Hamburg Algebra Group Seminar
String Nets Part 2

Last time:

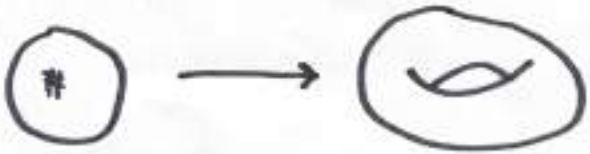
$$\mathbb{A} \longmapsto F_{\mathbb{A}}^{\text{string}}(\Sigma, \underline{V}) := \mathbb{C}\left[\mathrm{Graph}_{\mathbb{A}}(\Sigma, \underline{V})\right] / \mathrm{Null}_{\mathbb{A}}(\Sigma, \underline{V})$$

$$\underline{V} = \{ (b \in \partial\Sigma, V_b \in C) \}_{b \in B}$$

Functorial w.r.t. diffeomorphisms / isotopy.

1. Making string nets into a TQFT

Must handle topology-changing cobordisms, eg.



1 Generators and relations

⊕ Put TVBW and string nets in a unified framework, and prove

see: Goosen, Oriented 123-TQFTs via String-Nets and State-Sums, PhD thesis, 2018 → TVBW ≅ String nets as 123 TQFTS

Compare Kirillov, String-net model of TV invariants (used PLCW decompositions, no topology change)

⊖ • need parts decomp.

• uses generators-and-relations

2 Use Juhasz' surgery description of 2-3 TQFTs

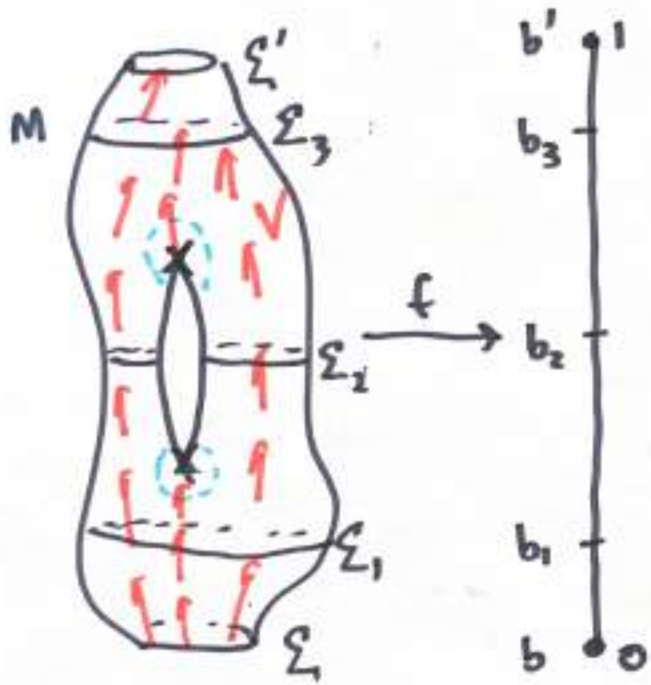
⊕ geometric, no parts decomp. ⊖ Harder to prove ≅ TVBW

2. Juhasz' surgery description of n-dim TQFTs

Goal: describe n-dim cobordisms in terms of:

(n-1)-dim data
- (n-1)-dim manifolds Σ
- diffeomorphisms between them up to isotopy
- embedded k-spheres $S^k \hookrightarrow \Sigma$

Let $M : \Sigma_1 \to \Sigma_1'$ be an n-dim cobordism:



Choose Morse function $f : M \to [0,1]$ and choose regular values

$$b = 0, b_1, b_2, \cdots, b_n, b' = 1$$

s.t. between each regular value there is either <u>one</u> or <u>zero</u> c.p's.

Define $\Sigma_i := f^{-1}(b_i)$. Choose a <u>gradient-like vector field</u> V for f.

We say ① $V(f) > 0$ away from c.p.

②

$$(M, f, \{b_i\}, V\}, U_i\}$$

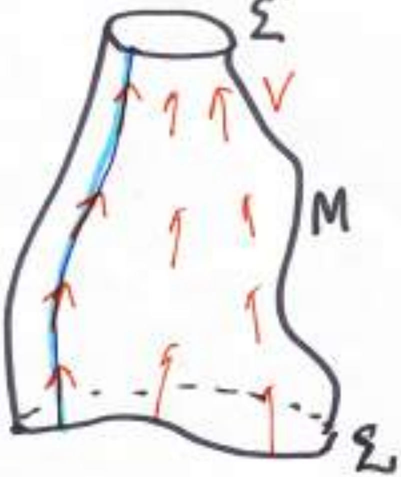
is <u>Morse data</u> for the cobordism.

