Invariants of ribbon graphs

Cósima Sakulski Otero

Abstract

This handout is meant to accompany one of the talks in the seminar on Hopf algebras, tensor categories and 3-manifold invariants (Summer term 2020) at Universität Hamburg. We plan to review the concept of strict monoidal categories and strict monoidal functors, that will lead to understand the utility of Mac Lane's coherence theorem. We will then focus on strict ribbon categories in order to explore their relation to ribbon graphs and define the category $\mathrm{Rib}_{\mathcal{V}}$, from which we can construct isotopy invariants.

1 Stritification and Mac Lane's coherence theorem

The main focus of this first part is to reach an understanding of the Mac Lane coherence theorem: it allows to extend any result obtained for strict monoidal categories to arbitrary monoidal categories. To do so, we will first review the concepts of strict monoidal categories and (strict) monoidal functors following [1].

Recall that in the previous seminar we defined a *monoidal category* $(\mathcal{C}, \otimes, I, a, l, r)$ as a category $\mathcal{C}$ which is endowed with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, with a unit of the tensor category I, an associativity constraint or associator a, a left unit constraint l and a right unit constraint r with respect to I such that the Pentagon Axiom and the Triangle Axiom were satistified.

Definition 1.1. A monoidal category is said to be *strict* if the associativity and unit constraints a, l and r are all identities of the category.

We will show that given a tensor category $\mathcal{C}$, one can construct a strict tensor category $\mathcal{C}^{str}$. To do so, we need to define first the main class of objects to relate monoidal categories. For clarity, we will use indices to denote which categories the structure elements belong to in the beginning, but we will drop them afterwards unless necessary.

Definition 1.2. (a) Let $\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, a_{\mathcal{C}}, l_{\mathcal{C}}, r_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, a_{\mathcal{D}}, l_{\mathcal{D}}, r_{\mathcal{D}})$ be tensor categories. A *tensor functor* or *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ where $F : \mathcal{C} \to \mathcal{D}$ is a functor, φ_0 is an isomorphism from $I_{\mathcal{D}}$ to $F(I_{\mathcal{C}})$ in $\mathcal{D}$, and

$$\varphi_2(U, V) : F(U) \otimes_{\mathcal{D}} F(V) \to F(U \otimes_{\mathcal{C}} V)$$

is a family of natural isomorphisms indexed by all couples (U, V) of objects of $\mathcal{C}$ such that the following diagrams

$$
\begin{array}{ccc}
(F(U)\otimes F(V))\otimes F(W) & \xrightarrow{a_{F(U),F(V),F(W)}} & F(U)\otimes (F(V)\otimes F(W)) \\
\Big\downarrow{\scriptstyle \varphi_2(U,V)\otimes\, \mathrm{id}_{F(W)}} & & \Big\downarrow{\scriptstyle \mathrm{id}_{F(U)}\otimes\varphi_2(V,W)} \\
F(U\otimes V)\otimes F(W) & & F(U)\otimes F(V\otimes W) \\
\Big\downarrow{\scriptstyle \varphi_2(U\otimes V,W)} & & \Big\downarrow{\scriptstyle \varphi_2(U,V\otimes W)} \\
F((U\otimes V)\otimes W) & \xrightarrow{F(a_{U,V,W})} & F(U\otimes (V\otimes W)),
\end{array}
$$

$$
\begin{array}{ccc}
I\otimes F(U) & \xrightarrow{l_{F(U)}} & F(U) \\
\Big\downarrow{\scriptstyle \varphi_0\otimes \mathrm{id}_{F(U)}} & & \Big\uparrow{\scriptstyle F(l_U)} \\
F(I)\otimes F(U) & \xrightarrow{\varphi_2(I,U)} & F(I\otimes U)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
F(U)\otimes I & \xrightarrow{r_{F(U)}} & F(U) \\
\Big\downarrow{\scriptstyle \mathrm{id}_{F(U)}\otimes\varphi_0} & & \Big\uparrow{\scriptstyle F(r_U)} \\
F(U)\otimes F(I) & \xrightarrow{\varphi_2(U,I)} & F(U\otimes I)
\end{array}
$$

commute for all objects (U, V, W) in $\mathcal{C}$. We say the functor is compatible with the associator, the left unit and the right unit constraints, respectively. The tensor functor $(F, \varphi_0, \varphi_2)$ is said to be *strict* if the isomorphism φ_0 and the natural transformation φ_2 are identities in $\mathcal{D}$.

