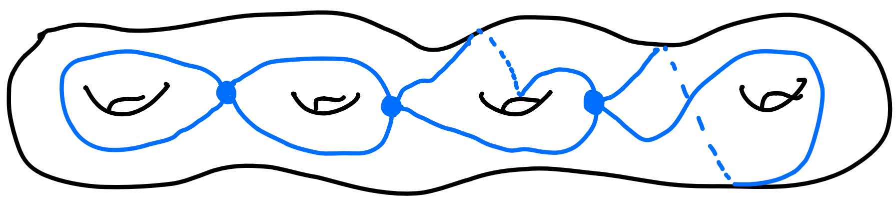
String nets and oriented 3-dimensional TQFTS

Part 1

Bruce Bartlett (Stellenbosch University)



Algebra group research seminar, University of Hamburg, 12 May 2020

Opening words

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- It's a bit more precise, and "mathematics-sounding".
- We can reserve the word "TQFT" for what it always referred to originally - a QFT with a topological action. So, eg. "The Chern-Simons TQFT gives rise to a bordism representation."

1. Introduction

The interesting thing about bordism representations (i.e. TQFTs) is often not so much the quantum invariants of manifolds, knots, etc. <u>themselves</u>, The interesting thing about bordism representations (ie. TQFTs) is often not so much the quantum invariants of manifolds, knots, etc. <u>themselves</u>, but rather the fact that these invariants can be computed in different ways, The interesting thing about bordism representations (ie. TQFTs) is often not so much the quantum invariants of manifolds, knots, etc. <u>themselves</u>, but rather the fact that these invariants can be computed in different ways, leading to <u>interesting equations</u> between these invariants.

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<u>Asymptotic expansion conjecture</u> The quantum Chern-Simons invariant Z_k of closed oriented 3-manifolds satisfies:

$$Z_k(M) \sim \frac{1}{2} e^{-\frac{3\pi i}{4}} \sum_{A \in \mathcal{M}^{\circ}_M} e^{2\pi i S_M(A)\cdot(k+2)} e^{\frac{-2\pi i I_A}{4}} \sqrt{\tau_M(A)}$$

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Defined using quantum groups … or skein theory.

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a sum over equivalence classes of irreducible flat SU(2) connections on M

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The classical Chern-Simons invariant of A:

$$S_M(A) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}\left(A \wedge dA + \tfrac{2}{3} A\wedge A\wedge A\right)$$

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the Atiyah-Patodi-Singer spectral flow $I_A(M)$

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There are two standard ways of building a 3-dimensional oriented bordism representations from categorical data.

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(2) Given a (semisimple) modular category D, we get a "once-extended, anomalous" bordism representation

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Uses surgery on knots, or the BHMV skein theory formalism.

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<u>Theorem</u> (Turaev-Virelizier 2010, Balsam-Kirillov 2010)

Given a spherical fusion category C, we have

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Theorem (Turaev-Virelizier 2010, Balsam-Kirillov 2010)

Given a spherical fusion category $\mathcal{C}$, we have

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- Formulated TVBW as a "3-2-1" theory (their formalism did not use the language of higher categories)
- Related TVBW to RT on these basic building blocks.

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Given a spherical fusion category C, he wrote down a generators-and-relations 1-2-3 bordism representation using the formalism of string nets:

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This refers to the finite presentation of the oriented 1-2-3 bordism bicategory found in:

B, Douglas, Schommer-Pries, Vicary. Extended 3-dimensional bordism as the theory of modular objects.

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In these talks, I want to explain this idea of "string nets" and how it can be used to formulate <u>both</u> TVBW theory (based on a spherical category C) and RT theory (based on a modular category D) in an entirely graphical way.

2. String nets for spherical fusion categories

A <u>fusion category</u> is a $\mathbb{C}$-linear semisimple category, with finitely simple objects, equipped with the structure of a rigid monoidal category. We also demand that the tensor unit $\mathbb{1}$ is simple.

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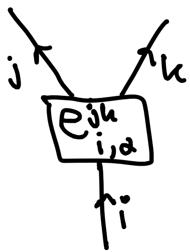
- 

basis for $\mathrm{Hom}(X_i, X_j \otimes X_k)$

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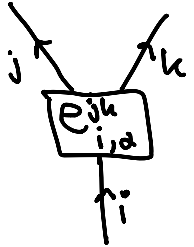
- 

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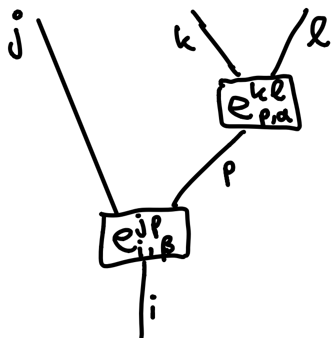
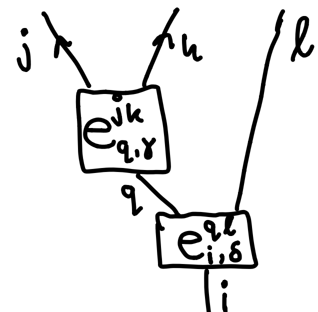
- 

