Invariants of ribbon graphs

Cósima Sakulski Otero

October 2, 2020

Seminar on Hopf algebras, tensor categories and 3-manifold invariants

Outline

1. Stritification and Mac Lane's coherence theorem
   - Monoidal functors and strict monoidal functors
   - Strict monoidal categories and Mac Lane's coherence theorem
2. Ribbon graphs
   - Coloring of ribbon graphs
   - Category of ribbon graphs and isotopy invariants

Monoidal functors

Definition 1.1. (a) Let $\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, a_{\mathcal{C}}, l_{\mathcal{C}}, r_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, a_{\mathcal{D}}, l_{\mathcal{D}}, r_{\mathcal{D}})$ be tensor categories. A *tensor functor* or *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ where $F : \mathcal{C} \to \mathcal{D}$ is a functor,

Monoidal functors

Definition 1.1. (a) Let $\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, a_{\mathcal{C}}, l_{\mathcal{C}}, r_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, a_{\mathcal{D}}, l_{\mathcal{D}}, r_{\mathcal{D}})$ be tensor categories. A *tensor functor* or *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ where $F : \mathcal{C} \to \mathcal{D}$ is a functor, φ_0 is an isomorphism from $I_{\mathcal{D}}$ to $F(I_{\mathcal{C}})$ in $\mathcal{D}$,

Monoidal functors

Definition 1.1. (a) Let $\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, a_{\mathcal{C}}, l_{\mathcal{C}}, r_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, a_{\mathcal{D}}, l_{\mathcal{D}}, r_{\mathcal{D}})$ be tensor categories. A *tensor functor* or *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ where $F : \mathcal{C} \to \mathcal{D}$ is a functor, φ_0 is an isomorphism from $I_{\mathcal{D}}$ to $F(I_{\mathcal{C}})$ in $\mathcal{D}$, and

$$\varphi_2(U, V) : F(U) \otimes_{\mathcal{D}} F(V) \to F(U \otimes_{\mathcal{C}} V)$$

is a family of natural isomorphisms for all pairs (U, V) of objects of $\mathcal{C}$ such that the following diagrams

Monoidal functors

Definition 1.1. (a) Let $\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, a_{\mathcal{C}}, l_{\mathcal{C}}, r_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, a_{\mathcal{D}}, l_{\mathcal{D}}, r_{\mathcal{D}})$ be tensor categories. A *tensor functor* or *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ where $F : \mathcal{C} \to \mathcal{D}$ is a functor, φ_0 is an isomorphism from $I_{\mathcal{D}}$ to $F(I_{\mathcal{C}})$ in $\mathcal{D}$, and

$$\varphi_2(U, V) : F(U) \otimes_{\mathcal{D}} F(V) \to F(U \otimes_{\mathcal{C}} V)$$

is a family of natural isomorphisms for all pairs (U, V) of objects of $\mathcal{C}$ such that the following diagrams

$$\begin{array}{ccc} I \otimes F(U) & \xrightarrow{l_{F(U)}} & F(U) \\ \Big\downarrow{\scriptstyle \varphi_0 \otimes \mathrm{id}_{F(U)}} & & \Big\uparrow{\scriptstyle F(l_U)} \\ F(I) \otimes F(U) & \xrightarrow{\varphi_2(I,U)} & F(I \otimes U), \end{array} \qquad \begin{array}{ccc} F(U) \otimes I & \xrightarrow{r_{F(U)}} & F(U) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{F(U)} \otimes \varphi_0} & & \Big\uparrow{\scriptstyle F(r_U)} \\ F(U) \otimes F(I) & \xrightarrow{\varphi_2(U,I)} & F(U \otimes I) \end{array}$$

Monoidal functors

and

$$\begin{array}{ccc}
(F(U)\otimes F(V))\otimes F(W) & \xrightarrow{a_{F(U),F(V),F(W)}} & F(U)\otimes (F(V)\otimes F(W)) \\
\Big\downarrow \varphi_2(U,V)\otimes \mathrm{id}_{F(W)} & & \Big\downarrow \mathrm{id}_{F(U)}\otimes\varphi_2(V,W) \\
F(U\otimes V)\otimes F(W) & & F(U)\otimes F(V\otimes W) \\
\Big\downarrow \varphi_2(U\otimes V,W) & & \Big\downarrow \varphi_2(U,V\otimes W) \\
F((U\otimes V)\otimes W) & \xrightarrow{F(a_{U,V,W})} & F(U\otimes (V\otimes W)),
\end{array}$$

commute for all objects (U,V,W) in $\mathcal{C}$.