(b) A *natural tensor transformation* $\eta : (F, \varphi_0, \varphi_2) \to (F', \varphi_0', \varphi_2')$ between tensor functors from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta : F \to F'$ such that the following diagrams commute for each couple (U,V) of objects in $\mathcal{C}$:

$$
\begin{array}{ccc}
 & & F(I) \\
 & {\scriptstyle \varphi_0}\nearrow & \\
I & & \Big\downarrow{\scriptstyle \eta(I)} \\
 & {\scriptstyle \varphi_0'}\searrow & \\
 & & F'(I)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
F(U)\otimes F(V) & \xrightarrow{\varphi_2(\mathrm{U,V})} & F(U\otimes V) \\
\Big\downarrow{\scriptstyle \eta(U)\otimes\eta(V)} & & \Big\downarrow{\scriptstyle \eta(U\otimes V)} \\
F'(U)\otimes F'(V) & \xrightarrow{\varphi_2'(U,V)} & F(U\otimes V).
\end{array}
$$

A natural tensor isomorphism is a natural tensor transformation that is also a natural isomorphism.

(c) A *tensor equivalence* between tensor categories is a tensor functor $F : \mathcal{C} \to \mathcal{D}$ such there exist a tensor functor $F' : \mathcal{D} \to \mathcal{C}$ and natural tensor isomorphisms $\eta : \mathrm{id}_{\mathcal{D}} \to FF'$ and $\theta : F'F \to \mathrm{id}_{\mathcal{C}}$.

If there exists a tensor equivalence as defined in (c), we say that $\mathcal{C}$ and $\mathcal{D}$ are *tensor equivalent*. Note the composition of any two tensor functors is as well a tensor functor. From (a), it is immediately clear that the identity functor is a strict tensor functor.

We now proceed to show how, given a tensor category $\mathcal{C}$, we can construct a strict tensor category $\mathcal{C}^{str}$ which is tensor equivalent to $\mathcal{C}$. Again, we are going to follow [1].

Let $\mathcal{S}$ be the class of all the finite sequences $S = (V_1, ..., V_k)$ of objects in $\mathcal{C}$, including the empty sequence $\emptyset$. The integer k is by definition the length of S. By convention, if $S = \emptyset$, $k = 0$. We define the following product

$$S * S' = (V_1, \ldots, V_k, V_{k+1}, \ldots, V_{k+n})$$

for any two sequences $S = (V_1, ..., V_k)$ and $S' = (V_{k+1}, ..., V_{k+n})$, obtained by plaacing S' after S. Note that by this definition, $S * \emptyset = S = \emptyset * S$.

To any sequence S of $\mathcal{S}$, we assign an object $F(V)$ of $\mathcal{C}$ defined by

$$F(\emptyset) = I, \quad F((V)) = V, \quad F(S * (V)) = F(S) \otimes V.$$

This can be seen explicitly as

$$F((V_1, V_2, \dots, V_{k-1}, V_k)) = F((V_1, V_2, \dots, V_{k-1})) \otimes V_k$$
$$= (F((V_1, V_2, \dots, V_{k-2})) \otimes V_{k-1}) \otimes V_k = ((\cdots (V_1 \otimes V_2) \otimes \cdots) \otimes V_{k-1}) \otimes V_k,$$

where the parentheses are being placed on the left side of the first element.

The category $\mathcal{C}^{str}$ has elements of $\mathcal{S}$, i.e. finite sequences of objects of $\mathcal{C}$, with morphisms given by

$$\mathrm{Hom}_{\mathcal{C}^{str}}(S, S') = \mathrm{Hom}_{\mathcal{C}}(F(S), F(S')).$$

Proposition 1.3. The categories $\mathcal{C}^{str}$ and $\mathcal{C}$ are equivalent.