$= \sum_{q,\gamma,\delta} F^{jk\ell;p\alpha\beta}_{i;\gamma q\delta}$


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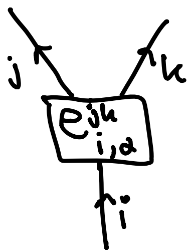
- 

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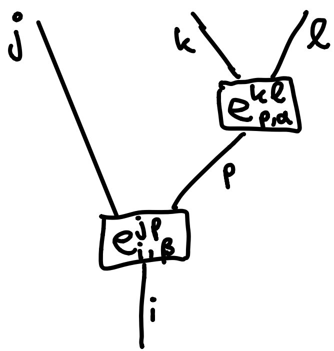
must satisfy pentagon + rigidity

- 

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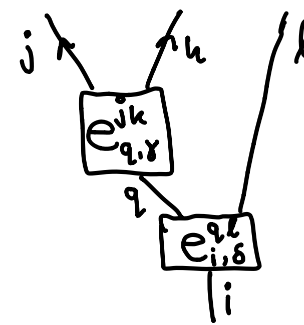

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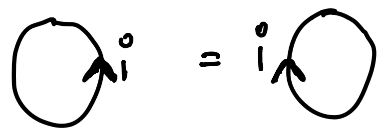

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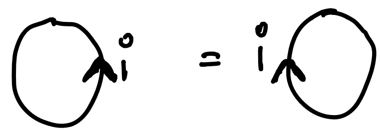

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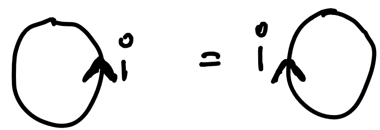

This means that a $\mathcal{C}$-labelled string diagram on the sphere can be unambiguously evaluated. A <u>spherical fusion category</u> is a fusion category equipped with a spherical structure.

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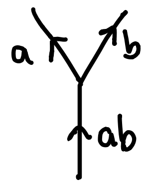
- Fusion rules 

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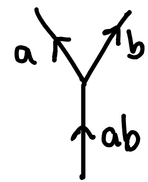

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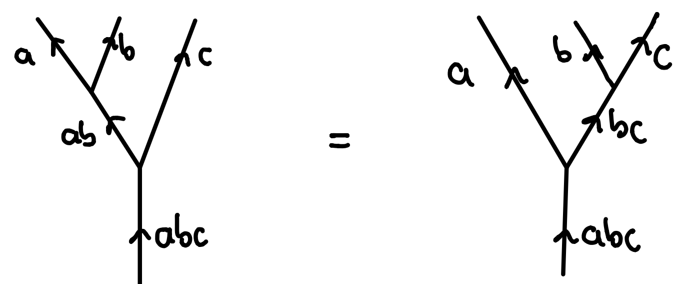

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a spherical structure on Vect[G]
=
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$G \longrightarrow \mathbb{Z}/2\mathbb{Z}$

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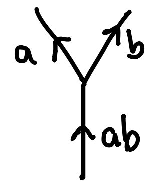

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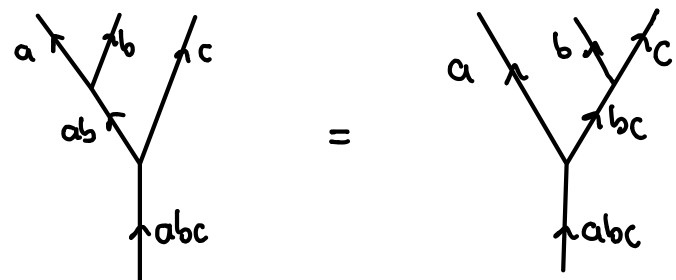

Example 1

$\mathrm{Vect}^{\omega}[G]$, G a finite group. $\omega \in Z^3(G, U(1))$

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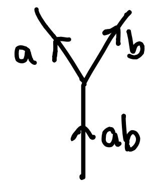
- Fusion rules



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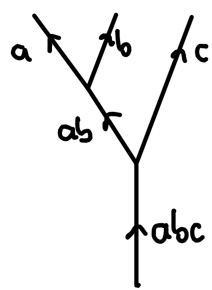
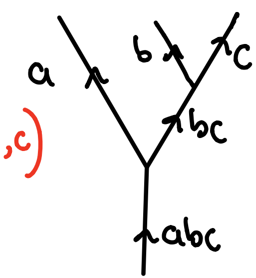


