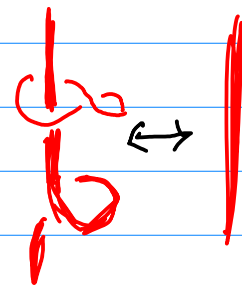
S2: Surgery invariants

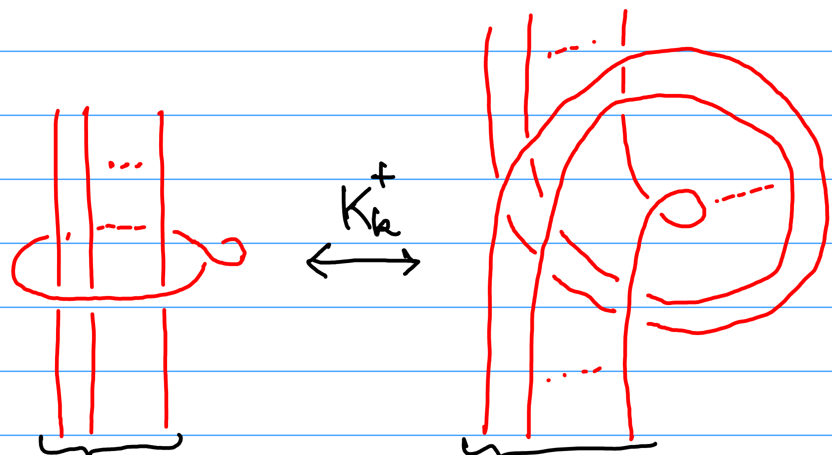
References:
[BK] Bakalov, Kirillov, Lectures on Tensor Categories and Modular Functors (AMS 2001)
[PS] Prasolov, Sossinsky, Knots, links, braids and 3-manifolds (AMS 1997)
[Tu] Turaev, Quantum Invariants of Knots and 3-Manifolds (de Gruyter, 2010)

1) Kirby calculus

Fenn–Rourke moves on links



in b.b.f.



k strands k strands $k \geq 0$

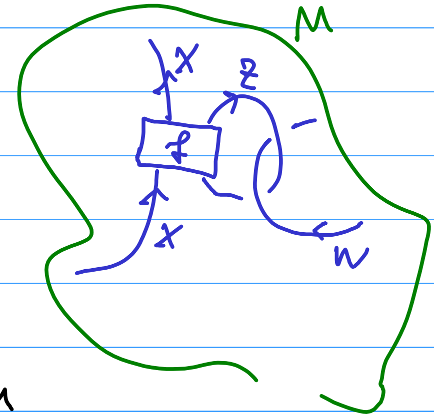
+ version with opposite crossings : K_k^-

<u>Thm</u> (Kirby, Fenn–Rourke 1978/79)

Let L, L' be framed links in S^3. Then $M_L \simeq M_{L'}$ iff L and L' are related by a finite sequence of Fenn–Rourke moves.

From [PS]: "... Unfortunately, our efforts to make the proof of the "hard part" of Kirby's theorem accessible enough for an introductory course did not meet with success, and we regretfully omit it ..."

With embedded ribbon graph:



M : cl. or. comp. conn. 3mf

C : ribbon cat

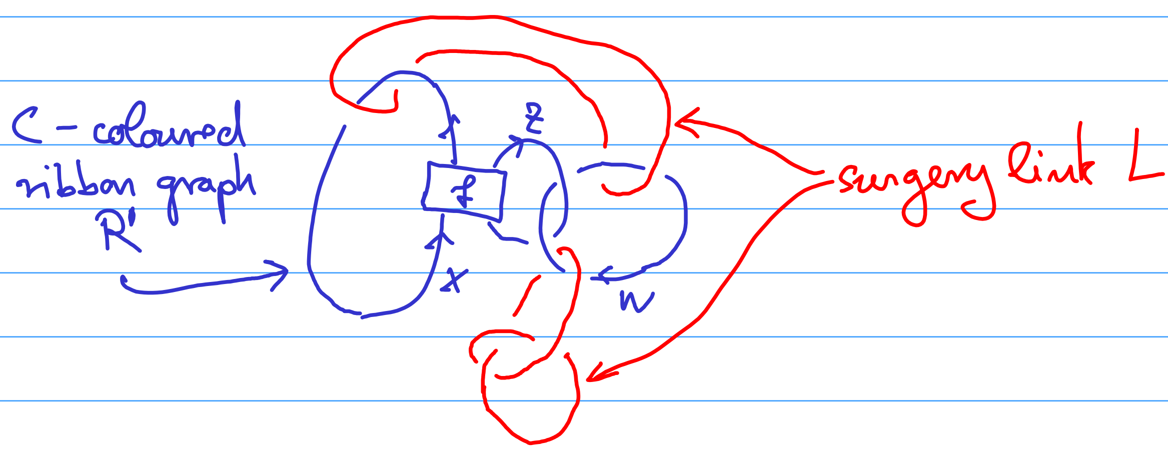
$R \subset M$: C-coloured ribbon graph

Suppose $M \xrightarrow[f]{\sim} M_L$ for some link L in S^3

1) By deforming R in M, make sure that:

$f(R)$ disjoint from tubular nbhood N_L of L

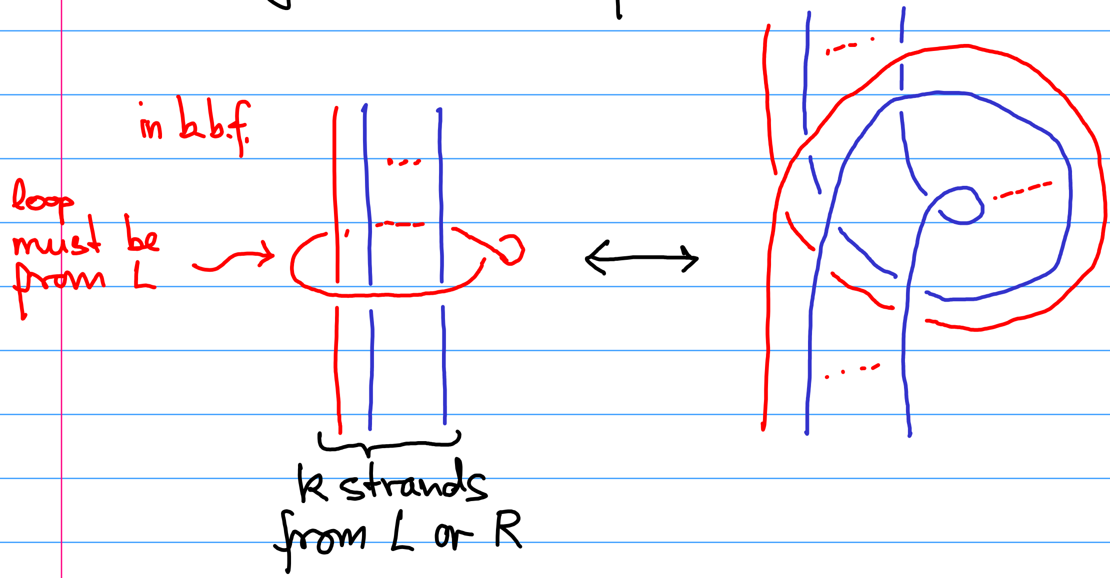
2) Get bichrome graph in S^3: $R' +$ surgery link



Cor. Every (M, R) is isom. to (M_L, R') for some surgery link L

Thm (Reshetikhin–Turaev '92)

$(M_L, R) \sim (M_{L'}, R')$ iff $\overbrace{L \cup R}^{\text{bichr. graph in } S^3}$ and $L' \cup R'$ are related by a finite sequence of the following moves:



& version with opposite crossings

Idea: Find a link inv. that does not change under FR-moves

2) Invariants from modular categories

$\mathcal{C}$: modular category : k-linear (alg. closed field)
finitely semisimple, $\mathbb{1}$ simple
ribbon

I : (finite) set of representatives of isocl. of simple objects in $\mathcal{C}$.

