3dTQFT from not necessarily semisimple modular categories

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1) Quick review of ssi Reshetikhin-Turaev TQFT

2) Make it non-ssi.

Example: symplectic fermions SF_N

Problem 1: surgery colour? $\rightsquigarrow$ Lyubashenko's coend

Problem 2: cylinders zero! $\rightsquigarrow$ Geer et al.'s modified traces

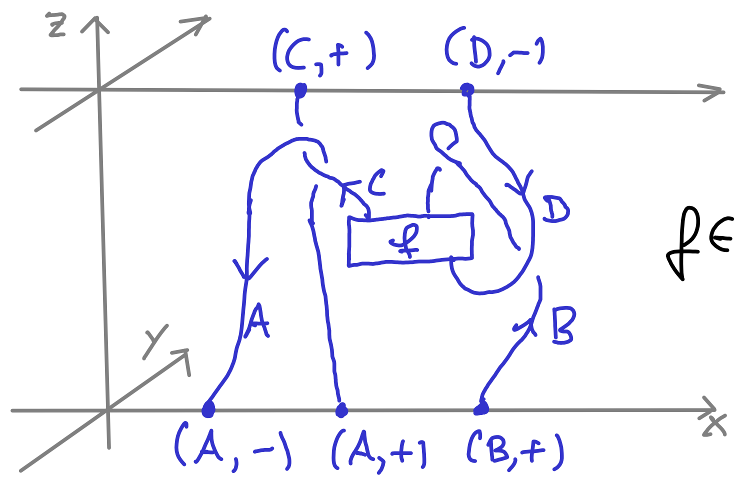
1) Review of RT TQFT

a) RT – functor

finitely ssi, $\mathbb{1}$ simple

$\mathcal{C}$: ribbon fusion cat. $/k$ ← alg. cl.

$\mathrm{Rib}_{\mathcal{C}}$: $\mathcal{C}$ – col. ribbon tangles (isotopy classes)



$f \in \mathcal{C}(D^*, C B^*)$

$F_{\mathcal{C}} : \mathrm{Rib}_{\mathcal{C}} \longrightarrow \mathcal{C}$ "make it a string diag."

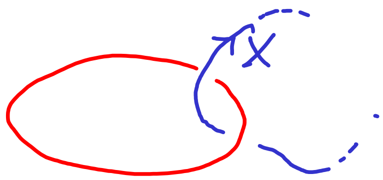
ribbon functor

In particular : closed tangle $\longmapsto$ number

$\Gamma \longmapsto F_{\mathcal{C}}(\Gamma)$

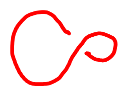
b) Surgery invariant

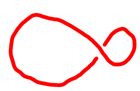
Define



$$:= \sum_{U \in \mathrm{Irr}\, C} \dim U$$

$$\langle M + \text{rib.} \rangle = D^{-\ell-1} F_C(\text{surg. link} + \text{rib.})$$

$\langle S^3 \rangle \rightsquigarrow$ $F_C($  $) = \sum_U (\dim U)^2 \cdot \theta_U =: \Delta_+$

$\langle S^3 \rangle \rightsquigarrow$ $F_C($  $) \quad \cdots \quad \theta_U^{-1} =: \Delta_-$

C <u>twist-nondeg.</u> if $\Delta_\pm \neq 0$.

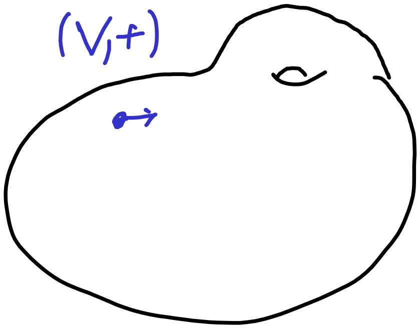
Then: $\mathrm{Dim}\, C := \sum_U (\dim U)^2 = \Delta_+ \Delta_-$

<u>Assume:</u> $\Delta_+ = \Delta_-$ (anomaly-free)