$= \omega(a,b,c)$



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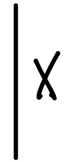
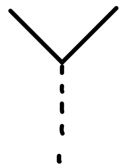
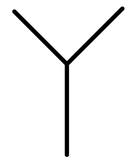
, 

- fusion rules  , 

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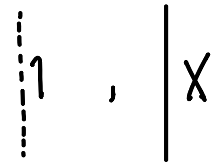

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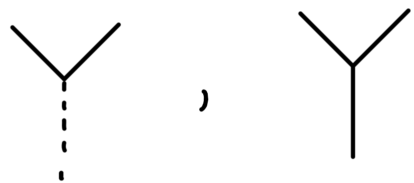

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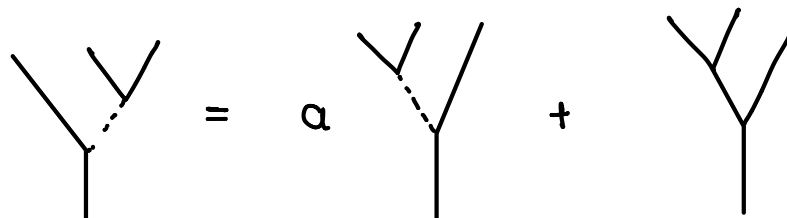
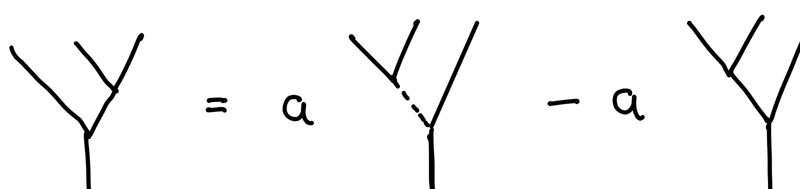


$a = -\frac{1}{2}(1+\sqrt{5})$



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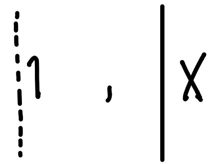

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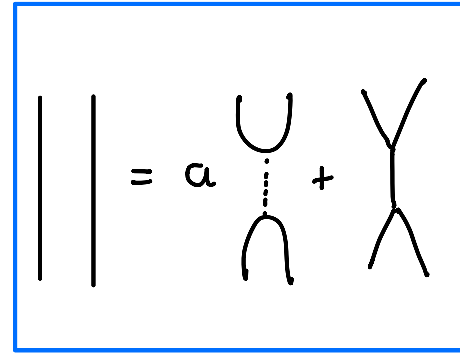

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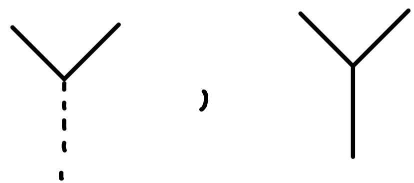
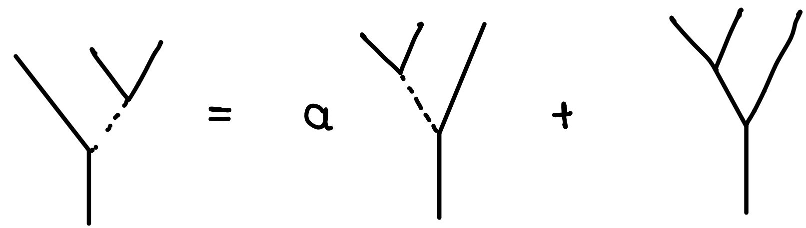


- associators



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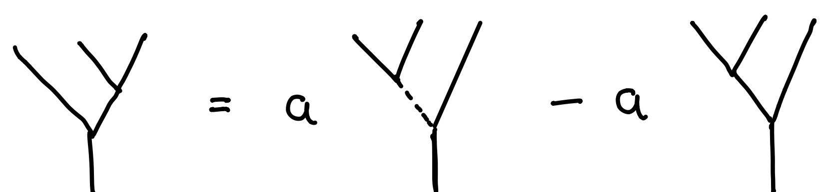

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(See also Goosen, Oriented 123-TQFTS via String-Nets and State-Sums, PhD thesis, 2018)

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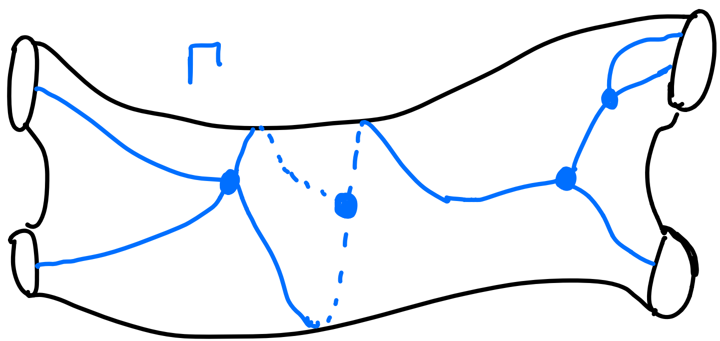

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outgoing orientation, in counterclockwise order (using orientation of Σ!) starting at ϵ_v.