$i, j \in I$ $\quad s_{ij} :=$  is a non-deg $|I| \times |I|$ matrix

'absolutely simple' : $\mathrm{End}(U) = K$ $\qquad K = \mathrm{End}(\mathbb{1})$

Ex K is alg. cl. field

$\Rightarrow$ simple $\qquad R$ ring $\qquad M$ R-module

R-mod $\qquad M$ simple $\Leftrightarrow$ only submod are 0 and M

$\mathcal{C}$ abelian $\qquad U \in \mathcal{C}$ simple

$\Leftrightarrow$ no nontriv. subobj

$V \xrightarrow{\text{mono}} U$

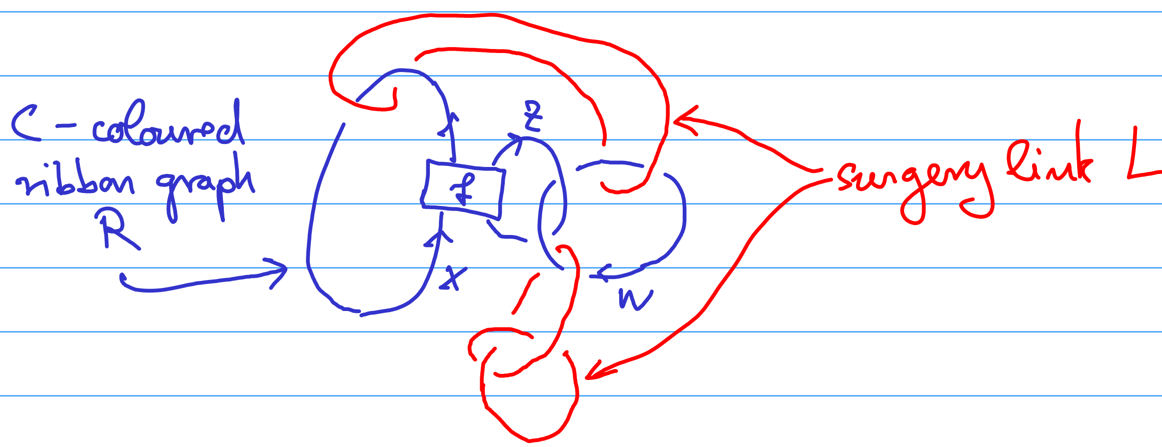
Had ribbon functor

$$F : \mathrm{Rib}_{\mathcal{C}} \longrightarrow \mathcal{C}$$

In particular, for closed diagram $\emptyset \xrightarrow{D} \emptyset$:

$$\underset{\mathrm{Inv}(D)}{\underset{\shortparallel}{F(D)}} \in \mathrm{End}(\mathbb{1}) = k \quad \text{, so get a number}$$

Bichrome ribbon graphs :



Extend Inv to bichrome case :

$$\mathrm{Inv}(R \sqcup L_1 \sqcup \dots \sqcup L_m)$$

$$:= \sum_{i_1, \dots, i_m \in I} \dim(i_1) \cdots \dim(i_m) \; \mathrm{Inv}\big(R \sqcup \vec{L}_1(i_1) \sqcup \dots \sqcup \vec{L}_m(i_m)\big)$$

pick some orientation
of L_1 and label by
simple object i_1

E.g.

$$\mathrm{ev}\big(\bigcirc^{n}\big) = \sum_{i \in I} \dim(i)\, \underbrace{\mathrm{ev}\big(\bigcirc_i^{\theta^n}\big)}_{= \theta_i^n \dim(i)}$$

$$= \sum_{i \in I} \dim(i)^2\, \theta_i^n$$

Set

$$\mathrm{ev}\big(\bigcirc^{\pm 1}\big) =: \Delta_{\pm} \quad \Big(= \sum_i \dim(i)^2\, \theta_i^{\pm 1} \Big)$$

<u>Lem</u> $\Delta_{\pm} \neq 0$ and $\Delta_+ \Delta_- = \sum_{i \in I} \dim(i)^2$

Pf: [BK] Sec. 3, uses modularity of C.

Choose $D \in k$ s.th. $D^2 = \Delta_+ \Delta_-$.

Set

$$\delta = \frac{D}{\Delta_-} = \frac{D^2}{\Delta_- D} = \frac{\Delta_+ \Delta_-}{\Delta_- D} = \frac{\Delta_+}{D} \text{ "}= \sqrt{\frac{\Delta_+}{\Delta_-}}\text{"}$$

Let M, R : 3mf with rib. gr.

$L \subset S^3$ link, $\tilde{R} \subset S^3$ ribbon gr. s.th.