- The tensor functor (F,φ_0,φ_2) is said to be *strict* if the isomorphism φ_0 and the natural transformation φ_2 are identities in $\mathcal{D}$.

Monoidal functors

(b) A *natural tensor transformation* $\eta : (F, \varphi_0, \varphi_2) \to (F', \varphi'_0, \varphi'_2)$ between tensor functors from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta : F \to F'$ such that for each couple (U, V) of objects in $\mathcal{C}$ the following hold:

$$\varphi'_0 = \eta(I) \circ \varphi_0$$

$$\eta(U \otimes V) \circ \varphi_2(U, V) = \varphi'_2 \circ \eta(U) \otimes \eta(V).$$

Monoidal functors

(b) A *natural tensor transformation* $\eta : (F, \varphi_0, \varphi_2) \to (F', \varphi_0', \varphi_2')$ between tensor functors from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta : F \to F'$ such that for each couple (U, V) of objects in $\mathcal{C}$ the following equalities hold:

$$\varphi_0' = \eta(I) \circ \varphi_0$$

$$\eta(U \otimes V) \circ \varphi_2(U, V) = \varphi_2' \circ \eta(U) \otimes \eta(V).$$

(c) A *tensor equivalence* between tensor categories is a tensor functor $F : \mathcal{C} \to \mathcal{D}$ such there exists a tensor functor $F' : \mathcal{D} \to \mathcal{C}$ and the natural tensor isomorphisms $\eta : \mathrm{id}_{\mathcal{D}} \to FF'$ and $\theta : F'F \to \mathrm{id}_{\mathcal{C}}$.

- If there exists a tensor equivalence as defined in (c), we say that $\mathcal{C}$ and $\mathcal{D}$ are *tensor equivalent*.

Stritification of categories

Definition 1.2. A monoidal category is said to be *strict* if the associativity and unit constraints a, l and r are all identities of the category.

Stritification of categories

Definition 1.2. A monoidal category is said to be ***strict*** if the associativity and unit constraints a, l and r are all identities of the category.

Given a tensor category $\mathcal{C}$, one can construct a strict tensor category $\mathcal{C}^{str}$. The main idea goes as follows.

Let $\mathcal{S}$ be the class of all the finite sequences $S = (V_1, ..., V_k)$ of objects in $\mathcal{C}$, including the empty sequence $\emptyset$, and define the following product

$$S * S' = (V_1, \ldots, V_k, V_{k+1}, \ldots, V_{k+n})$$

for any two sequences $S = (V_1, ..., V_k)$ and $S' = (V_{k+1}, ..., V_{k+n})$.

Stritification of categories

To any sequence S of $\mathcal{S}$, we assign an object $F(V)$ of $\mathcal{C}$ defined by

$$F(\emptyset) = I, \;\; F((V)) = V, \;\; F(S * (V)) = F(S) \otimes V.$$

Stritification of categories

To any sequence S of $\mathcal{S}$, we assign an object $F(V)$ of $\mathcal{C}$ defined by

$$F(\emptyset) = I, \;\; F((V)) = V, \;\; F(S * (V)) = F(S) \otimes V.$$

Then we can define the category $\mathcal{C}^{str}$ as the category with:

- elements of $\mathcal{S}$ as objects, i.e. finite sequences of objects of $\mathcal{C}$,
- morphisms given by $\mathrm{Hom}_{\mathcal{C}^{str}}(S, S') = \mathrm{Hom}_{\mathcal{C}}(F(S), F(S'))$.

Stritification of categories

To any sequence S of $\mathcal{S}$, we assign an object $F(V)$ of $\mathcal{C}$ defined by

$$F(\emptyset) = I, \;\; F((V)) = V, \;\; F(S * (V)) = F(S) \otimes V.$$

Then we can define the category $\mathcal{C}^{str}$ as the category with:

- elements of $\mathcal{S}$ as objects, i.e. finite sequences of objects of $\mathcal{C}$,
- morphisms given by $\mathrm{Hom}_{\mathcal{C}^{str}}(S, S') = \mathrm{Hom}_{\mathcal{C}}(F(S), F(S'))$.

Proposition 1.3. The categories $\mathcal{C}^{str}$ and $\mathcal{C}$ are equivalent.