From each <u>elementary cobordism</u>

$$M : \Sigma_1 \to \Sigma_1'$$

will <u>extract</u> <u>surgery data</u>. $\longleftrightarrow$

no critical points

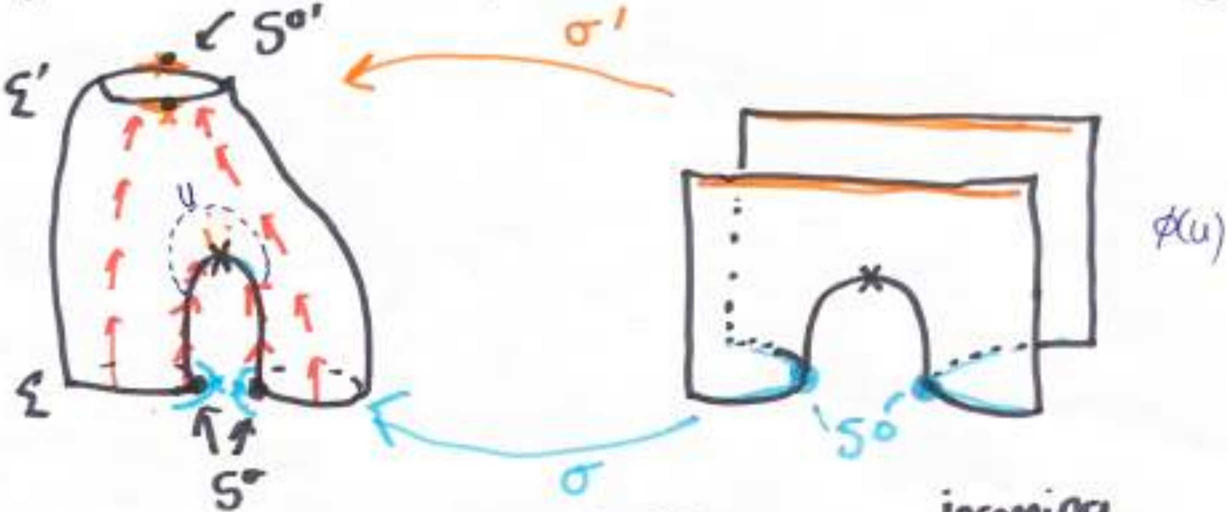


i.e. this is the surgery data for <u>no</u> critical points

Flow along (normalized) $V \rightsquigarrow$ diffeomorphism $\phi : \Sigma \to \Sigma'$

<u>Reference</u>: Juhasz : Defining and classifying TQFTs via surgery
Milnor : Lectures on the h-cobordism theorem

One critical point (of index $k+1$)



$\rightsquigarrow$ **characteristic framed sphere embeddings:**

$$\sigma : S^k \times \mathring{D}^{n-1-k} \hookrightarrow \Sigma \qquad S^\sigma := \sigma(S^k \times \{0\}) \subseteq \Sigma \text{ (incoming sphere)}$$

$$\sigma' : \mathring{D}^{k+1} \times S^{n-2-k} \hookrightarrow \Sigma' \qquad S^{\sigma'} := \sigma'(\{0\} \times S^{n-2-k}) \subseteq \Sigma' \text{ (outgoing sphere)}$$

and a diffeomorphism

$$\psi : \Sigma \setminus S^\sigma \xrightarrow{\cong} \Sigma' \setminus S^{\sigma'} .$$

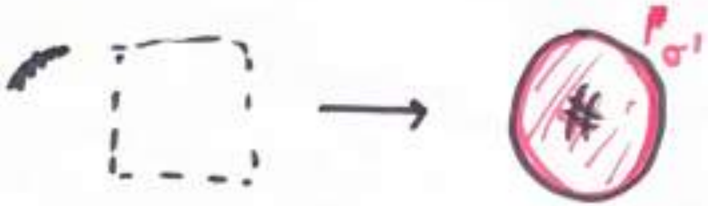
$$\sigma \equiv (\sigma, \sigma'; \psi)$$

surgery data for one critical point.

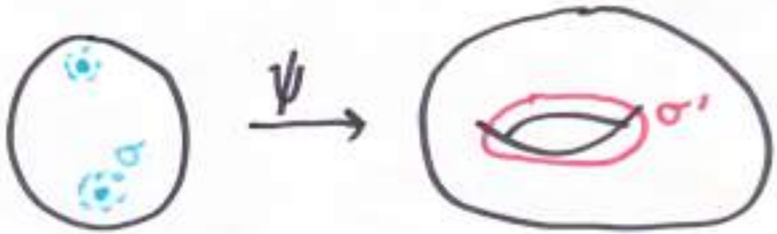
Examples (when $n=3$ i.e. $n-1=2$)

Surgery on a ...