$D := \sqrt{\mathrm{Dim}\, C} = \Delta_\pm$

c) 3d TQFT

$$Z : \mathrm{Bord}_3(\mathcal{C}) \longrightarrow \mathrm{Vect}$$

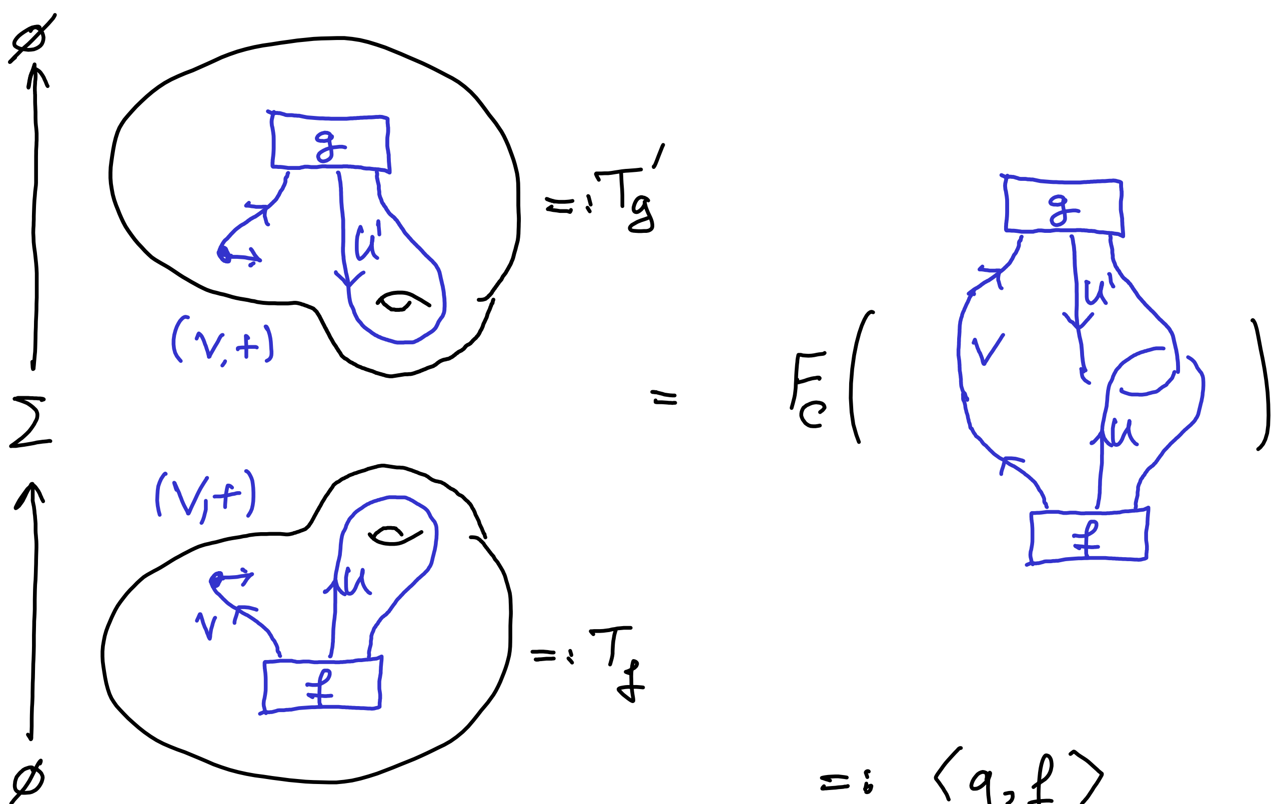


$\longmapsto \mathcal{C}(\mathbb{1}, V E)$

$$E = \bigoplus_{u} u u^* \simeq L = \bigoplus_{u} u^* u$$

"end"

Pairing with $\mathcal{C}(V L, \mathbb{1})$ via Heegard splitting of S^3



$=: \langle g, f \rangle$

C <u>modular</u> if all $\langle\, , \rangle$ are non-deg.