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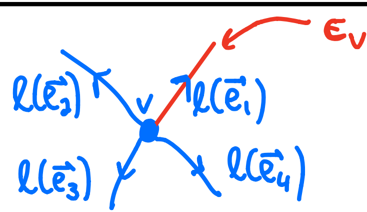

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$$\underline{V} = (B, \{V_b\})$$

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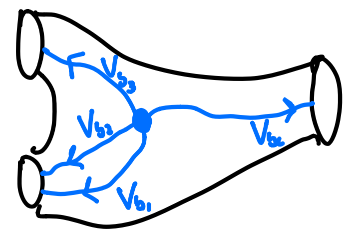

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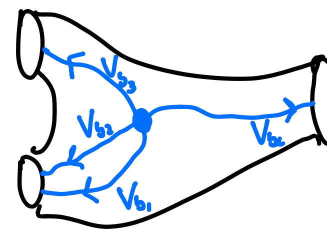
We write $\mathrm{Graph}(\Sigma, \underline{V})$ for the collection of all C-labelled graphs on Σ with boundary value $\underline{V}$.



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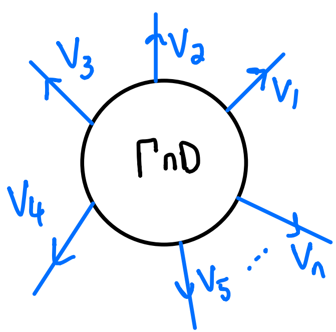

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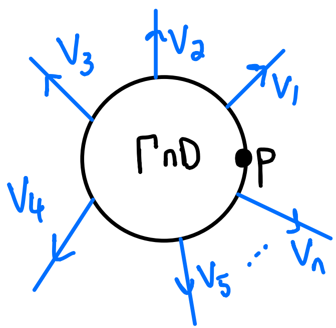

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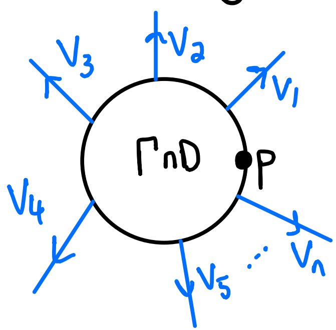

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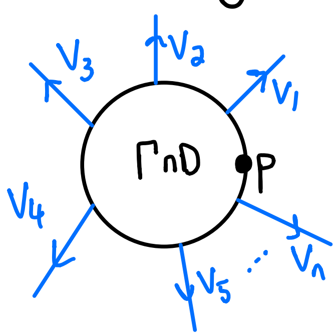

The <u>evaluation of Γ in D</u> is the resulting $\langle \Gamma \rangle_D \in \mathrm{Hom}(1, V_1 \otimes \cdots \otimes V_n)$.

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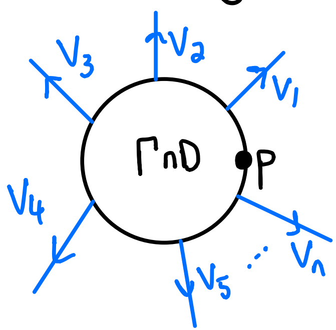

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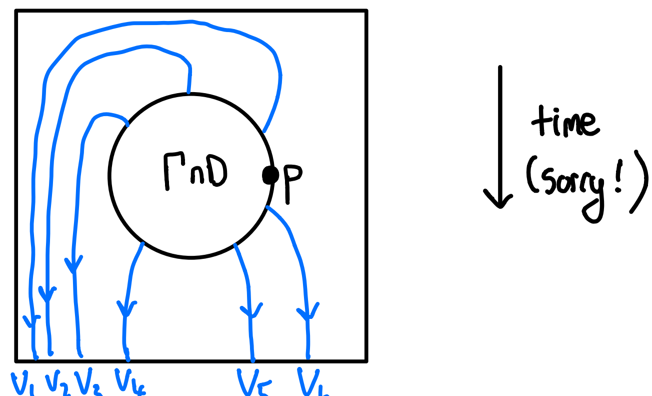