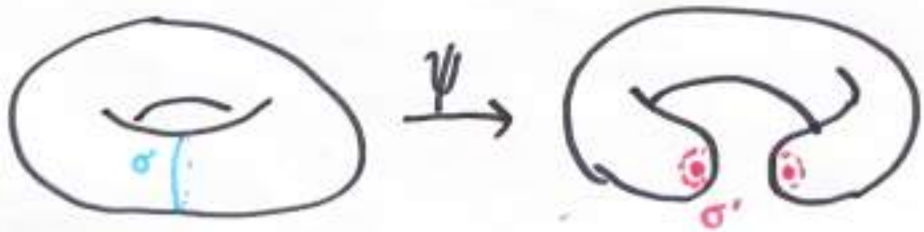
-1 sphere



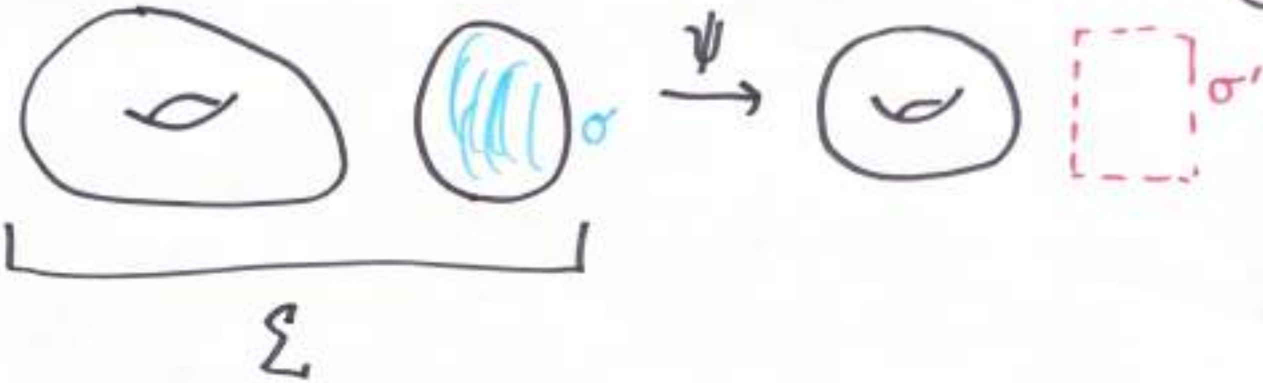
0-sphere



1-sphere



2-sphere



Theorem (Juhász) To give an $(n-1,n)$-dim oriented TQFT is the same as to give:

- A symmetric monoidal functor $F : (\underbrace{\mathrm{Man}_{n-1}}, \sqcup) \longrightarrow (\mathrm{Vect}, \otimes)$
  - oriented closed $n-1$ dim mflds
  - morphisms are isotopy classes of diffeomorphisms

- For each surgery 1-morphism $\underbrace{\sigma}_{\equiv (\sigma, \sigma', \psi)} : \Sigma \longrightarrow \Sigma'$, a linear map

$$Z(\sigma) : F(\Sigma) \longrightarrow F(\Sigma')$$

which satisfies:

- surgery-diffeomorphism naturality:

$$F(\phi) \circ Z(\sigma) = Z(\tilde{\sigma}) \circ F(\tilde{\phi})$$

- monoidal:

σ

$$\begin{tikzcd}
F(\Sigma_1) \otimes F(\Sigma_2) \arrow[r, "\Phi_{\Sigma_1,\Sigma_2}"] \arrow[d, "Z(\sigma)\otimes \mathrm{id}"'] & F(\Sigma_1 \sqcup \Sigma_2) \arrow[d, "Z(\sigma)"] \\
F(\Sigma_1') \otimes F(\Sigma_2) \arrow[r, "\Phi_{\Sigma_1',\Sigma_2}"'] & F(\Sigma_1' \sqcup \Sigma_2)
\end{tikzcd}$$

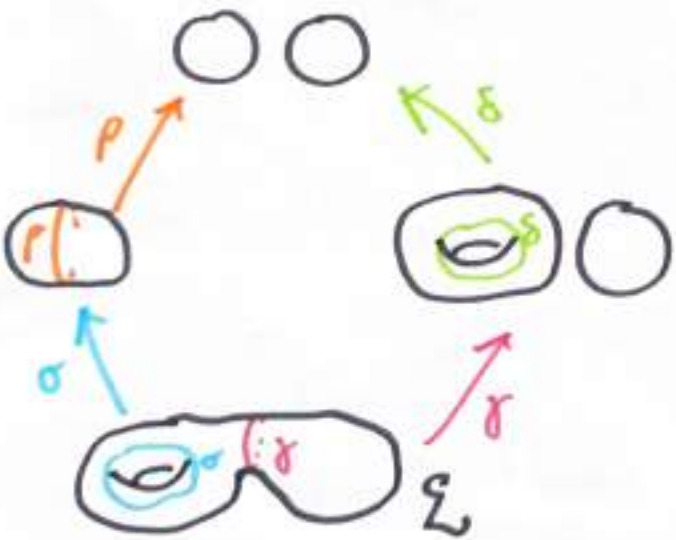
- parametrization invariance: $Z(\bar{\sigma}) = Z(\sigma)$

(11)

