T3) The universal construction

[Reference: [Co] Costantino, Notes on Topological Quantum Field Theories, Winter Braids Lecture Notes (2015), 1–45]

Quantization functors

Consider a cobordism category Bord, i.e. a category together with an empty object $\varnothing$ and the notions of disjoint union, orientation reversal and boundary.

Definition.
A functor $V\colon \mathrm{Bord} \to \mathrm{Vec}_{\mathbb{K}}$ satisfying $V(\varnothing) \cong \mathbb{K}$ is called *quantization functor*.

Remark.
It depends on the author, what the exact definition of a quantization functor is. In this handout we go along with the definition of [Co].

Remark.
Obviously, any monoidal functor from Bord to $\mathrm{Vec}_{\mathbb{K}}$ is also a quantization functor. In particular, TFTs are quantization functors.

Quantization functors

DEFINITION.
A quantization functor $V\colon \mathrm{Bord} \to \mathbb{K}$ is called *cobordism generated*, if for all objects Σ the associated vector space $V(\Sigma)$ is generated by the elements $V(M)(1)$ with $M \in \mathrm{Hom}(\varnothing, \Sigma)$, i.e.

$$V(\Sigma) = \mathrm{span}\,\{V(\mathrm{Hom}(\varnothing, \Sigma))(1)\}\,.$$

Invariants

Definition.

An *invariant (of n–dimensional manifolds)* is a map $\langle - \rangle$ from closed oriented smooth manifolds of dimension n to a field $\mathbb{K}$, which is constant on diffeomorphism classes.

Definition.

We say an invariant $\langle - \rangle$ is *multiplicative*, if we have

- $\langle M_1 \sqcup M_2 \rangle = \langle M_1 \rangle \langle M_2 \rangle$ for all closed n–dimensional oriented smooth manifolds M_1, M_2 and
- $\langle \varnothing \rangle = 1$.

The universal construction theorem

THEOREM.
Let $\mathrm{Bord_n}$ *be a cobordism category and* $\langle - \rangle : \mathrm{Hom}(\varnothing, \varnothing) \to \mathbb{K}$ *a multiplicative diffeomorphism invariant of n–dimensional manifolds, where* $\mathrm{Hom}(\varnothing, \varnothing)$ *is referred to as a* Hom*–space of the category* $\mathrm{Bord_n}$*.*

Then there exists a unique cobordism generated quantization functor $V : \mathrm{Bord_n} \to \mathrm{Vec}_{\mathbb{K}}$ *whose restriction to* $\mathrm{Hom}(\varnothing, \varnothing)$ *is the given invariant* $\langle - \rangle$*.*

The universal construction theorem – Proof

Proof.

Denote by

$$F(\Sigma) = \mathrm{span}\left\{\mathrm{Hom}(\varnothing, \Sigma)\right\}$$

the set freely generated by all cobordisms from $\varnothing$ to Σ.

Analogously,

$$F'(\Sigma) = \mathrm{span}\left\{\mathrm{Hom}(\Sigma, \varnothing)\right\}.$$

The universal construction theorem – Proof

Next, we want to define a pairing $\langle -, - \rangle_\Sigma : F'(\Sigma) \otimes F(\Sigma) \to \mathbb{K}$.

On basis elements $M_1 \in F(\Sigma), M_2 \in F'(\Sigma)$ the pairing is the invariant applied to the composition $M_2 \circ M_1$, i.e.

$$\langle M_2, M_1 \rangle_\Sigma := \langle M_2 \circ M_1 \rangle = \langle M_2 \sqcup_\Sigma M_1 \rangle .$$

Extending this definition linearly yields a pairing on $F'(\Sigma) \otimes F(\Sigma)$.

The universal construction theorem – Proof

We define a functor $V\colon \mathrm{Bord_n} \to \mathrm{Vec}_{\mathbb{K}}$ and we start by fixing V on objects via

$$V(\Sigma) := F(\Sigma)/\mathrm{Ann}(F'(\Sigma)),$$

where $\mathrm{Ann}(F'(\Sigma)) = \{x \in F(\Sigma) \mid \langle y, x\rangle_{\Sigma} = 0, \forall y \in F'(\Sigma)\}$.

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Similarly, we define a functor $V' \colon \mathrm{Bord_n^{op}} \to \mathrm{Vec}_{\mathbb{K}}$. On objects we set

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Remark that the pairing $\langle -, -\rangle_\Sigma : F'(\Sigma) \otimes F(\Sigma) \to \mathbb{K}$ descends to a pairing $\langle -, -\rangle_\Sigma : V'(\Sigma) \otimes V(\Sigma) \to \mathbb{K}$, which is non–degenerate by construction.