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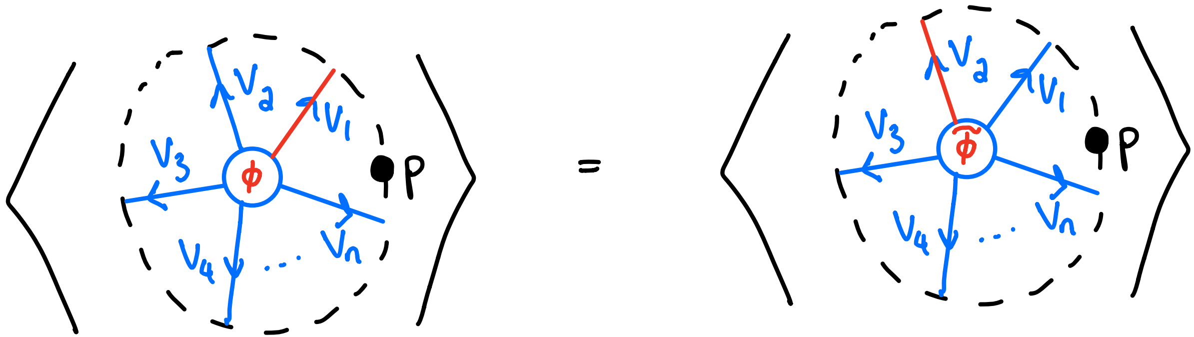

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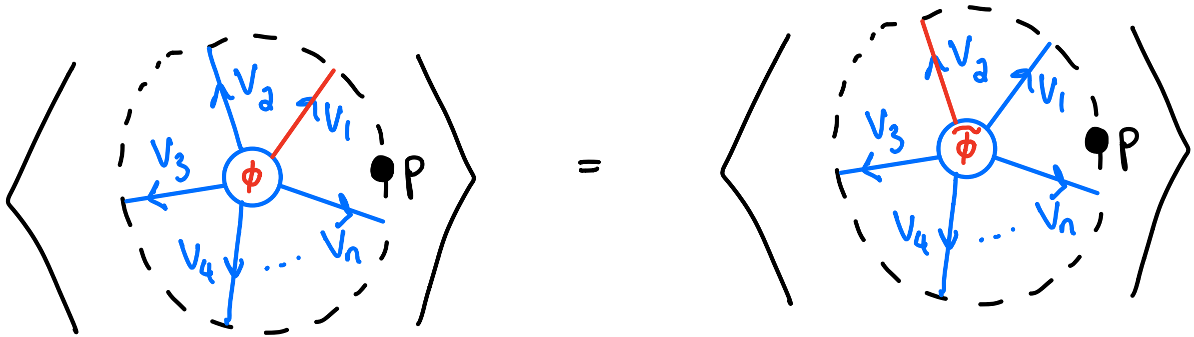

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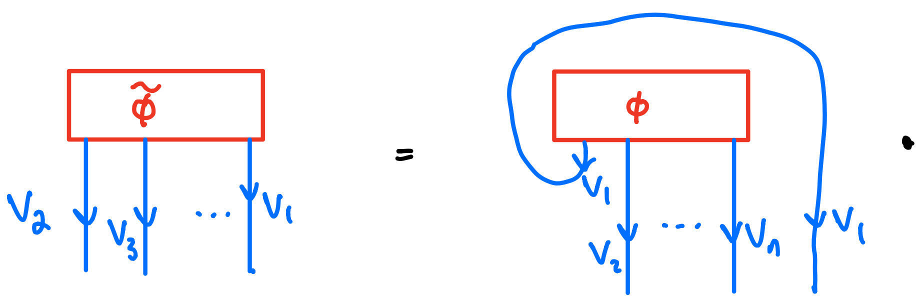

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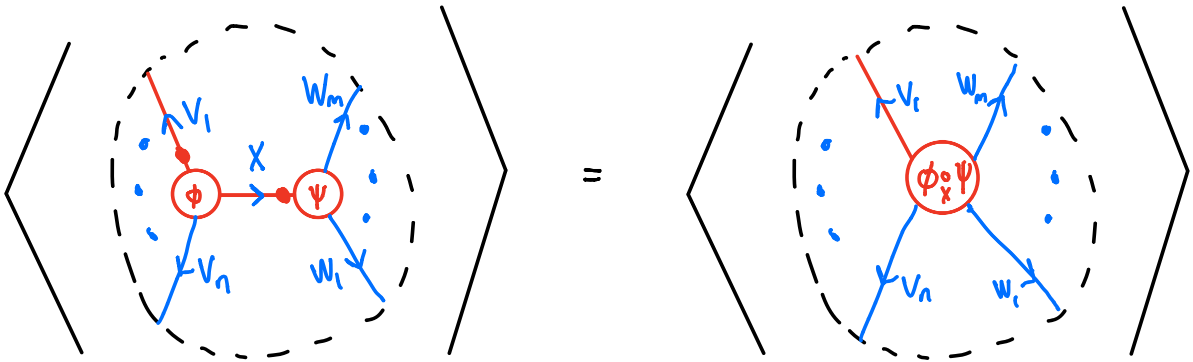

