MSc Seminar on Hopf Algebras, tensor categories and three-manifold invariants: The Reshetikhin-Turaev construction

Paulina Goedicke

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Abstract

In this session we will extend the notion of an n-dimensional topological quantum field theory (TQFT) defined in the last session to a 3D-TQFT for 3-bordisms with embedded ribbon graphs.

1 Motivation

Last session we defined n-dimensional TQFTs via a functor $\mathcal{Z}$ from the bordism category $\mathbf{Bord}_n$ to the category of vector spaces $\mathbf{Vect}_{\mathbb{K}}$. In this session we will construct an extended TQFT by replacing manifolds in $\mathbf{Bord}_n$ by manifolds with some additional structure.

2 Bordism category of decorated 3-manifolds

Definition 1.1: A 3-$manifold$ is a manifold that locally looks like $\mathbb{R}^3$.

Let C be a modular tensor category. The additional structure mentioned earlier is given by the notion of *decoration* and *colouring*.

Definition 1.2: We say that a conntected orientable surface is *decorated* if it is oriented and comes with a countable set of distinguished marked arcs where a marked arc is a a simple arc together with an object of C and a sign $v \in \{-1, 1\}$. We call these surfaces d-$surfaces$. Morphisms between d-surfaces are called d-homeomorphisms.

Definition 1.3: The *d-type* t of a d-surface of genus g and with m marked arcs $\langle W_i, v_i\rangle_{i\leq m}$ is a tuple $\langle g; \langle W_1, v_1\rangle, ..., \langle W_m, v_m\rangle\rangle$.

Recall that we call a ribbon graph *coloured over C* if it has the following additional structure:

i) Each band is directed.

ii) Each band is labeled (coloured) by an Object of C.

iii) Each coupon is labeled by a morphism of C

$$f : V_1^{\eta_1} \otimes ... \otimes V_m^{\eta_m} \to W_1^{\epsilon_1} \otimes ... \otimes W_n^{\epsilon_n}$$

where the V_i are the colours and the η_i the directions of the bands incident to the top edge and the W_i are the colours and the ϵ_i are the directions of the bands incident to the bottom edge.

This leads us to the following definition:

Definition 1.4: Let M be a 3-manifold whose boundary is endowed with a finite family of disjoint marked arcs.

A *ribbon graph in* M is an oriented surface Ω embedded in M and decomposed as a union of a finite number of directed annuli, directed bands and coupons such that Ω meets ∂M transversally along the distinguished arcs in ∂M which are bases of certain bands of Ω, other bases of bands lie on the bases of coupons, otherwise bands, coupons and annuli are disjoint. Moreover, the orientation of Ω induces on each arc of ∂M the orientation opposite to the given one.

Definition 1.5: A *decorated* 3-$manifold$ is a compact oriented 3-manifold with parametrized decorated boundary and with a v-coloured ribbon graph sitting in this 3-manifold.

Remark 1.6: By parametrized we mean that the boundary is homeomorphic to Σ_t where $\Sigma_t = \partial U_t$ is the so-called canonical surface of a d-type t. The latter can be constructed as follows: Let R_t be a ribbon graph with one coupon and $m+g$ bands. The first m bands are untwisted and unlinked. For $i \leq m$, the i-th band is labeled with the respective V_i, where the sign determines the orientation of the band. Moreover, there are g bands that form unknots with the coupon. We can then fix a closed regular neighbourhood U_t of R_t which is a handlebody of genus g. Except for the m bands that meet the boundary ∂U_t, R_t lies in the interior of U_t. We then set $\Sigma_t := \partial U_t$. With that we can also define decorated 3-bordisms:

Definition 1.7: A *decorated* 3-*bordism* is a triple $(M,\, \partial_- M,\, \partial_+ M)$ where $\partial_- M$ and $\partial_+ M$ are parametrized d-surfaces and M is a decorated 3-manifold with $\partial M = (-\partial_- M) \coprod \partial_+ M$.

Remark 1.8: In particular a d-homeomorphism of decorated 3-manifolds $M \to M'$ restricts to a d-homomorphism $\partial M \to \partial M'$ that commutes with the parametrizations.

We can construct a bordism category $\mathbf{Bord}_3$ where objects in $\mathbf{Bord}_3$ are given by classes of homeomorphisms of decorated 3-manifolds and morphisms are d-homeomorphisms of 3-manifolds.