• crossing: if • $S^\sigma \cap S^\gamma = \phi$ in Σ

• $\psi_\rho \circ \psi_\sigma = \psi_\delta \circ \psi_\gamma$

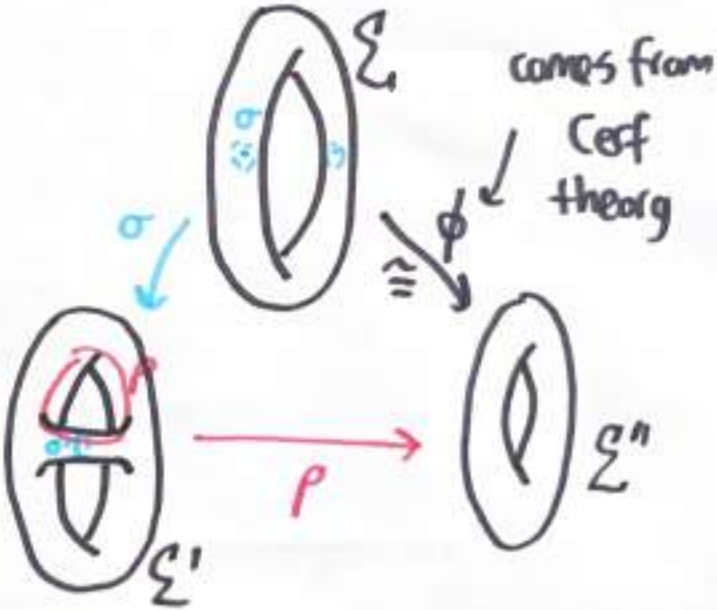
Then: $Z(\rho) \circ Z(\sigma) = Z(\delta) \circ Z(\gamma)$



• Cancellation: if

$S^\rho \cap S^{\sigma'} = 1\,pt$ in Σ'

$Z(\rho) \circ Z(\sigma) = F(\phi)$



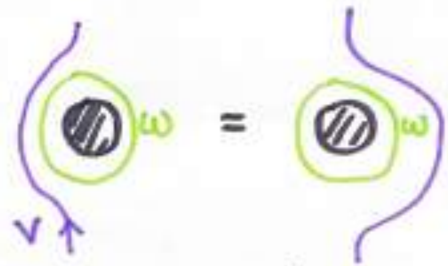
3. String-nets as a surgery-style TQFT . Given spherical fusion cat $\mathcal{A}$.

Ingredients:

① "Kirby color"

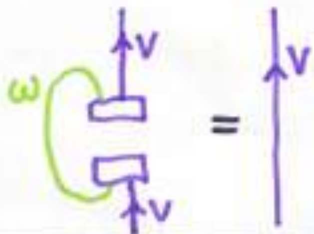
$$\omega := \sum_{X_i \in \mathrm{Irr}(\mathcal{A})} d_i \, X_i$$

$\rightsquigarrow$ cloaking



② "cutting strands"

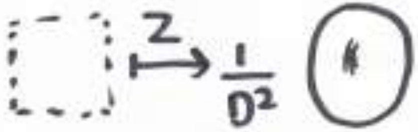
note:



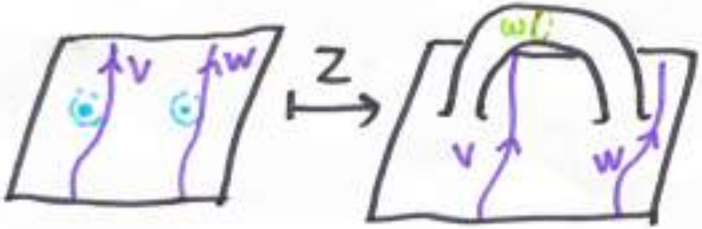
Theorem (B, in progress) The following assignments equip string-nets as a surgery-style TQFT:

Surgery on a...

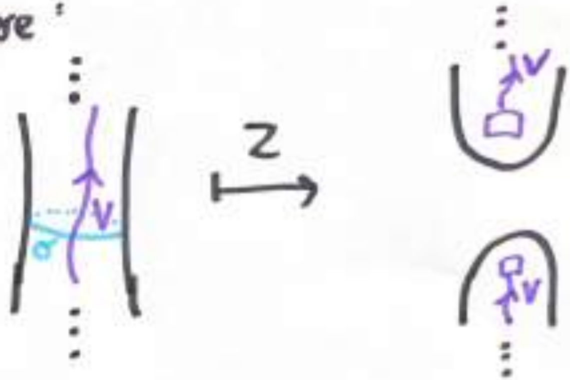
−1-sphere:



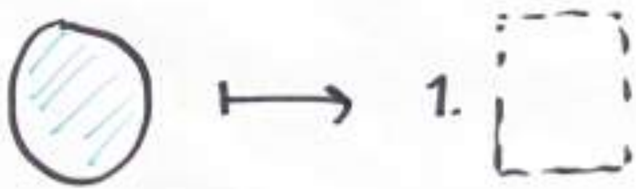
0-sphere:



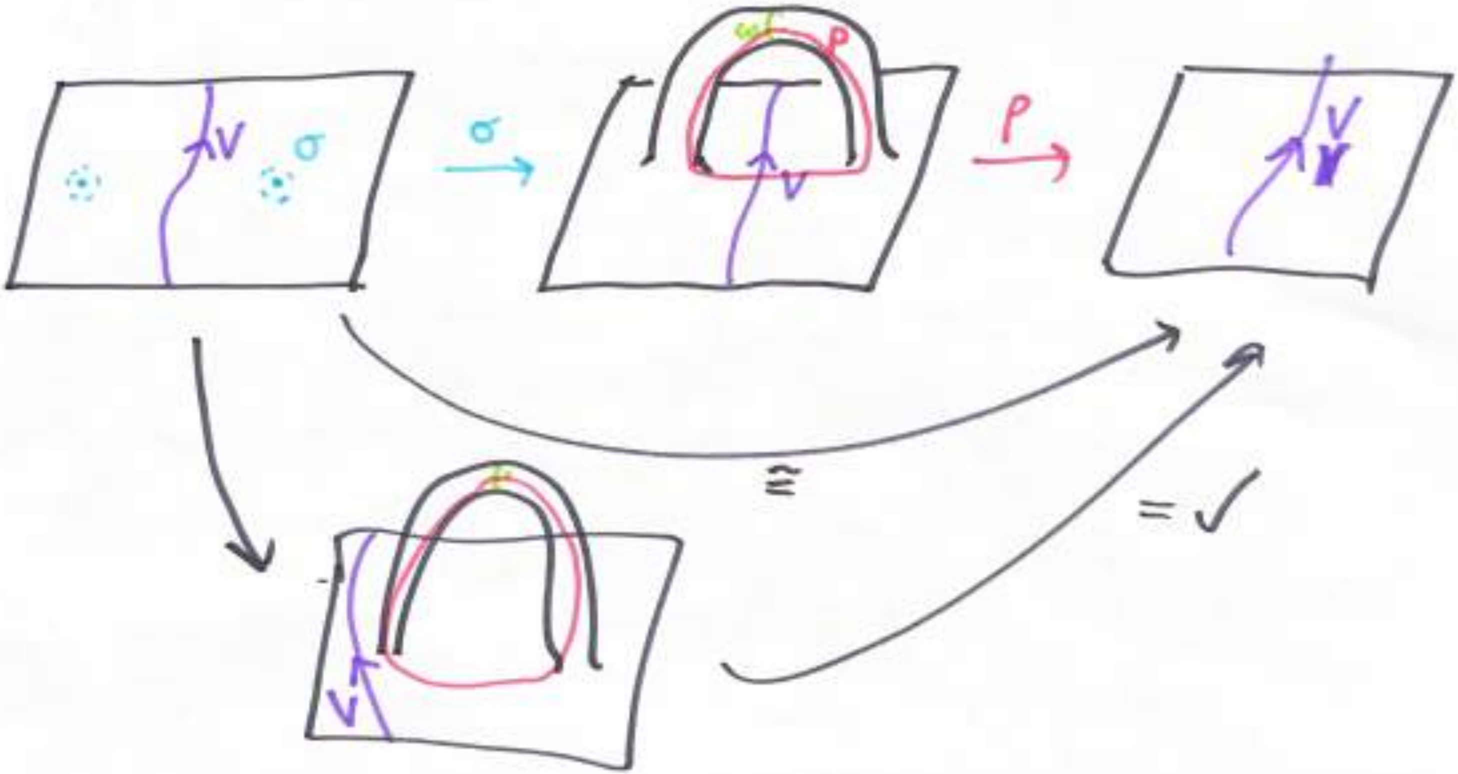
1-sphere:



2-sphere:



eg. checking cancellation:



4. 3d string nets from modular categories

Given a modular category C, can in principle describe RT TQFT also via string nets / internal string diagrams / skein theory:

$$Z_C(\Sigma) := \mathbb{C}[\mathrm{Graph}_C(M)] \big/ \mathrm{Null}_C(M) \qquad \partial M = \Sigma$$

eg. $Z\Big($

$\Big) = \operatorname{span}_{X_i \in \mathrm{Irr}(C)} \Big\{$

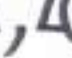
$\Big\}$

This viewpoint emerges naturally when considering a 123 TQFT: (17)

$$Z : \mathrm{Bord}_{123} \longrightarrow 2\mathrm{Vect}$$

$$S^1 \longmapsto \text{semisimple cat } \mathcal{C}$$

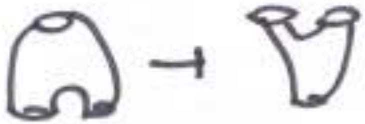
$\uparrow$  $\longmapsto$ $Z($$) : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$