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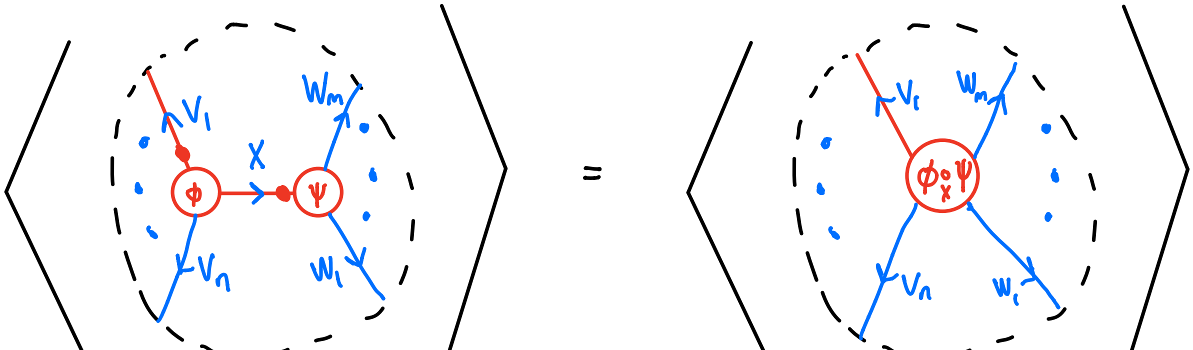

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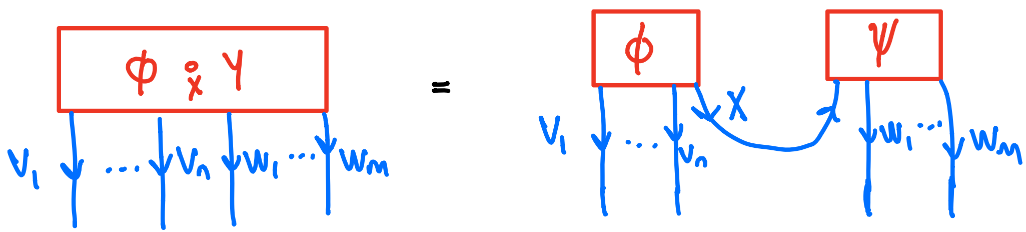

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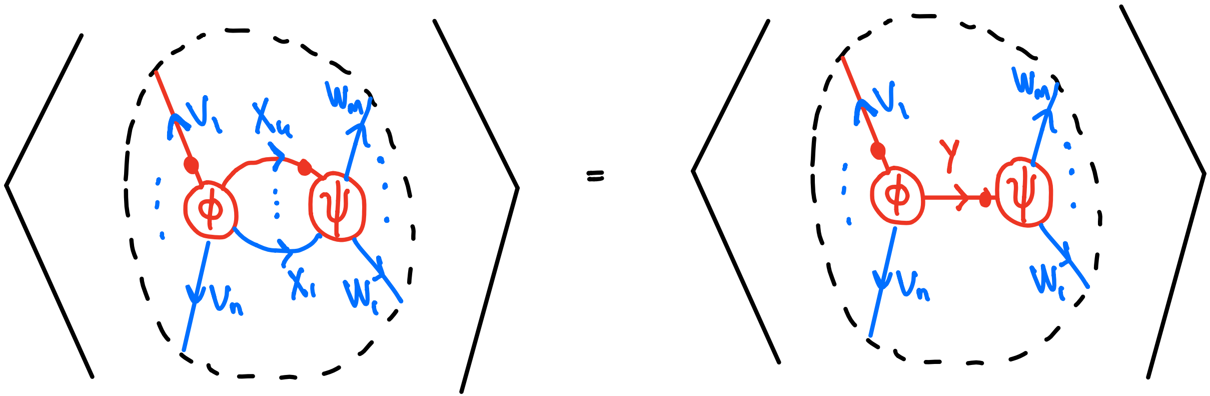

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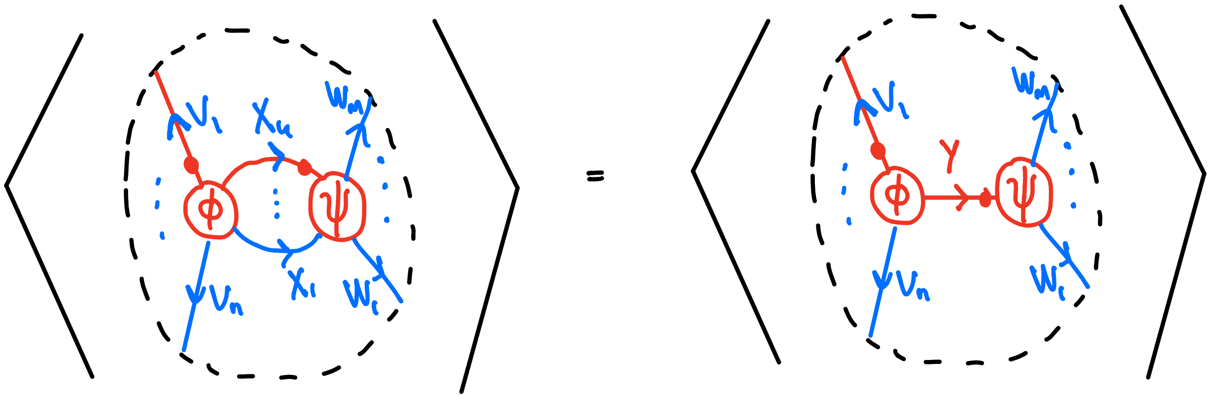