The universal construction theorem – Proof

We have to define the functors V and V' on morphisms. Let therefore $N \in \mathrm{Hom}(\Sigma_1, \Sigma_2)$ be a morphism in $\mathrm{Bord_n}$. For a basis element $M \in \mathrm{Hom}(\varnothing, \Sigma_1)$ of $V(\Sigma_1)$ we define

$$V(N)[M] := [N \circ M] = [N \sqcup_{\Sigma_1} M].$$

This defines a functor, since for any morphism $N' \in \mathrm{Hom}(\Sigma_2, \Sigma_3)$ we have

$$\begin{aligned} V(N' \circ N)[M] &= [(N' \circ N) \circ M] = [N' \circ (N \circ M)] \\ &= V(N')[N \circ M] = (V(N') \circ V(N))[M]. \end{aligned}$$

The universal construction theorem – Proof

Similarly, for a morphism $M \in \mathrm{Hom}(\Sigma_2, \varnothing)$ we define

$$V'(N)[M] := [M \circ N] = [M \sqcup_{\Sigma_2} N].$$

This defines a contravariant functor $\mathrm{Bord_n} \to \mathrm{Vec}_{\mathbb{K}}$, i.e. a functor $\mathrm{Bord}_\mathrm{n}^{\mathrm{op}} \to \mathrm{Vec}_{\mathbb{K}}$, since for any morphism $N' \in \mathrm{Hom}(\Sigma_0, \Sigma_1)$ we have

$$\begin{aligned} V'(N \circ N')[M] &= [M \circ (N \circ N')] = [(M \circ N) \circ N'] \\ &= V'(N')[M \circ N] = (V'(N') \circ V'(N))[M]. \end{aligned}$$

The universal construction theorem – Proof

Now that we have the two functors V and V', we observe that for all morphisms $N \in \mathrm{Hom}(\Sigma_1, \Sigma_2)$ and basis elements M_1 of $V(\Sigma_1)$ and M_2 of $V(\Sigma_2)$ the equality

$$\langle V'(N)(M_2), M_1 \rangle_\Sigma = \langle M_2 \circ N \circ M_1 \rangle = \langle M_2, V(N)(M_1) \rangle_\Sigma$$

holds and conclude that by linearity and non–degeneracy of $\langle -, - \rangle_\Sigma$ either of the functors V and V' is uniquely determined by the other one.

The universal construction theorem – Proof

To see that V is indeed a quantization functor, we need that the given invariant $\langle - \rangle$ is multiplicative.
We write $y, x \in F(\varnothing) = F'(\varnothing)$ in basis with $k_i, k'_j \in \mathbb{K}$ as $y = \sum_i k_i y_i$ and $x = \sum_j k'_j x_j$ and compute

$$\begin{aligned}
V(\varnothing) &= F(\varnothing)/\mathrm{Ann}(F'(\varnothing)) \\
&= F(\varnothing)/ \left\{x \in F(\varnothing) \mid \langle y, x\rangle_\varnothing = 0, \forall y \in F'(\varnothing)\right\} \\
&= F(\varnothing)/\{x \in F(\varnothing) \mid \sum_{i,j} k_i k'_j \langle y_i, x_j\rangle_\varnothing = 0, \forall y \in F'(\varnothing), \forall i, j\} \\
&= F(\varnothing)/\{x \in F(\varnothing) \mid \sum_{i,j} k_i k'_j \langle y_i \sqcup x_j\rangle = 0, \forall y \in F'(\varnothing), \forall i, j\} \\
&= F(\varnothing)/\{x \in F(\varnothing) \mid \sum_{i,j} k_i k'_j \langle y_i\rangle \langle x_j\rangle = 0, \forall y \in F'(\varnothing), \forall i, j\} \\
&= F(\varnothing)/\{x \in F(\varnothing) \mid \langle x_j\rangle = 0, \forall j\}.
\end{aligned}$$

The universal construction theorem – Proof

For a moment, denote by $\emptyset$ the empty manifold regarded as an element of $\mathrm{Hom}(\varnothing, \varnothing)$. Since $\langle - \rangle$ is multiplicative, we have

$$\langle \emptyset \rangle = 1 \in \mathbb{K}.$$

We can linearly extend $\langle - \rangle$ to $F(\varnothing) = \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \varnothing)\}$ and obtain, that the extension $\langle - \rangle_{ext}$ is surjective onto $\mathbb{K}$, since

$$\langle k\emptyset \rangle_{ext} = k\langle \emptyset \rangle = k \cdot 1 = k, \quad \forall k \in \mathbb{K}.$$

Further, the kernel of the $\langle - \rangle_{ext}$ is $\{x \in F(\varnothing) \mid \langle x_j \rangle = 0, \forall j\}$. Hence, by the isomorphism theorem,

$$V(\varnothing) = F(\varnothing)/\mathrm{ker}\ \langle - \rangle_{ext} \cong \mathrm{im}\ \langle - \rangle_{ext} = \mathbb{K}.$$

The universal construction theorem – Proof

It is left to show that V is cobordism generated. We have just seen, that the isomorphism $V(\varnothing) \cong \mathbb{K}$ is given by the invariant. Hence, the scalar $1 \in \mathbb{K}$ corresponds to the class $[\varnothing] \in V(\varnothing)$ of the empty manifold. Now by construction the functor V is cobordism generated, since for all Σ we have that

$$\begin{aligned} V(\Sigma) &= F(\Sigma)/\sim \\ &= \mathrm{span}\{\mathrm{Hom}(\varnothing, \Sigma)\}/\sim . \end{aligned}$$

But for every $N \in \mathrm{Hom}(\varnothing, \Sigma)$ we have by definition $V(N)(1) = V(N)[\varnothing] = [N \sqcup \varnothing] = [N]$, hence $V(\Sigma)$ is generated by $V(\mathrm{Hom}(\varnothing, \Sigma))$.

