Bordism category and definition of a TFT

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Seminar on Hopf algebras, tensor categories and three manifold invariants
October 06, 2020

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Vector spaces

Let $\mathbb{K}$ denote an arbitrary field, then there is a symmetric monoidal category $\mathrm{Vec}_{\mathbb{K}}$ described by

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2. **Morphisms:** $\mathbb{K}$-linear maps $f : V \to W$.
3. **Tensor product:** $V \otimes_{\mathbb{K}} W$ the usual tensor product with monoidal unit $\mathbb{K}$.
4. **Associativity constraints:** the canonical isomorphisms

$$(U \otimes_{\mathbb{K}} V) \otimes_{\mathbb{K}} W \to U \otimes_{\mathbb{K}} (V \otimes_{\mathbb{K}} W),\ (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$$

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5. **Braiding:** the canonical isomorphisms

$$U \otimes_{\mathbb{K}} V \to V \otimes_{\mathbb{K}} U,\ u \otimes v \mapsto v \otimes u$$

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1. **Objects:** oriented closed $(n-1)$-dimensional smooth manifolds M.
2. **Morphisms:** A morphism between manifolds M and N is an equivalence class of bordisms $W = (B; M, N)$. Where a bordism is an n-dimensional oriented smooth manifold B with boundary $M^* \sqcup N$.

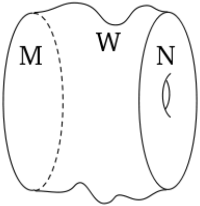
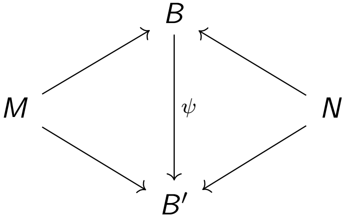


Figure 1: Bordism between manifolds M and N

An equivalence of bordisms is an orientation-preserving diffeomorphism $\psi : B \to B'$ such that the following diagram commutes



3. **Composition:** Is given by gluing bordisms along their common boundary and the identity of an object M is the cylinder $M \times [0,1]$.

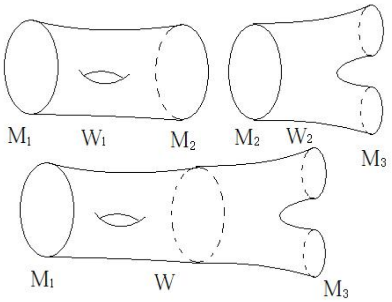


Figure 2: Composition of bordisms $W = W_1 \sqcup_{M_2} W_2$

Remark

A priori $W_1 \sqcup_{M_2} W_2$ is defined as a topological space. To get the smooth structure we consider "collars" on the $(n-1)$-manifolds, i.e., $M_2 \times (-\epsilon, \epsilon)$, the boundary parametrisations are defined on $M_2 \times (-\epsilon, 0]$ and $M_2 \times [0, \epsilon)$.

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4. **Tensor Product:** It is given by the disjoint union of the manifolds and the unit is the empty manifold $\emptyset$. The disjoint union of manifolds is associative.
5. **Braiding:** Given two $(n-1)$-manifolds M and N, the cylinder provides a braiding in Bord_n

$$\beta_{M,N} : M \sqcup N \to N \sqcup M$$

Bordism category

An n-dimensional manifold B with boundary $M^* \sqcup N$, it can be seen as

- A morphism $M \to N$.
- A morphism $\emptyset \to M^* \sqcup N$ or $\emptyset \to N \sqcup M^*$.
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If B is an n-dimensional manifold without boundary it can be seen as a morphism $\emptyset \to \emptyset$.

Similarly, a diffeomorphism $\phi : M \to M'$ between $(n-1)$-dimensional manifolds induces an isomorphism in Bord_n via the cylinder

$$M \times [0,1] \xleftarrow{\iota} M \times \{1\} \overset{\phi}{\cong} M'$$

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Topological Field Theories

Definition (Topological Field Theory)

An n-dimensional *TFT* is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_n \to \mathsf{Vec}_{\mathbb{K}}$$

We will see in detail what does this data entail.

1. **Assignment on objects:**
   For every manifold M we obtain a vector space $\mathcal{Z}(M)$.
2. **Assignment on morphisms:**
   For every bordism $B : M \to N$ there is a linear map

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3. **Functoriality:**
   For every $M \in \mathrm{Bord}_n$, the morphism $\mathcal{Z}(M \times [0,1]) = \mathrm{id}_{\mathcal{Z}(M)}$.
   For bordisms $B_1 : M \to N$ and $B_2 : N \to L$,

$$\mathcal{Z}(B_2 \sqcup_N B_1) = \mathcal{Z}(B_2) \circ \mathcal{Z}(B_1) : \mathcal{Z}(M) \to \mathcal{Z}(L)$$

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4. **Monoidal structure:**
   - $\mathcal{Z}(\emptyset) \cong \mathbb{K}$.
   - $\mathcal{Z}(M \sqcup N) \cong \mathcal{Z}(M) \otimes_{\mathbb{K}} \mathcal{Z}(N)$ for $M, N \in \mathrm{Bord}_n$.