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A formal linear combination $\Gamma = c_1 \Gamma_1 + \cdots + c_n \Gamma_n$ of $\mathcal{C}$-labelled graphs is called a <u>null combination</u> if

$$\langle \Gamma \rangle_D = 0$$

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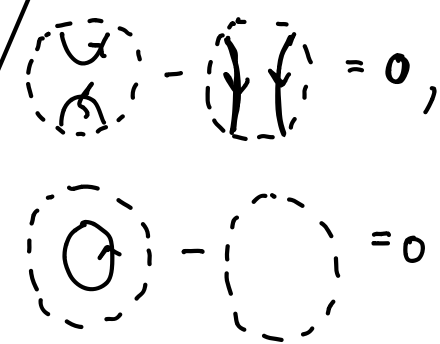

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> Indeed, string nets is morally the same thing as <u>factorization homology</u>.

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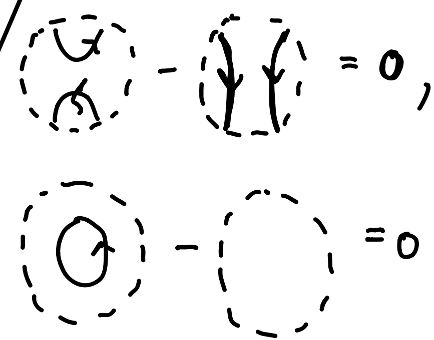

The string-net spaces are monoidal with respect to disjoint union:

$$Z^{\text{String}}(\Sigma_1 \sqcup \Sigma_2) \cong Z^{\text{String}}(\Sigma_1) \otimes Z^{\text{String}}(\Sigma_2)$$

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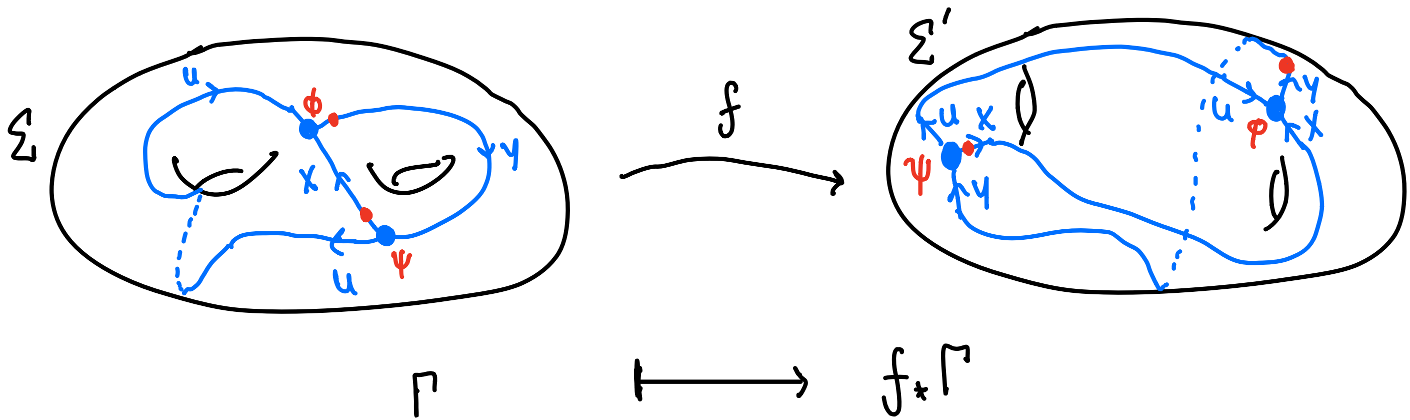
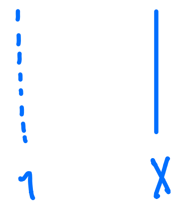


Example For the Yang-Lee category with simple objects



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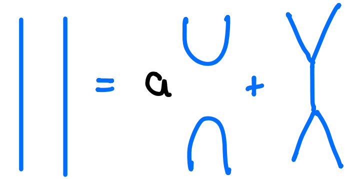
$\mathbf{1}$ $\qquad X$

(recall eg.