3 Construction of a TQFT

We are now ready to define $\mathcal{Z}(N)$ for N in $\mathbf{Bord}_3$. Recall that the functor $\mathcal{Z}$ sends to every manifold N in $\mathbf{Bord}_n$ a $\mathbb{K}$-vectorspace $\mathcal{Z}(N)$ and to every bordism M in $\mathbf{Bord}_n$ from $\partial_- M$ to ∂_+ a $\mathbb{K}$-linear map $\mathcal{Z}(M) : \mathcal{Z}(\partial_- M) \to \mathcal{Z}(\partial_+ M)$.

Observation 2.1: We start by defining the space of states. For each d-type t we can define a projective $\mathbb{K}$-module Ψ_t via

$$\Phi(t;i) = W_1^{v_1} \otimes ... \otimes W_m^{v_m} \otimes \bigotimes_{r=1}^{g} (V_{i_r} \otimes V_{i_r}^*) \tag{1}$$

and setting

$$\Psi_t = \bigoplus_{i \in I^g} \mathrm{Hom}(\mathbb{I}, \Phi(t;i)). \tag{2}$$

We then define $\mathcal{Z}(N)$ to be the non-ordered tensor product of Ψ_t where t runs over types t of the components of N.

Next we assign to every 3-bordism a $\mathbb{K}$-homomorphism (where τ is the operator invariant of M)

$$\tau(M) = \tau(M, \partial_- M, \partial_+ M) : \mathcal{Z}(\partial_- M) \to \mathcal{Z}(\partial_+ M) \tag{3}$$

such that:

i) For any connected component Σ of $\partial_- M$ of type $t = t(\Sigma)$, glue U_t, to M along the given parametrization $\partial U_t = \Sigma_t \to \Sigma$. These gluings are performed with respect to all components of $\partial_- M$.

ii) In the same way we glue U_t^- to M along the d-homeomorphism $\partial U_t^- \to -\Sigma$ for every component Σ of $\partial_+ M$ of type $t = t(\Sigma)$.

These gluings lead to a closed oriented 3-manifold $\tilde{M}$ with embedded ribbon graph $\tilde{\Omega}$ where $\tilde{\Omega}$ can be obtained by gluing Ω, the given ribbon graph in M, and the ribbon graphs in the standard handlebodies.
The colouring of the extension of $\tilde{\Omega}$ over Ω is not unique. Fixing a colouring we can apply the topological invariant τ of v-coloured ribbon graphs and get a certain $\tau(\tilde{M}, \tilde{\Omega}, y) \in \mathbb{K}$. This yields an element of $\mathbb{K}$. The assignment is polylinear with respect to the colours of coupons and thus yields a $\mathbb{K}$-homomorphism

$$\mathcal{Z}(\partial_- M) \otimes (\mathcal{Z}(\partial_+ M))^* \to \mathbb{K}. \tag{4}$$

The action of $\tau(M)$ is now defined as the composition of the adjoint transpose $\mathcal{Z}(\partial_- M) \to \mathcal{Z}(\partial_+ M)$ with the endomorphism $\eta(\partial_+ M) : \mathcal{Z}(\partial_+ M) \to$

$\mathcal{Z}(\partial_+ M)$ which is defined on the summands $\mathrm{Hom}(\mathbb{I}, \Phi(t;i))$ by multiplication with $(\mathrm{rk}(C))^{1-g}\Pi_{n=1}^{g}\dim(i_g)$ and on non-connected surfaces Σ_1,Σ_2 such that $\eta(\Sigma_1 \coprod \Sigma_2) = \eta(\Sigma_1) \otimes \eta(\Sigma_2)$ and $\eta(\emptyset) = \mathrm{id}_{\mathbb{K}}$.

Theorem 2.2: The function $\tau(M) = \tau(M, \partial_- M, \partial_+ M) : \mathcal{Z}(\partial_- M) \to \mathcal{Z}(\partial_+ M)$ extends the functor $\mathcal{Z}$ to a non-degenerate TQFT.

This implies in particular that we get a TQFT $(\mathcal{Z}, \tau)$ based on parametrized d-surfaces and decorated 3-manifolds.

In proving this theorem one has to check that the axioms for a TQFT are satisfied. The main point is here to explicitly show functoriality where the idea is to use a geometric technique that enables us to present decorated 3-bordisms by ribbon graphs in $\mathbb{R}^3$ and to express the operator invariants of 3-bordisms through operator invariants of ribbon graphs.

References

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