Some properties

Notice that given an n-manifold B with boundary $\partial B = M$, from the bordism $B : \emptyset \to M$ we obtain a map $\mathcal{Z}(B) : \mathbb{K} \to \mathcal{Z}(M)$.

Moreover, in the case that B has no boundary we obtain a map

$$\mathcal{Z}(B) : \mathbb{K} \to \mathbb{K}$$

and therefore an invariant.

Topological Field Theories

Proposition

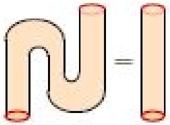
Let $\mathcal{Z} : Bord_n \to Vec_{\mathbb{K}}$ be a TFT, then for every $M \in \mathrm{Bord}_n$ we have that $\mathcal{Z}(M)$ is a finite dimensional vector space. Additionally $\mathcal{Z}(M^*) \cong \mathcal{Z}(M)^*$.

Proof: Let $M \in \mathrm{Bord}_n$, and consider the cylinder $M \times [0,1]$, which can be viewed as a morphism

$$\mathsf{ev}_M : M^* \sqcup M \to \emptyset, \qquad \mathsf{coev}_M : \emptyset \to M \sqcup M^*$$

Then the following equation holds due to diffeomorphism invariance

$$\mathsf{ev}_M \sqcup \mathsf{id}_{M^*} \circ \mathsf{id}_{M^*} \sqcup \mathsf{coev}_M = \mathsf{id}_{M^*} \tag{1}$$



Proof: (continuation)
Now denote by $U := \mathcal{Z}(M)$ and $V := \mathcal{Z}(M^*)$, then we obtain, by applying $\mathcal{Z}$, the following linear maps

$$\langle -,-\rangle := \mathcal{Z}(\mathrm{ev}_M) : V \otimes_{\mathbb{K}} U \to \mathbb{K}, \qquad f := \mathcal{Z}(\mathrm{coev}_M) : \mathbb{K} \to U \otimes_K V$$

and therefore equation (1) becomes the identity

$$(\langle -,-\rangle \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes f) = \mathrm{id}_V \tag{2}$$

Now consider $1 \in \mathbb{K}$, then we can write $f(1) = \sum_{i=1}^n u_i \otimes v_i$ with $u_i \in U$ and $v_i \in V$. For $v \in V$ equation (2) implies that

$$v = \sum_{i=1}^n \langle v, u_i \rangle v_i$$

thus V is finite-dimensional. Additionally $V \to U^*, v \mapsto \langle v, - \rangle$ is an iso.

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Revisit of Bord$_1$:

Objects: Oriented 0-dimensional closed manifolds correspond to disjoint unions of $\bullet_+$ and $\bullet_-$.

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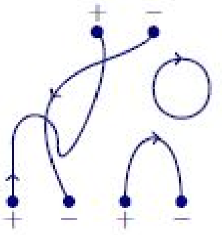
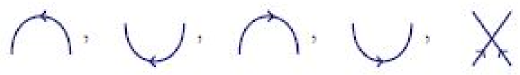
Morphisms: Are diffeomorphism classes of lines connecting the oriented points. For example



is a morphism between $\bullet_+ \sqcup \bullet_- \sqcup \bullet_+ \sqcup \bullet_-$ and $\bullet_+ \sqcup \bullet_-$.

1d TFT's

In general every bordism in Bord_1 is generated by gluing and taking disjoint unions of



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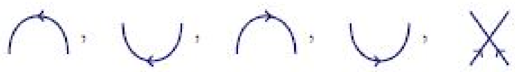

subject to the relations coming from the invariance under diffeomorphism

Theorem

There is one-to-one correspondence between 1d TFT's $\mathcal{Z} : \mathrm{Bord}_1 \to \mathrm{Vec}_{\mathbb{K}}$ and finite-dimensional vector spaces, via $\mathcal{Z} \mapsto \mathcal{Z}(\bullet_+)$.

1d TFT's

Conversely, given a finite-dimensional space define $\mathcal{Z}(\bullet_+) := V$ and $\mathcal{Z}(\bullet_-) := V^*$. For the 0-manifold $M := \bullet_+ \sqcup \bullet_+ \sqcup \bullet_- \sqcup \bullet_-$, the space $\mathcal{Z}(M) = V \otimes_{\mathbb{K}} V \otimes_{\mathbb{K}} V^* \otimes_{\mathbb{K}} V^*$ is assigned.