$\Leftrightarrow$ 

$=: s_{u,v}$ is non-deg $(u, v \in \mathrm{Irr}\, C)$

For $\Sigma \xrightarrow{M} \Sigma'$ define $Z(M)$ via

$$\langle g, Z(M) f \rangle := \langle T'_g \circ M \circ T_f \rangle$$

2) Make it non-ssi

<u>Example</u>: SF_N finite ribbon cat$/\mathbb{C}$

$\Vert$ $\mathbb{C}$-lin.

$$\underbrace{\mathrm{Rep}_{\mathrm{SVec}}\, Gr_{2N}}_{\text{grade } 0} \oplus \underbrace{\mathrm{SVec}}_{\text{grade } 1}$$

simple: $\mathbb{1} = \mathbb{C}^{1|0}$ $\qquad T = \mathbb{C}^{1|0}$

$\Pi\mathbb{1} = \mathbb{C}^{0|1}$ $\qquad \Pi T$

"twisted"

$$T \otimes T = Gr_{2N}$$

$$\dim \mathbb{1} = 1$$
$$\Pi\mathbb{1} = -1$$
$$T = 0$$
$$\Pi T = 0$$

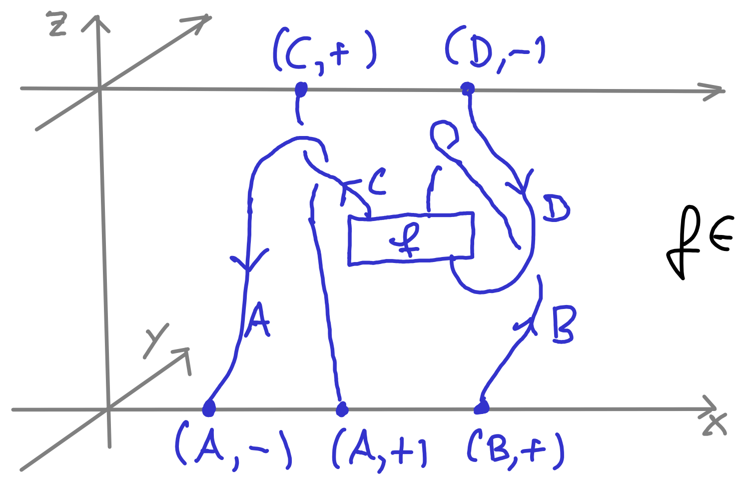
$$\theta_{\mathbb{1}} = \mathrm{id}$$
$$\theta_{\Pi\mathbb{1}} = \mathrm{id}$$
$$\theta_T = \beta^{-1}\, \mathrm{id}_T$$
$$\theta_{\Pi T} = -\beta^{-1}\, \mathrm{id}_T$$

$$\beta^4 = (-1)^N$$

a) RT – functor

C : ribbon ~~fusion~~ finite cat. $/k$ ← alg. cl.

Rib_C : C – col. ribbon tangles (isotopy classes)



$f \in C(D^*, C B^*)$

$$F_C : \mathrm{Rib}_C \xrightarrow[\text{ribbon functor}]{} C$$

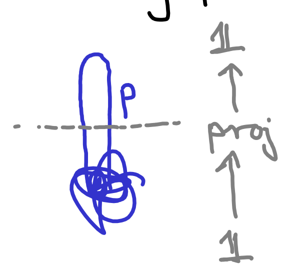
"make it a string diag."

In particular : closed tangle $\longmapsto$ number

$\Gamma \longmapsto F_C(\Gamma)$ ✓

⚠ If ≥ 1 ribbons in Γ labelled by projective, then

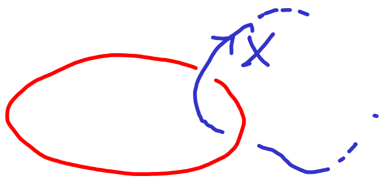
$F_C(\Gamma) = 0$

$\Gamma =$ 

gives 0 if C non-ssi

b) Surgery invariant

Define

 $:= \sum_{U \in \mathrm{Irr}\, C} \dim U$ 

$$\langle M + \text{rib.} \rangle = D^{-\ell-1} E_C(\text{surg. link} + \text{rib.})$$

E.g. SF_N:

assume: no coupons

- label T, ΠT occur $\rightsquigarrow 0$
- cat. traces only see $Gr(C)$

  $\mathrm{Rep}_{SVec}(Gr_{2N}) \rightsquigarrow SVec$

- SVec is sym.

c) 3d TQFT

$\mathcal{C}$ <u>modular</u> if all $\langle\ ,\ \rangle$ are non-deg.

