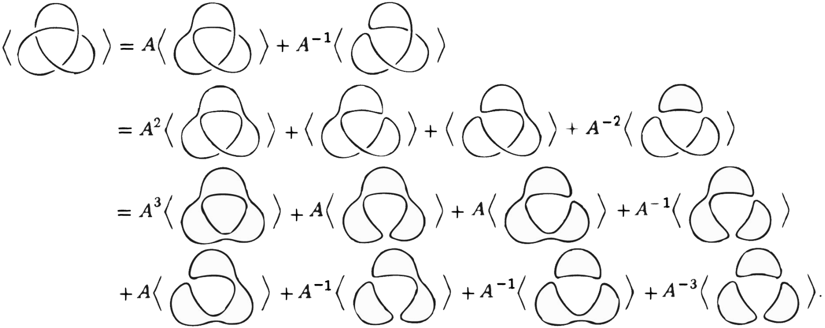

3. The Kaufmann bracket

Since a diagram without crossings is the disjoint union of loops we obtail the value of the bracket recursively by (II) and (III).

$$\langle D \rangle = (-A^2 - A^{-2})(-A^5 - A^{-3} + A^{-7}).$$

3. The Kaufmann bracket

To obtain an isotopy invariant of links from the Kauffman bracket of their diagrams, it is sufficient to show the invariance of the Kauffman bracket under the R1, R2 and R3 moves.

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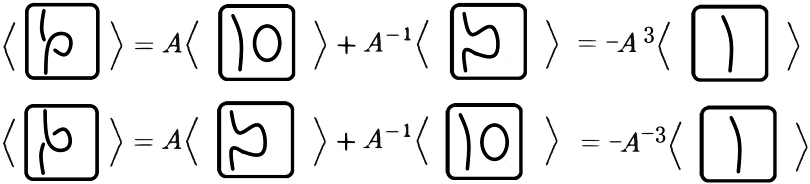

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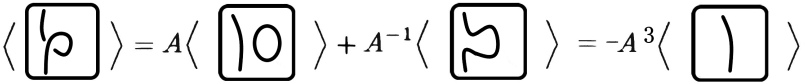
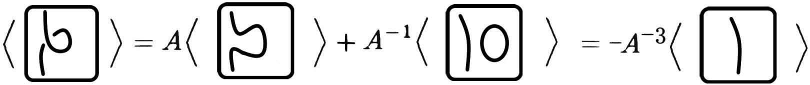
To obtain an isotopy invariant of links from the Kauffman bracket of their diagrams, it is sufficient to show the invariance of the Kauffman bracket under the R1, R2 and R3 moves. Unfortunately, $\langle D \rangle$ does change under the R1 move on a diagram D as can be seen by the following calculation.





R2 and R3 do not change $\langle D \rangle$, we omit the calculation. To deal with R1, we modify $\langle D \rangle$ by *writhe* of D.

4. Deal with R1 by writhe

Definition 4.1

For an oriented diagram D we define the *writhe* of D by

$$\begin{aligned} w(D) = &\ (\text{the number of positive crossings of } D) \\ &-(\text{the number of negative crossings of } D). \end{aligned}$$

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Definition 4.1

For an oriented diagram D we define the *writhe* of D by

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Theorem 4.2

Let L be an oriented link, and D an oriented diagram of L. Then,

$$(-A^3)^{-w(D)}\langle D\rangle$$

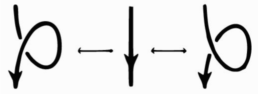
is invariant under the R1, R2 and R3 moves, where $\langle D\rangle$ is the Kauffman bracket of D with its orientation forgotten. In particular, it is an isotopy invariant of L.

4. Deal with R1 by writhe

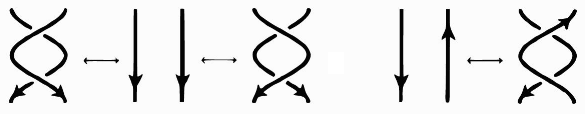
Proof.

It is sufficient to check the invariance under the moves.

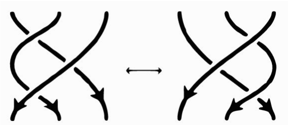
$\overrightarrow{R1}$:



$\overrightarrow{R2}$:



$\overrightarrow{R3}$:



□

4. Deal with R1 by writhe

Definition 4.3 (Jones polynomial)

Putting $A^2 = t^{-1/2}$ we define the quotient of the invariant by $(-A^2 - A^{-2})$ as the *Jones polynomial* $V_L(t)$ of an oriented link L, i.e., we set

$$V_L(t) = \frac{(-A^3)^{-w(D)}}{(-A^2 - A^{-2})} \langle D \rangle \Big|_{A^2 = t^{-1/2}} \in \mathbb{Z}[t^{1/2}, t^{-1/2}].$$

4. Deal with R1 by writhe

Example 4.4

For the trivial knot K_0 the Jones polynomial $V_{K_0}(t) = 1$.

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For the trivial knot K_0 the Jones polynomial $V_{K_0}(t) = 1$.
Let L be the Trefoil knot with orientation and D its oriented diagram. Then the Jones polynomial is

$$\begin{aligned} V_L(t) &= \frac{(-A^3)^{-w(D)}}{(-A^2 - A^{-2})} \langle D \rangle \Big|_{A^2 = t^{-1/2}} \\ &= (-A^3)^{-3}(-A^5 - A^{-3} + A^{-7}) \Big|_{A^2 = t^{-1/2}} \\ &= t + t^3 - t^4. \end{aligned}$$

4. Deal with R1 by writhe

Theorem 4.5

The Jones polynomial satisfies the following relation, called the *skein relation* of the Jones polynomial,

$$t^{-1}V_{L_+} - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t).$$

Here, L_+, L_-, and L_0 are three oriented links, which are identical except for a ball, where they have a positive, a negative or no crossing, respectively.