$\leftrightarrow$ vector spaces $\mathrm{Hom}_{\mathcal{C}}\big(A, Z($$)(B \boxtimes C)\big)$

$$\equiv \mathrm{Hom}_{\mathcal{C}}(A, B \otimes C)$$

$\equiv \Big\{$  $\Big\}$

Similarly, because



(via )

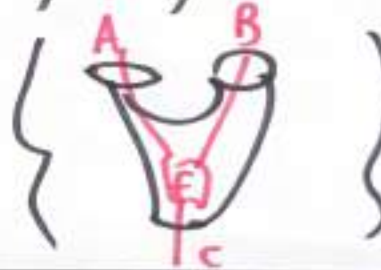
we have :

$$\mathrm{Hom}_{C\boxtimes C}\big(A\boxtimes B,\ Z($$  $$)(C)\big)$$

$$\cong \mathrm{Hom}_C\big(Z($$  $$)(A\boxtimes B),\ C\big)$$

$$\cong \mathrm{Hom}_C(A\boxtimes B,\ C)$$

$\cong$



See:
B, Douglas, Schommer-Pries, Vicary, Modular categories as representations of the 3d bordism 2-category

I'm not sure how to state an exact analogue of Juhasz' theorem in this setting. But see:

(2012) Borodzik, Nemethi, Ranicki. Morse theory for manifolds with boundary.

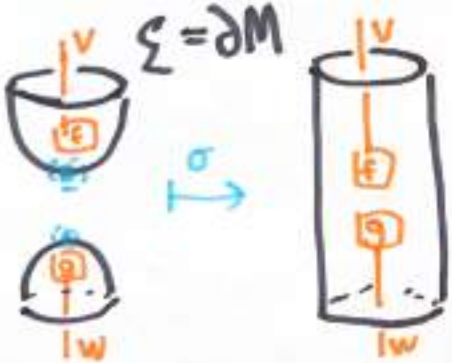
<u>However</u>, it is possible to state what the surgery maps should <u>do</u>, as this is essentially what we did with generators-and-relations.

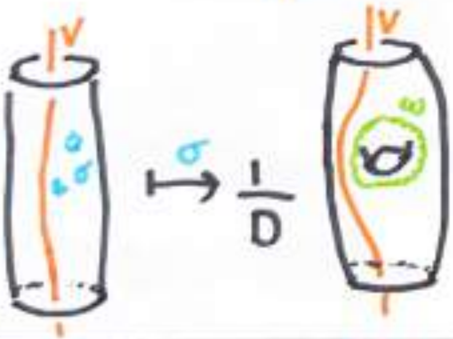
3d surgery style TQFT based on a modular category $\mathcal{C}$

surgery on a...

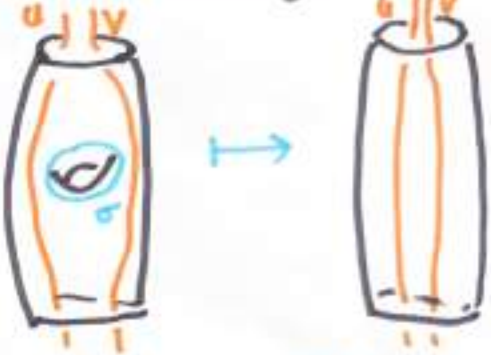
−1 sphere:

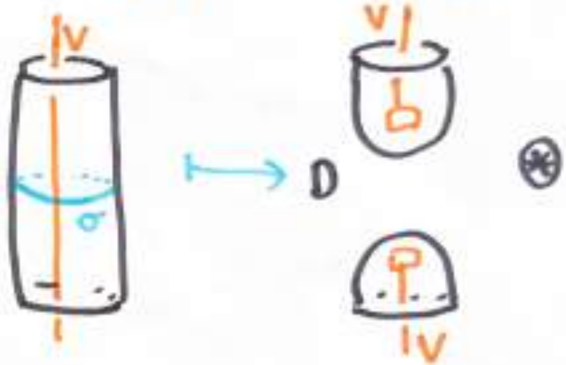
0-sphere:



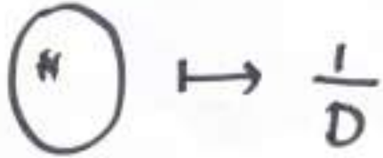


1-sphere:





2-sphere:

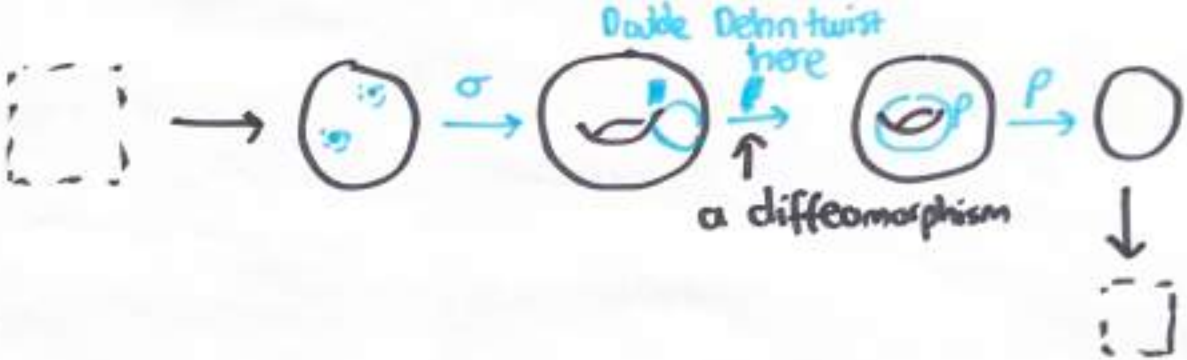


<u>Example</u> Let's compute $Z(\underbrace{\mathbb{RP}^3}_{\text{closed 3-manifold}})$ in both approaches.

We first need a surgery description of the cobordism $\phi \xrightarrow{\mathbb{RP}^3} \phi$.

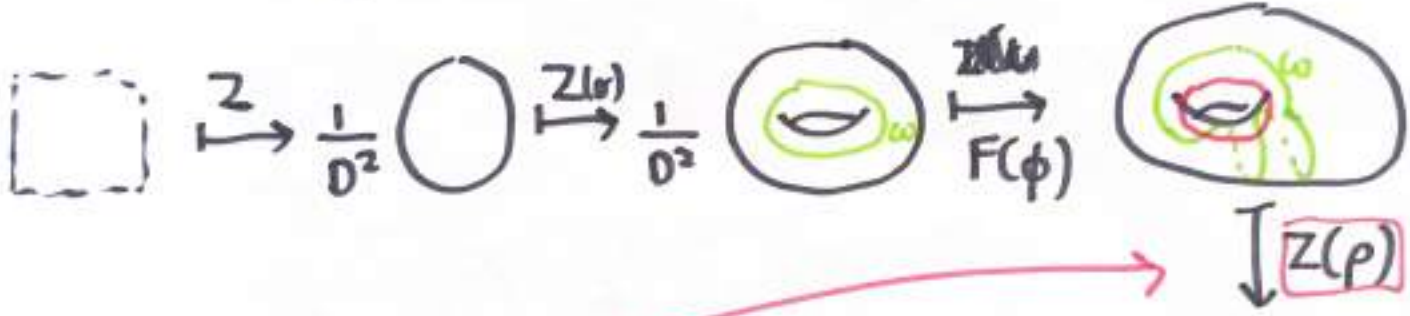
I should do this from first principles, using a Morse function and a gradient-like vector field. But I'm going to cheat. I know it is:

$\mathbb{RP}^3$ =



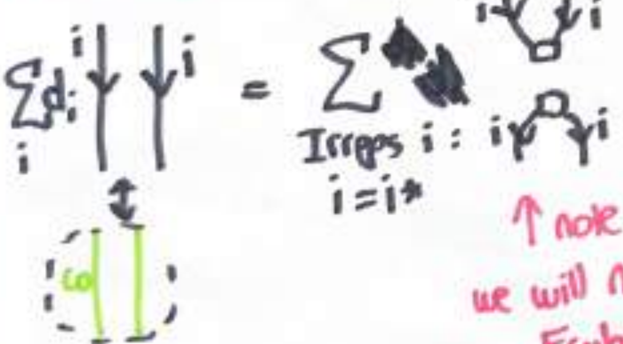
Let $\mathcal{A}$ be a spherical fusion category.

We compute, using 2d string nets labelled by objects of $\mathcal{A}$:



$$\frac{1}{D^2}\sum_{\substack{V_i \in Irr(A) \\ i = i^*}}$$

Must cut a double ω strand:



↑ note arrows: not natural directions.

we will need the Frobenius-Schur indicators

$$\frac{1}{D^2}\sum_{\substack{V_i \in Irr(A) \\ i = i^*}} FS(V_i)\dim(V_i)$$

As a check on our calculations, let's verify for well-known spherical fusion categories.

Take $A = \mathrm{Rep}(G)$

Then we've computed $Z(\mathbb{RP}^3) = \frac{1}{|G|} \sum_{\substack{\text{Irreps } V_i \text{ of } \mathrm{Rep}(G) \\ V_i \cong V_i^*}} FS(V_i) \dim(V_i)$ ①

On the other hand, we know that $A = \mathrm{Vect}[G]$ and $A = \mathrm{Rep}(G)$ give equivalent TQFTs. Using $A = \mathrm{Vect}[G]$, we get

$$Z(\mathbb{RP}^3) = \frac{1}{|G|} \sum_{\substack{g \in G \\ g^2 = e}} 1 \quad ②$$

And we know from representation theory that ① = ② (check...)

As a final check for the spherical fusion cat calculation, we know that TVBW based on $\mathrm{Vect}[G]$ = Dijkgraaf-Witten.