$\Leftrightarrow$  $=: s_{u,v}$ is non-deg $(u, v \in \operatorname{Irr} \mathcal{C})$

E.g. SF_N: $s_{u,T} = 0$ (all u)

In fact $\quad \mathcal{C}$ non-ssi modular cat. $/\mathbb{C}$

$\Rightarrow$ exists simple proj.

$\Rightarrow$ $s_{u,v}$ deg.

$P_{\mathbb{1}} \twoheadrightarrow \mathbb{1}$ $\quad \exists\, \mathbb{1} \hookrightarrow P_{\mathbb{1}}$

Lyubashenko's invariant

$\mathcal{C}$: finite ribbon cat, unimodular

$L = \int^{X \in \mathcal{C}} X^* X$ coend $\quad \left(\mathcal{C} \text{ ssi} : L = \bigoplus_{u \in \mathrm{Irr}} u^* u \right)$

dinatural transf. $\iota_X : X^* X \longrightarrow L$

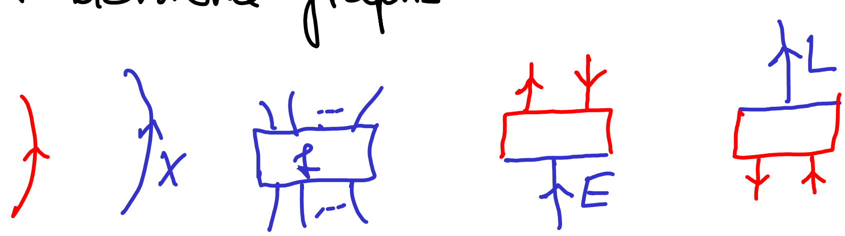
L is Hopf alg in $\mathcal{C}$

Pick non-zero integral $\Lambda : \mathbb{1} \longrightarrow L$ (unique up to scalar)

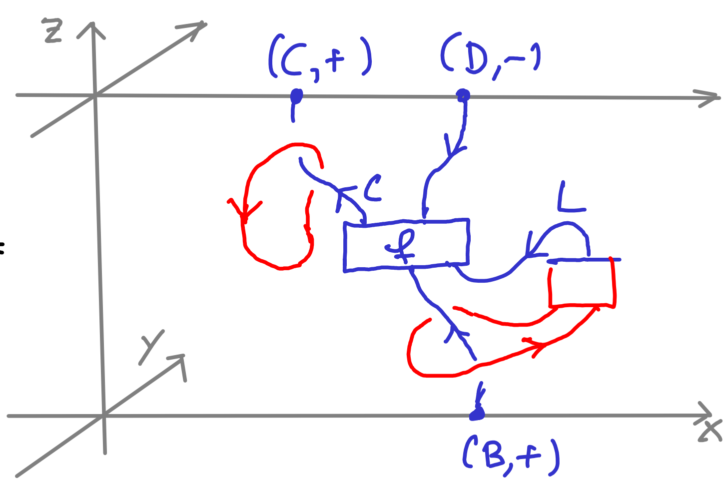
$\left(\mathcal{C} \text{ ssi} : \Lambda = \sum_{u \in \mathrm{Irr}} \dim(u) \cdot \right.$  $\left. \right)$

a) LRT – functor

Rib_Λ : bichrome graphs

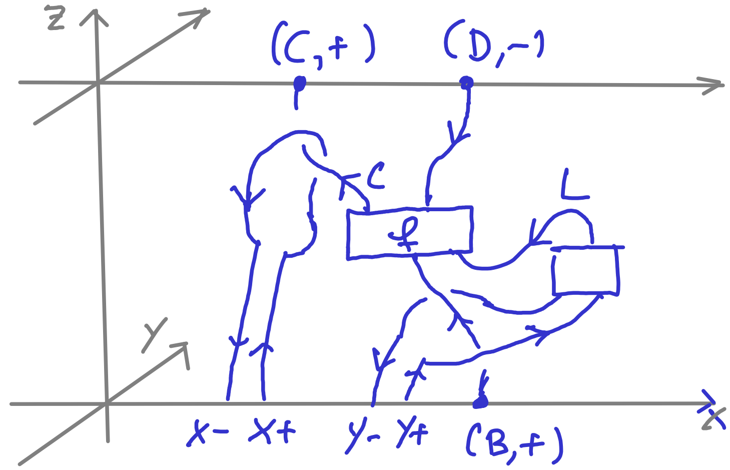


E.g.