Proof. We can extend the map F to a functor from $\mathcal{C}^{str}$ to $\mathcal{C}$, which is the identity on morphisms, hence fully faithful. Note also any single object in $\mathcal{C}$ is isomorphic to the image under F os a sequence of length one, so F is essentially surjective. This defines a functor $G : \mathcal{C} \to \mathcal{C}^{str}$ such that $FG = \mathrm{id}_{\mathcal{C}}$ and $GF = \mathrm{id}_{\mathcal{C}^{str}}$ via $\theta(S) = \mathrm{id}_{F(S)} = GF(S) \to S$. $\square$

Now that it is clear that we can obtain one category from the other, it suffices to identify $S \otimes S' = S * S'$ in order to endow $\mathcal{C}^{str}$ with the structure of a strict tensor category. For the morphisms we can define the following natural isomorphism

$$\varphi(S, S') : F(S) \otimes F(S') \to F(S * S')$$

for any pair in $\mathcal{C}^{str}$. Set $\varphi(\emptyset, S) = l_S$, $\varphi(S, \varphi) = r_S$ and

$$\varphi(S, (V)) = \mathrm{id}_{F(S) \otimes V} : F(S) \otimes V \to F(S \otimes (V)),$$

$$\varphi(S, S' * (V)) = (\varphi(S, S') \otimes \mathrm{id}_V) \circ a^{-1}_{F(S), F(S'), V}.$$

We state the following lemma involving more than two objects in $\mathcal{C}^{str}$ without proving it.

Lemma 1.4. If S, S', S'' are objects on $\mathcal{C}^{str}$, we have

$$\varphi(S, S' * S'') \circ (\mathrm{id}_S \otimes \varphi(S', S'')) \circ a_{F(S), F(S'), F(S'')}$$
$$= \varphi(S * S', S'') \circ (\varphi(S, S') \otimes \mathrm{id}_{S''}).$$

Note the equality can be naturally extended from (S, S', S'') to $(S, S', S'' * (V))$.

The same way we identified a tensor product between elements, we can now define the tensor product of two morphisms of $\mathcal{C}^{str}$. If $f : F(S) \to F(T)$ and $f' : F(S') \to F(T')$ are any pair of morphisms in $\mathcal{C}$, we define the tensor product $f * f'$ in $\mathcal{C}^{str}$ by the commutative diagram

$$\begin{array}{ccc} F(S) \otimes F(S') & \xrightarrow{\varphi(S,S')} & F(S * S') \\ \downarrow{\scriptstyle f \otimes f'} & & \downarrow{\scriptstyle f * f'} \\ F(T) \otimes F(T') & \xrightarrow{\varphi(T,T')} & F(T * T'). \end{array}$$

Theorem 1.5. Equipped with this tensor product $\mathcal{C}^{str}$ is a strict tensor category. The categories $\mathcal{C}$ and $\mathcal{C}^{str}$ are tensor equivalent.

We refer to [1] for the proof. Theorem 1.5 implies Mac Lane's coherence theorem which asserts that every diagram in a monoidal category made up of the associativity and unitality constraints commutes, as mentioned in [3]; that is, in a tensor category any diagram built from the constraints a, l, r, and the identities by composing and tensoring, is commutative. This can also be shown by interchanging in an unique way the order of parenthesis of the tensor products in the Pentagon Axiom. Interestingly enough, this establishes an equivalence between any monoidal category and a certain strict monoidal category. For our case, this means that the results obtained for strict ribbon categories directly extend to arbitrary ribbon categories.

2 Ribbon graphs

In this section we will briefly explain coloring of ribbon graphs by a given ribbon category $\mathcal{V}$ and see how their isotopy classes form a monoidal category, denoted by $\mathrm{Rib}_{\mathcal{V}}$. Moreover, in the upcoming seminars we will see how ribbon graphs allow to build 3-manifold invariants and to study 3-dimensional TFTs.