$\square$

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In general we have for

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where $F(\Sigma \sqcup \Sigma')$ contains in particular connected manifolds connecting Σ and Σ'.

These are a priori not contained in

$$V(\Sigma) \otimes V(\Sigma') = (F(\Sigma)/\sim) \sqcup (F(\Sigma')/\sim).$$

This can only happen, when all of said manifolds connecting Σ and Σ' are divided out of $F(\Sigma \sqcup \Sigma')$ by $\sim$.

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Further, the universal construction is in such a way that the functor V extends the given invariant, i.e. the invariant obtained from V by looking at the values $V(M)(1)$ for $M \in \mathrm{Hom}(\varnothing, \varnothing)$ is precisely the invariant we started with.

Consequences

Concrete: If we have a manifold $M \in \mathrm{Hom}(\varnothing, \varnothing)$, then $V(M)\colon V(\varnothing) \to V(\varnothing)$ is the map given by $[N] \mapsto [N \sqcup M]$, in particular $V(M)[\varnothing] = [M]$. Since the isomorphism $V(\varnothing) \cong \mathbb{K}$ is given by the invariant, we get that $V(M)\colon \mathbb{K} \to \mathbb{K}$ is given by $k \mapsto k\langle M\rangle$.

Application in dimension 1

EXAMPLE.

Let Z be a one–dimensional TFT, i.e. a symmetric monoidal functor $\mathbb{Z}\colon \mathrm{Bord}_1 \to \mathbb{K}$.

We know that $Z(\bullet_+) = V$ and $Z(\bullet_-) = V^*$ for some finite–dimensional $\mathbb{K}$–vector space V.

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$$V = Z(\bullet_+) = \mathrm{span}\,\{\mathrm{Hom}(\varnothing, \bullet_+)\}\,.$$

But every one–dimensional closed oriented manifold with boundary has to have an even number of boundary points, hence

$$\mathrm{Hom}(\varnothing, \bullet_+) = \varnothing.$$

Thus,

$$V = \mathrm{span}\,\{\varnothing\} = \{0\}.$$

Analogously, we get $Z(\bullet_-) = V^* = \{0\}$.

Application in dimension 1

Further, Z is as a TFT a symmetric monoidal functor and hence

$$Z(\bullet_+ \sqcup \bullet_-) = Z(\bullet_+ \otimes \bullet_-) \cong Z(\bullet_+) \otimes Z(\bullet_-) = \{0\} \otimes \{0\} \cong \{0\}.$$

In fact, monoidality already implies $Z(\Sigma) = \{0\}$ for all objects Σ in Bord_1 (except $Z(\varnothing) = \mathbb{K}$).

Hence, Z is on all non–trivial objects trivial and so it is on all morphism sets other than $\mathrm{Hom}(\varnothing, \varnothing)$.

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Hence, Z is on all non–trivial objects trivial and so it is on all morphism sets other than $\mathrm{Hom}(\varnothing, \varnothing)$.

But morphisms in $\mathrm{Hom}(\varnothing, \varnothing)$ are just disjoint unions of circles.

Since Z is monoidal, it is enough to show that Z is trivial on the morphism $S^1 \in \mathrm{Hom}(\varnothing, \varnothing)$ to show that Z is trivial on all morphisms and objects other than $\varnothing$. We compute

$$Z(S^1) = Z(\mathrm{ev} \circ \mathrm{coev}) = Z(\mathrm{ev}) \circ Z(\mathrm{coev}),$$

which is a composition of trivial maps.

Application in dimension 1

Thus we have proven that any cobordism generated 1–dimensional TFT is trivial and we conclude, that non–trivial 1–dimenional TFTs cannot be obtained by the universal construction.

Application in dimension 1

Remark.
One can ask the question, how the functor obtained from the universal construction in dimension 1 looks like.
We see that if Σ is a 0–dimensional manifold consisting of an odd number of points, then $\mathrm{Hom}(\varnothing, \Sigma) = \varnothing$ and hence,

$$V(\Sigma) = \mathrm{span}\{\varnothing\} = \{0\}.$$

For an object with an even number of points, the construction becomes more subtle and the result will in general be non–trivial.

Application to Reshetikhin–Turaev invariants

REMARK.
One can show that the Reshetikhin–Turaev construction is cobordism generated and indeed obtained by the universal construction theorem applied to the Reshetikhin–Turaev invariants.