And

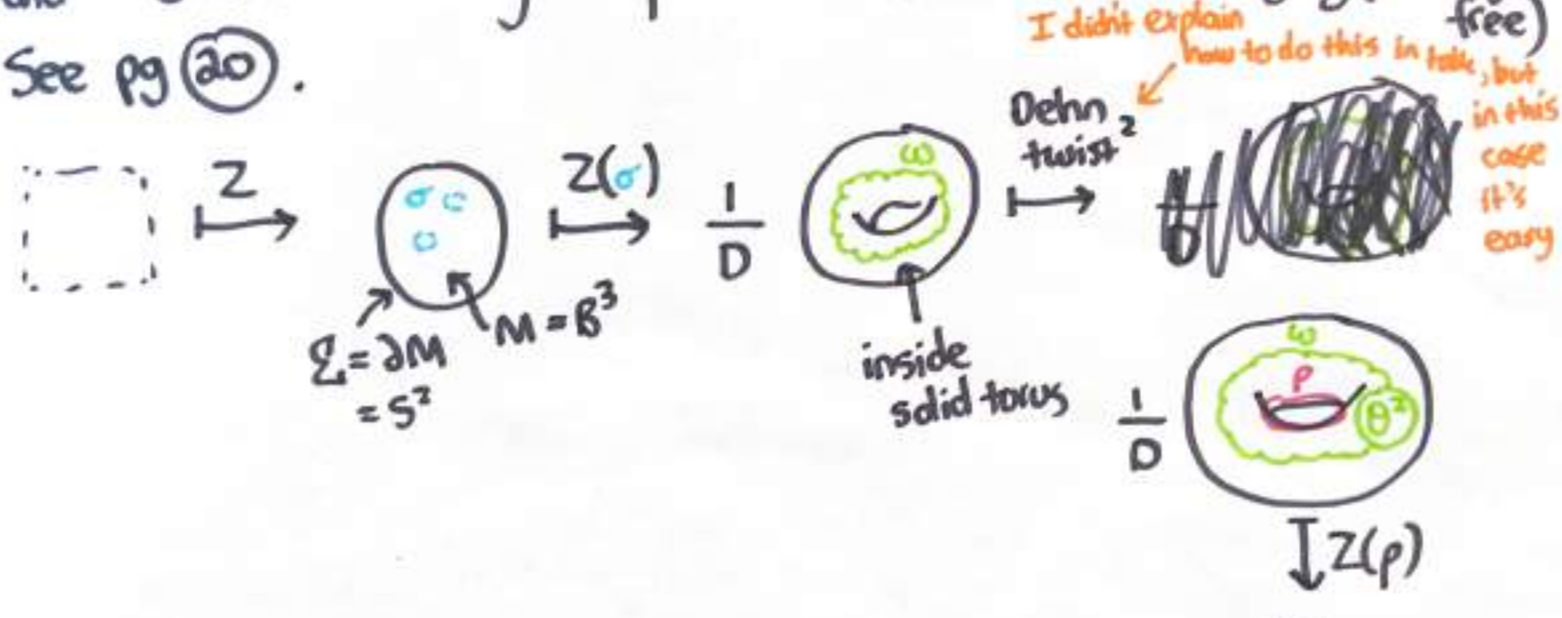
$$Z^{DW}(\mathbb{RP}^3) = \frac{1}{|G|} \# \left\{ \begin{array}{c} \text{Principal } G\text{-bundles} \\ \text{on } \mathbb{RP}^3 \end{array} \right\} / \sim$$

$$= \frac{1}{|G|} \mathrm{Hom}(\underbrace{\pi_1 \mathbb{RP}^3}_{\mathbb{Z}/2\mathbb{Z}}, G)$$

$$= \frac{1}{|G|} \sum_{\substack{g \in G \\ g^2 = e}} 1 \quad = 2 .$$

Finally, let's compute $Z(\mathbb{RP}^3)$ using the surgery approach and C-labelled string nets, C = modular category (anomaly-free)

See pg 20.



$$③\ \frac{1}{D^2} \sum_{i \in \mathrm{Irr}(C)} d_i \theta_i^2 =$$

As a check on our calculations, when $\mathcal{A} = \mathrm{Vect}[G]$, then

$C = Z(\mathcal{A}) = \mathrm{Rep}(\Lambda G)$ [reps of Drinfeld double]

And

exercise! (I might have made an error)

$$\frac{1}{|G|^2} \sum_{V_i \in \mathrm{Irreps}(\Lambda G)} \dim(V_i)\, \theta_{V_i}^2 = \frac{1}{|G|} \sum_{W_i \in \mathrm{Irreps}(G)} FS(W_i) \dim(W_i)$$

remember, for $\mathrm{Rep}(\Lambda G)$
the twist is defined as

$$\theta_{[g],U} = \frac{\mathrm{tr}_U(g)}{\dim U}$$

$[g]$ a conjugacy class in G

U an irrep of centralizer of g