$(M,R) \simeq (M_L, \tilde{R})$

Set

$$\tau(M,R) := D^{-1-|L|} \, \delta^{-\sigma(L)} \, \mathrm{Inv}(L \sqcup \tilde{R})$$

components of L

signature of L
:= signature of the intersection form on H_2 of a bounding 4 mfld for M_L

Thm (Reshetikhin–Turaev '91, Turaev '94)

$\tau(M,R)$ is a 3mf invariant

3) Some steps in proof

Only consider $\delta = 1$ "anomaly free modular cat." (*)

Then $\Delta_- = \Delta_+ = \mathcal{D}$ and

$$\tau(M, R) = \mathcal{D}^{-1-|L|} \operatorname{inv}(L \sqcup \tilde{R})$$

E.g.

$$\tau(S^3, \phi) = \mathcal{D}^{-1} \cdot \operatorname{inv}(\phi) = \mathcal{D}^{-1}$$

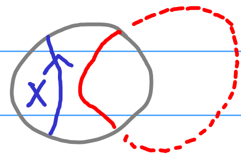
$$\tau(S^3, \phi) = \mathcal{D}^{-2} \operatorname{inv}(\bigcirc^{\pm 1}) = \mathcal{D}^{-2} \Delta_{\pm} \overset{(*)}{=} \mathcal{D}^{-1}$$

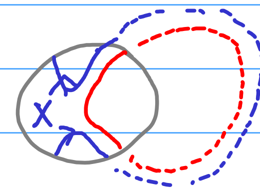
$$\tau(S^2 \times S^1, \phi) = \mathcal{D}^{-2} \underbrace{\operatorname{inv}(\bigcirc^{0})}_{} = \mathcal{D}^{-2} \mathcal{D}^{2} = 1$$

$$= \sum_i \dim(i) \underbrace{\operatorname{inv}(\bigcirc_i)}_{\dim(i)}$$

Lem. (Edge slides)

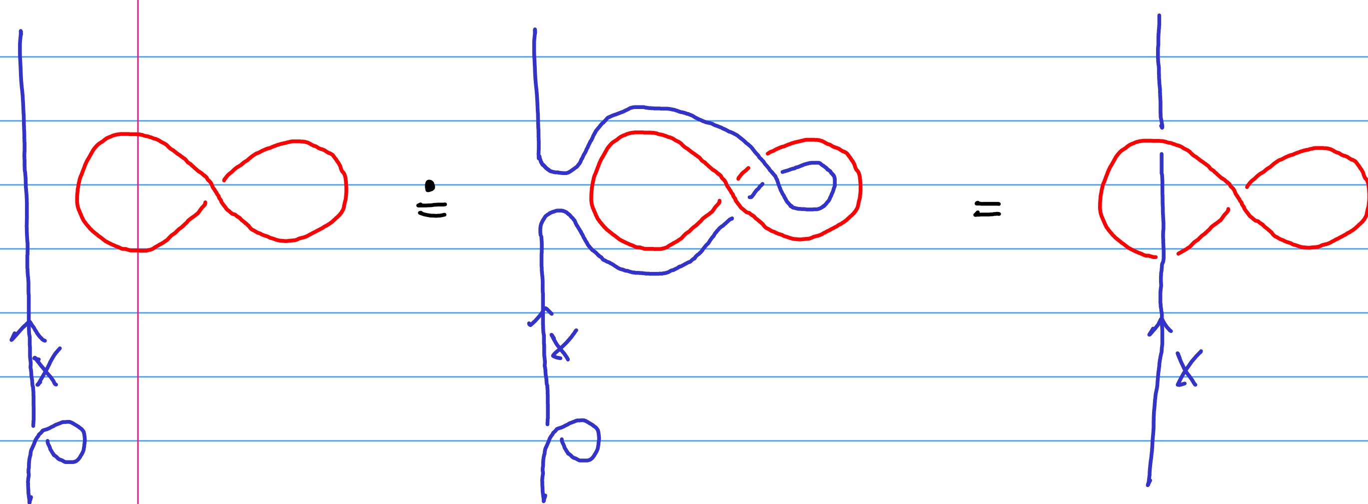
can knot in any way

Inv(

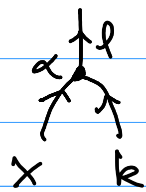
) = Inv(

)

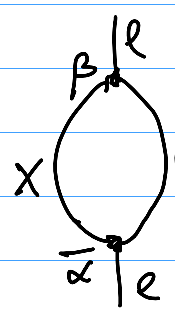
This implies inv. under FR, because



Let $\{\alpha\}$ be basis of $\mathrm{Hom}(X \otimes k, \ell)$ $\quad k, \ell \in I$



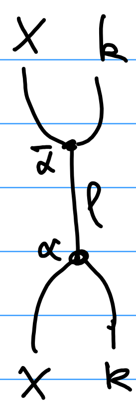
Dual basis $\{\bar{\alpha}\}$ of $\mathrm{Hom}(\ell, X \otimes k)$ s.th.