Stritification of categories

To any sequence S of $\mathcal{S}$, we assign an object $F(V)$ of $\mathcal{C}$ defined by

$$F(\emptyset) = I, \;\; F((V)) = V, \;\; F(S * (V)) = F(S) \otimes V.$$

Then we can define the category $\mathcal{C}^{str}$ as the category with:

- elements of $\mathcal{S}$ as objects, i.e. finite sequences of objects of $\mathcal{C}$,
- morphisms given by $\mathrm{Hom}_{\mathcal{C}^{str}}(S, S') = \mathrm{Hom}_{\mathcal{C}}(F(S), F(S'))$.

Proposition 1.3. The categories $\mathcal{C}^{str}$ and $\mathcal{C}$ are equivalent.

- It suffices to identify $S \otimes S' = S * S'$ in order to endow $\mathcal{C}^{str}$ with the structure of a strict tensor category.

Stritification of categories

- We define the following natural isomorphism

$$\varphi(S,S') : F(S) \otimes F(S') \to F(S * S')$$

for any pair in $\mathcal{C}^{str}$. Set $\varphi(\emptyset, S) = l_S$, $\varphi(S,\varphi) = r_S$ and

$$\varphi(S,(V)) = \mathrm{id}_{F(S)\otimes V} : F(S) \otimes V \to F(S \otimes (V)),$$

$$\varphi(S, S' * (V)) = (\varphi(S,S') \otimes \mathrm{id}_V) \circ a^{-1}_{F(S),F(S'),V}.$$

Stritification of categories

- We define the following natural isomorphism

$$\varphi(S,S') : F(S) \otimes F(S') \to F(S * S')$$

for any pair in $\mathcal{C}^{str}$. Set $\varphi(\emptyset, S) = l_S$, $\varphi(S, \varphi) = r_S$ and

$$\varphi(S,(V)) = \mathrm{id}_{F(S)\otimes V} : F(S) \otimes V \to F(S \otimes (V)),$$

$$\varphi(S, S' * (V)) = (\varphi(S,S') \otimes \mathrm{id}_V) \circ a^{-1}_{F(S),F(S'),V}.$$

- If $f : F(S) \to F(T)$ and $f' : F(S') \to F(T')$ are any pair of morphisms in $\mathcal{C}$, we define the tensor product $f * f'$ in $\mathcal{C}^{str}$ by the commutative diagram

$$\begin{array}{ccc} F(S) \otimes F(S') & \xrightarrow{\varphi(S,S')} & F(S * S') \\ \downarrow{\scriptstyle f \otimes f'} & & \downarrow{\scriptstyle f * f'} \\ F(T) \otimes F(T') & \xrightarrow{\varphi(T,T')} & F(T * T'). \end{array}$$

Mac Lane's coherence theorem

Theorem 1.4. Equipped with this tensor product $\mathcal{C}^{str}$ is a strict tensor category. The categories $\mathcal{C}$ and $\mathcal{C}^{str}$ are tensor equivalent.

Mac Lane's coherence theorem

Theorem 1.4. Equipped with this tensor product $\mathcal{C}^{str}$ is a strict tensor category. The categories $\mathcal{C}$ and $\mathcal{C}^{str}$ are tensor equivalent.

- Theorem 1.4 implies Mac Lane's coherence theorem which asserts that in a tensor category any diagram built from the constraints a, l, r, and the identities by composing and tensoring, is commutative.
- Interestingly enough, this establishes an equivalence between monoidal categories and strict monoidal categories which, in our case, means that we can work with strict ribbon categories without loss of generality.

Ribbon graphs: definitions

Before starting with a formal definition, we mention the basic concepts to talk about ribbon graphs:

- A *band* is the square $[0,1] \times [0,1]$ or any homeomorphic image of it, whose intervals $[0,1] \times 0$ and $[0,1] \times 1$ are called *bases* of the band;
- the image of the band $(1/2) \times [0,1]$ is called the *core* of the band;
- an *annulus* is the cylinder $S^1 \times [0,1]$ or a homeomorphic image of it;
- a *coupon* is a band with a distinguished base.

A band or an annulus is said to be *directed* if its core is oriented, and this orientation of the core itself is called the *direction*.

Ribbon graphs: definitions

Definition 2.1. Let k, l be non-negative integers. A *ribbon (k,l)-graph* in $\mathbb{R}^3$ is an oriented surface Ω embedded in the strip $\mathbb{R}^2 \times [0,1]$ and decomposed into a union of a finite number of annuli, bands, and coupons such that:

(i) Ω meets the planes $\mathbb{R}^2 \times 0$, $\mathbb{R}^2 \times 1$ orthogonally along the following segments which are bases of certains bands of Ω:

$$\{[i-(1/10), i+(1/10)] \times 0 \times 0 \mid i = 1, \dots, k\},$$

$$\{[j-(1/10), j+(1/10)] \times 0 \times 1 \mid j = 1, \dots, l\},$$

called the *boundary intervals* of the graph.