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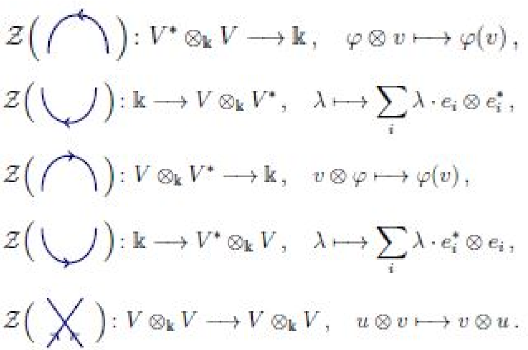
If we consider a basis $\{e_i\}$ of V, to the bordism generators we assign the linear maps



$$\mathcal{Z} : \ \mathsf{Bord}_2 \to \mathsf{Vec}_{\mathbb{K}}$$

Revisit of Bord$_2$:

Objects: Oriented 1-dimensional closed manifolds correspond to disjoint unions of the circle S^1.

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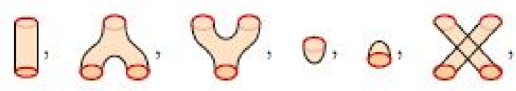

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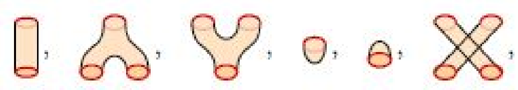
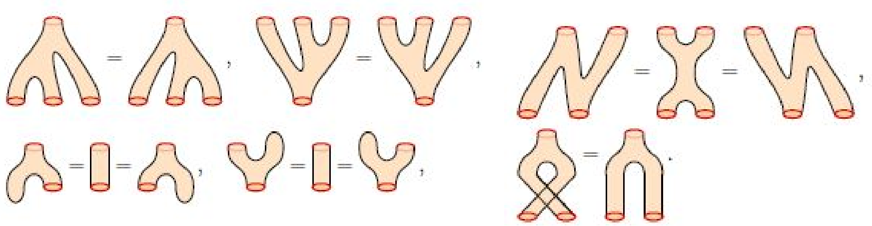
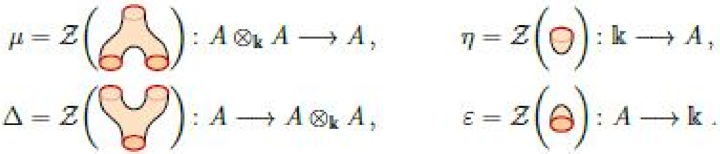


under the relations



We obtain a finite-dimensional space $A := \mathcal{Z}(S^1)$. Which comes together with linear maps coming from the bordism generators



This maps provide a Frobenius algebra structure on A.

2d TFT's

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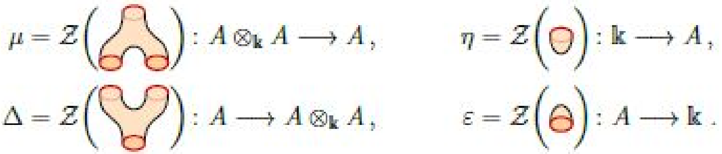


This maps provide a Frobenius algebra structure on A.

Definition

A *Frobenius algebra* is a vectors space A together with

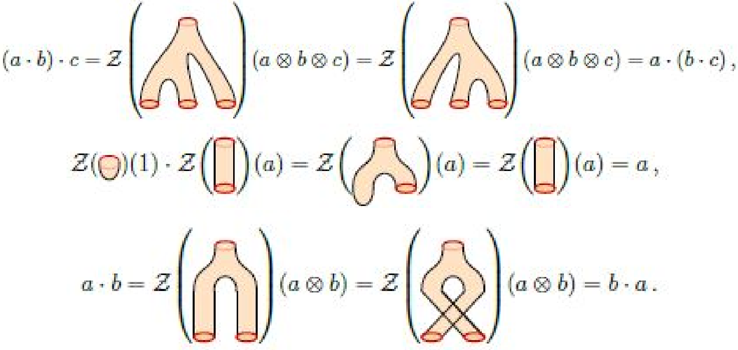
- An associative algebra structure (A, μ, η).
- A coassociative coalgebra structure (A, Δ, ϵ).

fulfilling the Frobenius property, i.e.,

$$(\mu \otimes id) \circ (id \otimes \Delta) = \Delta \circ \mu = (id \otimes \mu) \circ (\Delta \otimes id)$$

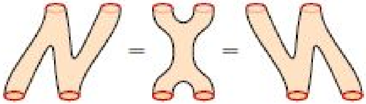
2d TFT's

The maps μ and η fulfill the corresponding relations and determine an algebra structure on A which is commutative.



Similarly, the maps Δ and ϵ determine a coalgebra structure on A.

The Frobenius property is fulfilled by using functoriality of $\mathcal{Z}$ on the relation



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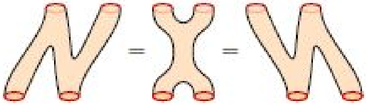

Theorem

There is a one-to-one correspondence between 2-dimensional TFT's $\mathcal{Z} : Bord_2 \to Vec_{\mathbb{K}}$ *and commutative Frobenius algebras A.*