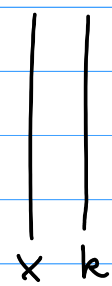
$= \delta_{\alpha,\beta} \cdot id_\ell$

<u>Lem 1</u>

$\sum_{\ell \in I, \alpha}$

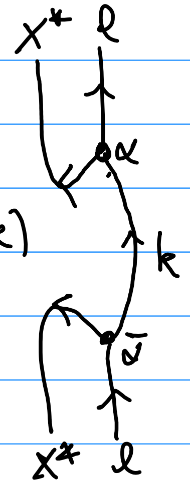


$=$

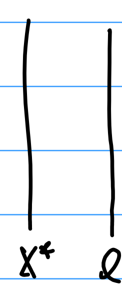


<u>Lem 2</u>

$\sum_{k \in I, \alpha} \dim(k)$

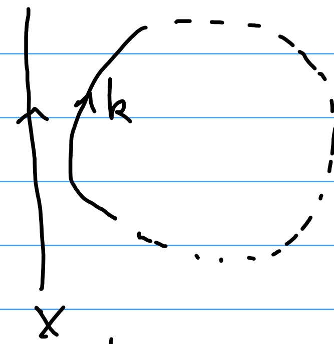


$= \dim(\ell)$



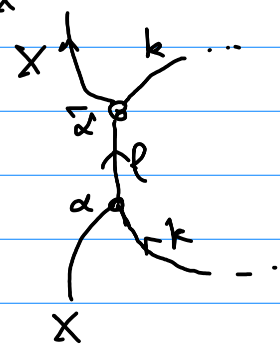
Pf of edge slide :