Ribbon graphs: definitions

Definition 2.1. Let k, l be non-negative integers. A *ribbon (k,l)-graph* in $\mathbb{R}^3$ is an oriented surface Ω embedded in the strip $\mathbb{R}^2 \times [0,1]$ and decomposed into a union of a finite number of annuli, bands, and coupons such that:

(i) Ω meets the planes $\mathbb{R}^2 \times 0$, $\mathbb{R}^2 \times 1$ orthogonally along the following segments which are bases of certains bands of Ω:

$$\{[i-(1/10), i+(1/10)] \times 0 \times 0 \mid i = 1, \dots, k\},$$

$$\{[j-(1/10), j+(1/10)] \times 0 \times 1 \mid j = 1, \dots, l\},$$

called the *boundary intervals* of the graph.

(ii) other bases of bands lie on the bases of the coupons – otherwise bands, coupons and annuli are disjoint;

Ribbon graphs: definitions

Definition 2.1. Let k, l be non-negative integers. A *ribbon (k,l)-graph* in $\mathbb{R}^3$ is an oriented surface Ω embedded in the strip $\mathbb{R}^2 \times [0,1]$ and decomposed into a union of a finite number of annuli, bands, and coupons such that:

(i) Ω meets the planes $\mathbb{R}^2 \times 0$, $\mathbb{R}^2 \times 1$ orthogonally along the following segments which are bases of certains bands of Ω:

$$\{[i-(1/10), i+(1/10)] \times 0 \times 0 \mid i = 1, \dots, k\},$$

$$\{[j-(1/10), j+(1/10)] \times 0 \times 1 \mid j = 1, \dots, l\},$$

called the *boundary intervals* of the graph.

(ii) other bases of bands lie on the bases of the coupons – otherwise bands, coupons and annuli are disjoint;

(iii) the bands and annuli are directed.

Ribbon graphs: standard position

- Coupons, bands and annuli go parellel to $\mathbb{R} \times 0 \times \mathbb{R}$, with their bases parallel to $\mathbb{R} \times 0 \times 0$,
- the cores of bands and annuli do not overlap coupons and are allowed to have only double transversal crossings.

After this deformation, we draw the projections into the plane $\mathbb{R} \times 0 \times \mathbb{R}$ taking into account overcrossings and undercrossing.

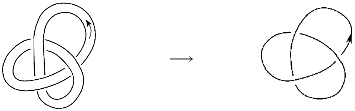

Figure 1: Example of the *standard position* of the trefoil.

Up to isotopy, we can always recover the original ribbon graph.

Ribbon graphs: coloring

Let $\mathcal{V}$ be a strict monoidal category with duality. A ribbon graph is said to be *colored over $\mathcal{V}$* if the bands are colored with its objects and the coupons with its morphisms.

- More precisely, let $(V_1, \ldots, V_m)$ be the colors of the bands incident to the bottom base and $(W_1, \ldots, W_n)$ to the top. We denote as $\epsilon_1, \ldots, \epsilon_m \in \{-1, +1\}$ and $\nu_1, \ldots \nu_n \in \{-1, +1\}$, the numbers that indicate the directions of the band, so that $\epsilon_i = 1, \nu_j = -1$ means they are going *out* of the coupon and $\epsilon_i = -1, \nu_j = 1$ means they are going *in*.

Ribbon graphs: coloring

Let $\mathcal{V}$ be a strict monoidal category with duality. A ribbon graph is said to be *colored over* $\mathcal{V}$ if the bands are colored with its objects and coupons with its morphisms.

- More precisely, let $(V_1, \ldots, V_m)$ be the colors of the bands incident to the bottom base and $(W_1, \ldots, W_n)$ to the top. We denote as $\epsilon_1, \ldots, \epsilon_m \in \{-1, +1\}$, $\nu_1, \ldots \nu_n \in \{-1, +1\}$, the numbers that indicate the directions of the band, so that $\epsilon_i = 1, \nu_j = -1$ means they are going *out* of the coupon and $\epsilon_i = -1, \nu_j = 1$ means they are going *in*.
- A color of the coupon is any morphism of the form

$$f : V_1^{\epsilon_1} \otimes \cdots \otimes V_m^{\epsilon_m} \to W_1^{\nu_1} \otimes \cdots \otimes W_n^{\nu_n},$$

  where the objects in $\mathcal{V}$ are $V^{+1} = V$ and $V^{-1} = V^*$.