Before starting with a formal definition, we mention the basic concepts to talk about ribbon graphs:

- A *band* is the square $[0,1] \times [0,1]$ or any homeomorphic image of it, whose intervals $[0,1] \times 0$ and $[0,1] \times 1$ are called *bases* of the band;
- the image of the band $(1/2) \times [0,1]$ is called the *core* of the band;
- an *annulus* is the cylinder $S^1 \times [0,1]$ or a homeomorphic image of it;
- a *coupon* is a band with a distinguished base.

A band or an annulus is said to be *directed* if its core is oriented, and this orientation of the core itself is called the *direction.*

Definition 2.1. Let k, l be non-negative integers. A *ribbon* (k,l)*-graph* in $\mathbb{R}^3$ is an oriented surface Ω embedded in the strip $\mathbb{R}^2 \times [0,1]$ and decomposed into a union of a finite number of annuli, bands, and coupons such that:

(i) Ω meets the planes $\mathbb{R}^2 \times 0$, $\mathbb{R}^2 \times 1$ orthogonally along the following segments which are bases of certains bands of Ω:

$$\{[i-(1/10), i+(1/10)] \times 0 \times 0 \mid i = 1, \dots, k\},$$
$$\{[j-(1/10), j+(1/10)] \times 0 \times 1 \mid j = 1, \dots, l\},$$

called the *boundary intervals* of the graph.

(ii) other bases of bands lie on the bases of the coupons – otherwise bands, coupons and annuli are disjoint;

(iii) the bands and annuli are directed.

A ribbon (k,l)-graph can be considered an oriented compact surface in $\mathbb{R}^3$ decomposed into the pieces defined above with k inputs and l outputs. The choice of orientation of a ribbon graph is equivalent to a choice of a preferred side of Ω. For simplicity, we may fix the right-handed orientation in $\mathbb{R}^3$.

We are mainly interested in using ribbon graphs because we can generalize knot diagrams by plane pictures of ribbon graphs. Following the explanation in [2], the plan is to *deform* the graph in $\mathbb{R}^2 \times [0,1]$ in such a way that ends up very close the plane $\mathbb{R} \times 0 \times \mathbb{R}$, considered the standard position. By convenience, the bases of the coupons should be oriented parallel to the horizontal line $\mathbb{R} \times 0 \times 0$, with the distinguished base in the lower position, forming rectangles parallel to $\mathbb{R} \times 0 \times \mathbb{R}$ altogether. The orientation of coupons is induced by the one of the surface Ω , for which, taking into account our fixation, should go counterclockwise in $\mathbb{R} \times 0 \times \mathbb{R}$. The bands and annuli are supposed to go parallel to this planes, and the projections of their corresponding cores can only have double transversal crossings and should not overlap with those of the coupons'. The resulting picture is the diagram of the ribbon graph.

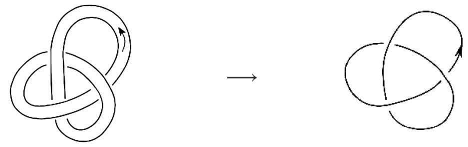

Figure 1: Example of the *deformation* explained above for a trefoil graph into its diagram.

The theory of ribbon graphs generalizes the more familiar theory of framed oriented links in $\mathbb{R}^3$, already discussed in the previous seminar. In fact, to each ribbon $(0,0)$-graph Ω consisting of annuli, we can associate the link of circles in $\mathbb{R}^3$ formed by their oriented cores.

Once we have set these relations to diagrams of ribbon graphs, we may consider to *color* them by a monoidal category. In particular, let $\mathcal{V}$ be a strict monoidal category with duality. A ribbon graph is said to be *colored over* $\mathcal{V}$ if the bands are colored with its objects and coupons with its morphisms. More precisely, let $(V_1, \ldots, V_m)$ be the colors of the bands incident to the bottom base and $(W_1, \ldots, W_n)$ the colors of the ones incident to the top. We denote as $\epsilon_1 \ldots, \epsilon_m \in \{-1, +1\}$ and analogously, $\nu_1, \ldots \nu_n \in \{-1, +1\}$, the numbers that indicate the directions of the band, so that $\epsilon_i = 1, \nu_j = -1$ means they are going *out* of the coupon and $\epsilon_i = -1, \nu_j = 1$ means they are going *in*. A color of the coupon is an arbitrary morphism