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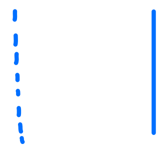

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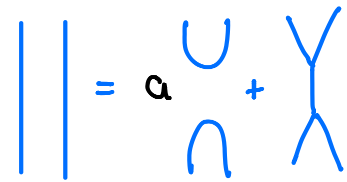
(recall eg.  , $a = -\frac{1}{2}(1+\sqrt{5})$)

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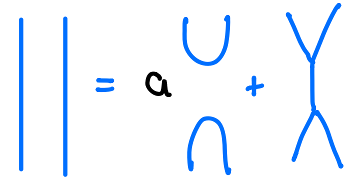

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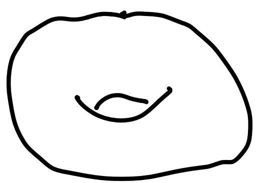
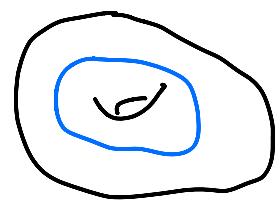


(empty)

s_3 =



s_2 =

s_4 =

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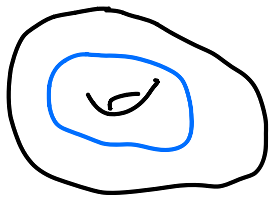
$S_3 =$ 

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Do right handed Dehn-twist θ here

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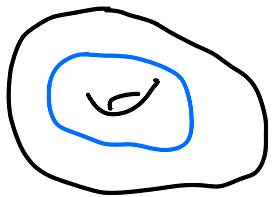
$S_3 =$ 

$\longmapsto$ 

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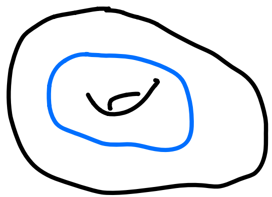
$S_3 =$ 

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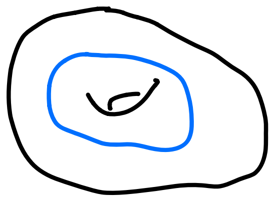
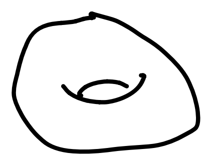
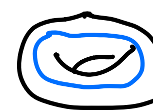
$S_3 =$ 

$\longmapsto$  $= (a+1)$  $-$  $-$ 

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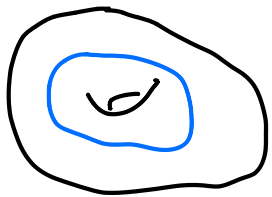
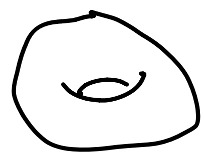
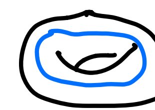
$S_3 =$ 

$\longmapsto$  $=$ $(a+1)$  $-$  $-$ 

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So, $f_*(S_3) = (a+1)S_1 - S_2 - S_3 - aS_4$.

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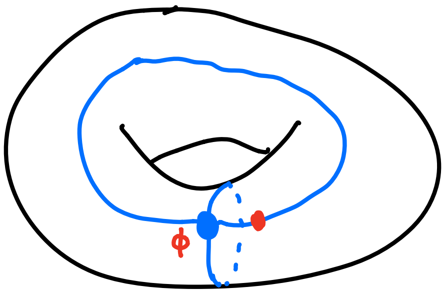

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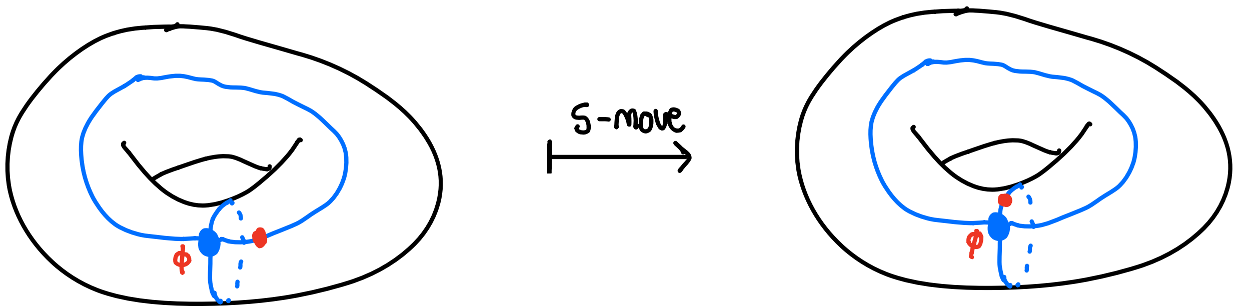

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