5. Deal with R1 by framing links

Definition 5.1

A *framed link* is the image of an embedding of a disjoint union of annuli $S^1 \times [0,1]$ into $\mathbb{R}^3$.

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Since framed knots are orientable manifolds they can be projected onto $\mathbb{R}^2$ such that the annuli are "flat". We say such diagram is obtained by *blackboard framing*.

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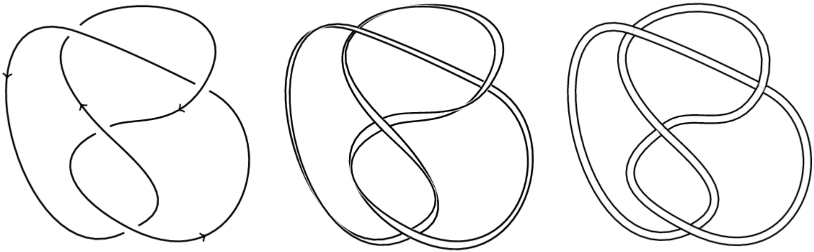
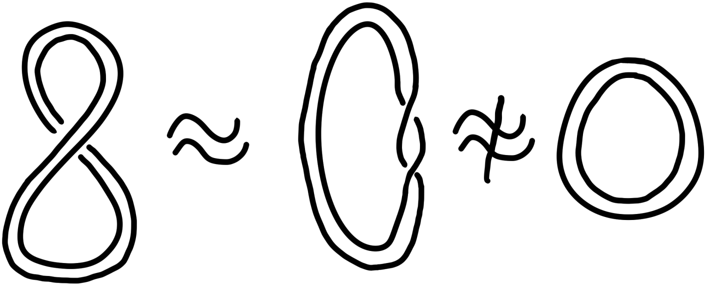


(A) Knot Diagram (B) Framed Knot (C) Blackboard Framing

5. Deal with R1 by framing links

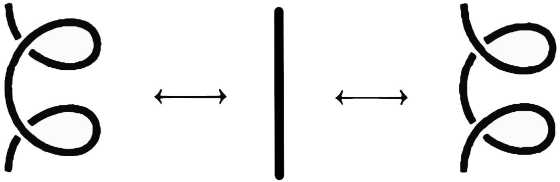
Remark 5.3
The R1 move is not trivial on framed links.



5. Deal with R1 by framing links

Theorem 5.4

Let L and L' be two framed links, and D and D' diagrams of them by blackboard framings. Then, L is isotopic to L' if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the $\mathcal{R}1$ (See figure below), R2 and R3 moves.



5. Deal with R1 by framing links

Proof.

It is trivial to show that, if D is related to D' by a sequence of the moves, then L is isotopic to L'.

Conversely, suppose that L is isotopic to L'. Then, by forgetting the framings D and D' are related by a sequence of isotopies of $\mathbb{R}^2$ and R1, R2 and R3 moves by Theorem 1.6.

Now we modify the sequence: We ignore the R1 move at fisrt and drag the R1-loops of a knot together. Then we can resolves pairs of negative and positive R1 moves by $\mathcal{R}1$. Since L is isotopic to L' they have the same number of "twists". $\square$

5. Deal with R1 by framing links

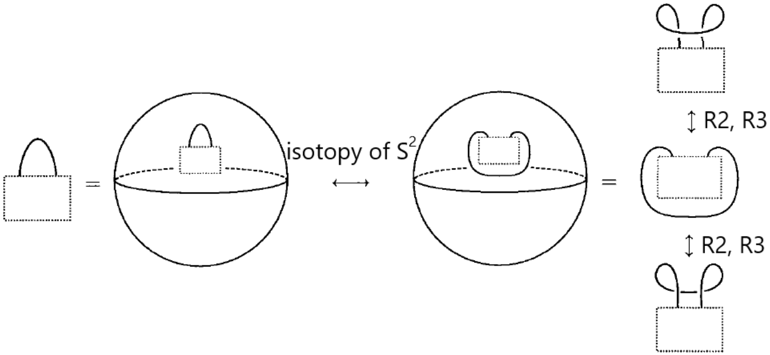
Corollary 5.5

Let L and L' be two framed links, and D and D' diagrams of them by blackboard framings. Further, we regard D and D' as diagrams on $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Then, L is isotopic to L' if and only if D is related to D' by a sequence of isotopies of S^2 and R2 and R3 moves.

5. Deal with R1 by framing links

Proof.

We obtain the $\mathcal{R}1$ move by pulling a loop around the Sphere followed by a sequence of R2 and R3 moves. □



5. Deal with R1 by framing links

Theorem 5.6

Let L be a framed link, and let D be a diagram of L by blackboard framing. Then, the Kauffman bracket $\langle D \rangle$ is an isotopy invariant of L. We denote the invariant also by $\langle L \rangle$ and call it the Kauffman bracket of a framed link L.

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Proof.

As shown before, $\langle D \rangle$ is invariant under the R2 and R3 moves on a diagram D. Hence, by Corollary 5.5 it is an isotopy invariant of a framed link L. □

Fun facts

It is known that there are nontrivial links with Jones polynomial equal to that of the corresponding unlinks by the work of Morwen Thistlethwaite.

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It is an open problem, if there is a none trivial knot with Jones polynomial equal to 1.