Ribbon graphs: the category

The v-colored ribbon graphs over $\mathcal{V}$ may be regarded as a strict monoidal category denoted by $\text{Rib}_{\mathcal{V}}$:

- objects are finite sequences $\eta = ((V_1, \epsilon_1), \ldots, (V_m, \epsilon_m))$,
- morphisms are (isotopy types of) v–colored ribbon graphs

The tensor product in $\text{Rib}_{\mathcal{V}}$ acts on the objects by juxtaposition, while the morphisms are placed next to each other without overlapping.

Composition of morphisms is basically obtained by putting one colored ribbon graph on top of the other and gluing them.

Operator invariant

Theorem 2.2. Let $\mathcal{V}$ be a strict ribbon category with braiding c, twist θ, and compatible duality ($*$, b, d). There exists a unique covariant functor $F = F_{\mathcal{V}} : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$ preserving the tensor product and satisfying the following conditions:

(1) F transforms any object $(V, +1)$ into V and any object $(V, -1)$ into V^*;

Operator invariant

Theorem 2.2. Let $\mathcal{V}$ be a strict ribbon category with braiding c, twist θ, and compatible duality ($*$, b, d). There exists a unique covariant functor $F = F_{\mathcal{V}} : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$ preserving the tensor product and satisfying the following conditions:

(1) F transforms any object $(V, +1)$ into V and any object $(V, -1)$ into V*;

(2) for any objects V, W of $\mathcal{V}$, we have

$$F(X^+_{V,W}) = c_{V,W}, \;\; F(\varphi_V) = \theta_V, \quad F(\cap_V) = b_V, \;\; F(\cup_V) = d_V;$$

Operator invariant

(3) for any elementary v-colored ribbon graph Γ, we have $F(\Gamma) = f$ where f is the color of the only coupon of Γ.

Moreover, the functor F has the following properties:

$$F(X^-_{V,W}) = (c_{W,V})^{-1},\ F(Y^+_{V,W}) = (c_{W,V*})^{-1},\ F(Y^-_{V,W}) = c_{V*,W},$$

$$F(Z^+_{V,W}) = (c_{W*,V})^{-1} \quad F(Z^-_{V,W}) = c_{V,W*},$$

$$F(T^+_{V,W}) = c_{V*,W*}, \quad F(T^-_{V,W}) = (c_{W*,V*})^{-1}, \quad F(\varphi'_V) = (\theta_V)^{-1}.$$

Operator invariant

(3) for any elementary v-colored ribbon graph Γ, we have $F(\Gamma) = f$ where f is the color of the only coupon of Γ.

Moreover, the functor F has the following properties:

$$F(X^-_{V,W}) = (c_{W,V})^{-1}, \ F(Y^+_{V,W}) = (c_{W,V*})^{-1}, \ F(Y^-_{V,W}) = c_{V*,W},$$

$$F(Z^+_{V,W}) = (c_{W*,V})^{-1} \quad F(Z^-_{V,W}) = c_{V,W*},$$

$$F(T^+_{V,W}) = c_{V*,W*}, \quad F(T^-_{V,W}) = (c_{W*,V*})^{-1}, \quad F(\varphi'_V) = (\theta_V)^{-1}.$$

- The term *operator invariant* is meant to recall the following properties of F:

$$F(\downarrow_V) = \mathrm{id}_V, \ F(\uparrow_V) = \mathrm{id}_{V^*} \quad \text{and} \quad F(\Omega\Omega') = F(\Omega)F(\Omega')$$

for any pair of composable ribbon graphs. Moreover, by definition, note F is a functor that preserves the tensor product, $F(\Omega \otimes \Omega') = F(\Omega) \otimes F(\Omega')$.

Final remarks

- The results we may obtain for strict ribbon categories can be extended to ribbon categories according to Mac Lane's coherence theorem.
- We related the topology of ribbon graphs with the algebra of ribbon categories through coloring by objects and morphisms.
- Braiding, twist and duality are the elementary structures to build up a consistent theory of isotopy invariants.
- The functor F in theorem 2.2 can be regarded as a TFT in Euclidean 3-space and a fundamental tool for the construction of invariants of 3-manifolds.

Next: examples of invariants of ribbon graphs.