$$f : V_1^{\epsilon_1} \otimes \cdots \otimes V_m^{\epsilon_m} \to W_1^{\nu_1} \otimes \cdots \otimes W_n^{\nu_n},$$

where the objects in $\mathcal{V}$ are $V^{+1} = V$ and $V^{-1} = V^*$. The theory of diagrams extends naturally from here: it suffices to attach an object of $\mathcal{V}$ to the cores of the bands and annuli respectively, and colors to all the coupons.

As it turns out, v-colored ribbon graphs over $\mathcal{V}$ can be organized into a strict monoidal category, denoted by $\text{Rib}_{\mathcal{V}}$, whose objects are finite sequences $\eta = ((V_1, \epsilon_1), \ldots, (V_m, \epsilon_m))$ and the morphisms are (an isotopy type of) v–colored ribbon graphs over the same category $\mathcal{V}$. The tensor product in $\text{Rib}_{\mathcal{V}}$ acts on the objects by juxtaposition, while the morphisms are placed next to each other without overlapping. Composition of morphisms is basically obtained by putting one colored ribbon graph on top of the other and gluing them. Note the identity morphisms consist only of untwist unlinked vertical bands without annuli or coupons. This endows $\text{Rib}_{\mathcal{V}}$ with the structure of a strict monoidal category.

One may ask if there is a deeper relation between the category or ribbon graphs $\text{Rib}_{\mathcal{V}}$ and the stric monoidal category $\mathcal{V}$ used to color it. So the next step is to define the so called *operator invariant* $F(\Omega)$ of a given graph Ω. We can already indicate that it is a covariant functor that preserves the tensor product as defined in the beginning of this seminar.

In order to fully understand this, we first need to specify the morphisms in $\text{Rib}_{\mathcal{V}}$ used to define this covariant functor. The colors of the strings are objects of $\mathcal{V}$ and the name of each morphisms is indicated below.

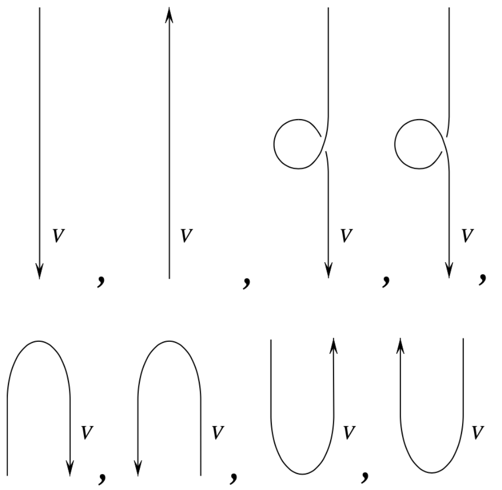

Figure 2: From left to right, the morphisms are referred to as $\downarrow_V, \uparrow_V, \varphi_V, \varphi'_V, \cap_V, \cap^-_V, \cup_V$, and $\cup^-_V$, respectively.

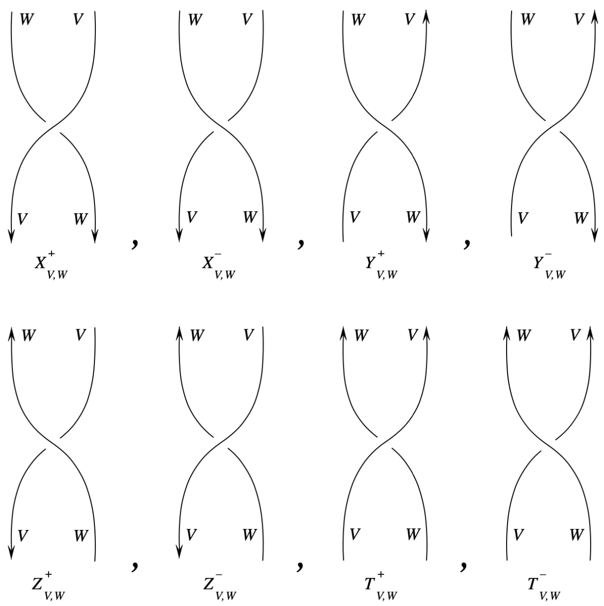

Figure 3: Second set of morphisms needed to define F. The names are indicated below each one of the graph diagrams.