$$\sum_{k \in I} \dim(k)$$

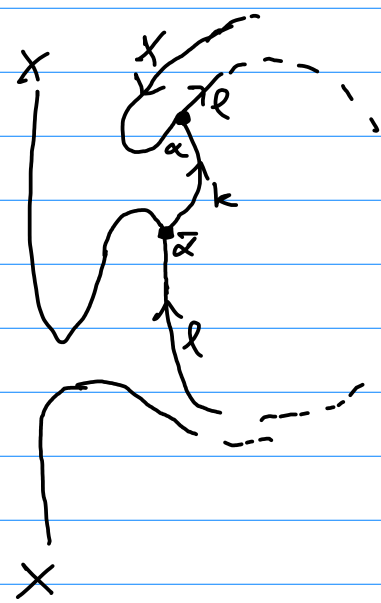


Lem 1

$$= \sum_{k,\ell,\alpha} \dim k$$

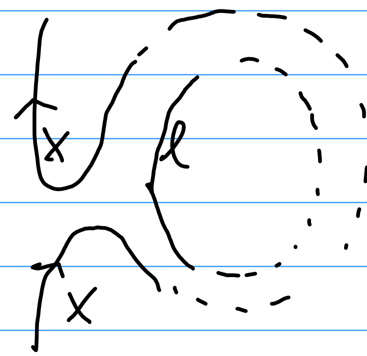


$$= \sum_{k,\ell,\alpha} \dim k$$



Lem 2

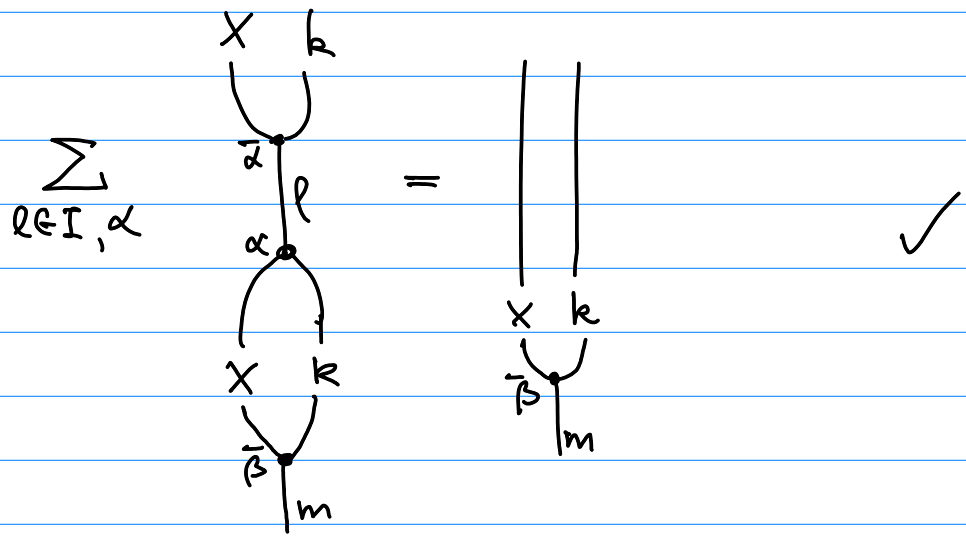
$$= \sum_{\ell} \dim \ell$$



$\square$

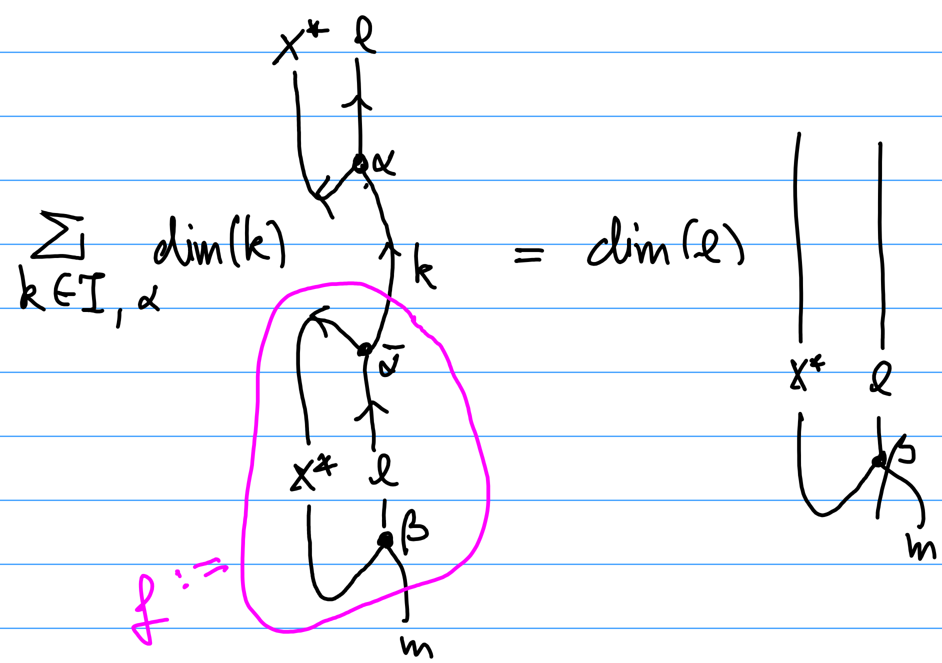
Pf of Lem 1:

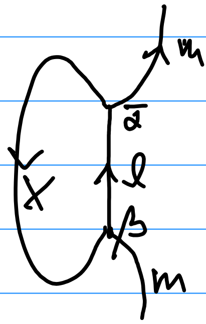
Enough to show for all m, β:



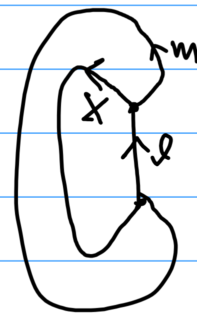
Pf of Lem 2:

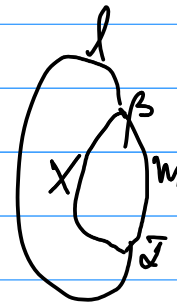
Enough to show for all m, β:



$f = \delta_{k,m}$ 

$\Rightarrow f = \lambda \cdot \mathrm{id}_m$ with

$\dim m \cdot \lambda = \mathrm{tr} f = \delta_{k,m}$ 

$= \delta_{k,m}$ 

$\Rightarrow f = \delta_{k,m}\, \delta_{\alpha,\beta} \dfrac{\dim \ell}{\dim m}$

✓