Finally, we are ready to comment more on this functor. From [2]:

Theorem 2.3. Let $\mathcal{V}$ be a strict ribbon category with braiding c, twist θ, and compatible duality ($*$, b, d). There exists a unique covariant functor $F = F_{\mathcal{V}} : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$ preserving the tensor product and satisfying the following conditions:

(1) F transforms any object $(V, +1)$ into V and any object $(V, -1)$ into V^*;

(2) for any objects V, W of $\mathcal{V}$, we have

$$F(X^+_{V,W}) = c_{V,W}, \quad F(\varphi_V) = \theta_V, \quad F(\cap_V) = b_V, \quad F(\cup_V) = d_V;$$

(3) for any elementary v-colored ribbon graph Γ, we have $F(\Gamma) = f$ where f is the color of the only coupon of Γ.

Moreover, the functor F has the following properties:

$$F(X^-_{V,W}) = (c_{W,V})^{-1}, \quad F(Y^+_{V,W}) = (c_{W,V^*})^{-1}, \quad F(Y^-_{V,W}) = c_{V^*,W},$$

$$F(Z^+_{V,W}) = (c_{W^*,V})^{-1} \quad F(Z^-_{V,W}) = c_{V,W^*},$$

$$F(T^+_{V,W}) = c_{V^*,W^*}, \quad F(T^-_{V,W}) = (c_{W^*,V^*})^{-1}, \quad F(\varphi'_V) = (\theta_V)^{-1}.$$

Sketch of proof. The first part of the proof consists of fully describing the category of $\mathcal{V}$-colored ribbon graphs $\mathrm{Rib}_{\mathcal{V}}$ in terms of generators and relations between these. The ribbon graphs in figures 2 and 3 already define a complete system of relations between morphisms, so we can use the axioms of a strict monoidal category to obtain any relation between these by composing and tensoring. This allows us to define the value of F for any ribbon graph.

The second part of the proof is focused on proving that, although every ribbon graph admits different expressions of this kind, they can also be obtained from each other by elementary local transformations. These local transformations are the Reidemeister moves described in the previous seminar, and the word associated with each diagram can be shown to be transformed to another equivalent one, in the sense that can be obtained by the relations built in the first part of the proof. The invariance of the functor is thus proved by using these local transformations.

Remark 2.4. The term *operator invariant* is meant to recall the following properties of F:

$$F(\downarrow_V) = \mathrm{id}_V, \quad F(\uparrow_V) = \mathrm{id}_{V^*} \quad \text{and} \quad F(\Omega\Omega') = F(\Omega)F(\Omega')$$

for any pair of composable ribbon graphs. Moreover, by definition, note F is a functor that preserves the tensor product, $F(\Omega \otimes \Omega') = F(\Omega) \otimes F(\Omega')$.

Remark 2.5. If the graphs Ω and Ω' are isotopic, $F(\Omega) = F(\Omega')$. This amplifies the graphical calculus used up to this point, with the addition of isotopy invariance of the morphisms in the category $\mathrm{Rib}_{\mathcal{V}}$. These invariants can be regarded as a generalization of the Jones polynomial seen in the last seminar and in fact, more examples will be discussed in the upcoming talks due to its importance in 3-manifold invariants and TFTs.

3 References

[1] C. Kassel, *Quantum Groups.* Graduate Texts in Mathematics 155, Springer, Berlin (1995).

[2] V. Turaev, *Quantum invariants of knots and 3-manifolds.* de Gruyter Studies in Mathematics 18, 2nd edition, De Gruyter Berlin/New York (2010).

[3] V. Turaev and A. Virelizier, *Monoidal Categories and Topological Field Theory.* Progress in Mathematics 322, Springer (2017).