$T :=$ 

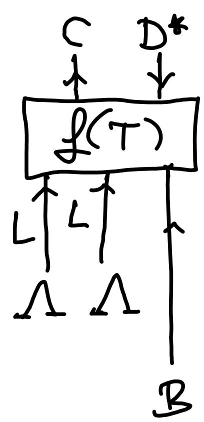
$F_\Lambda : \mathrm{Rib}_\Lambda \longrightarrow \mathcal{C}$ ribbon fun.

E.g. $F_\Lambda(T) : B \longrightarrow CD^*$

$\mathrm{dinat}\,(\eta_T)_{X,Y} = F_{\mathcal{C}}\Big($  $\Big)$

$: \underbrace{X^* X}_{\text{dinat.}} \underbrace{Y^* Y} B \longrightarrow CD^*$

$\rightsquigarrow \exists!\ f(T) : L L B \longrightarrow CD^*$

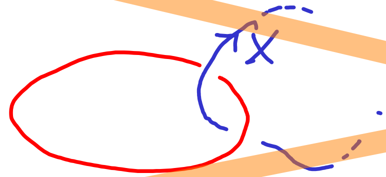
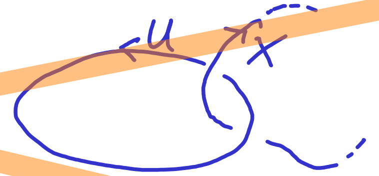
Then $F_\Lambda(T) =$ 

In particular: closed bichrome graph $\Gamma \longmapsto$ number

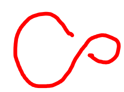
$\longmapsto F_\Lambda(\Gamma)$

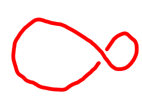
b) Surgery invariant

Define

 $\cancel{:= \sum_{U \in \mathrm{Irr}\, C} \dim U}$ 

$$\underbrace{\langle M + \text{rib.} \rangle}_{=:\, \mathrm{Lyu}(M + \text{rib})} = D^{-\ell - 1} F_\Lambda(\text{surg. link} + \text{rib.})$$

$\langle S^3 \rangle \rightsquigarrow F_\Lambda($  $)$ $\cancel{= \sum_U (\dim U)^2 \cdot \theta_U} =: \Delta_+$

$\langle S^3 \rangle \rightsquigarrow F_\Lambda($  $)$ $\cancel{\cdots\ \theta_U^{-1}} =: \Delta_-$

C twist-nondeg. if $\Delta_\pm \neq 0$.

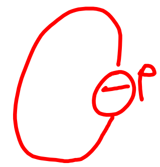
~~Then:~~ $\cancel{\mathrm{Dim}\, C := \sum_U (\dim U)^2 = \Delta_+ \Delta_-}$

Assume: $\Delta_+ = \Delta_-$ (anomaly-free)

$D := \cancel{\sqrt{\mathrm{Dim}\, C} =}\ \Delta_\pm$

Thm (Lyubashenko '95) $\mathrm{Lyu}(M + \text{rib.})$ is an invariant.

E.g, lens space

$$\mathrm{Lyu}(L_{1,p}) = F_\Lambda\Big($$  $$\Big)$$

$$= \varepsilon \circ T^p \circ \Lambda$$

ε: counit of L

E.g. SF_N (unimodular ✓ anomaly-free for $N \in 2\mathbb{Z}$, $\beta = \pm 1$)

L? Λ? ($SF_N \simeq \mathrm{Rep}_{\mathrm{vec}}$(quasi Hopf Q_N)
$L = (Q_N^*)_{\mathrm{coadj}}$)

$$\Delta_\pm = (\pm 2)^N \beta^2$$

$$\mathrm{Lyu}(L_{1,p}) = \overset{\text{indep. of } p}{(\mathrm{const})}\; p^N$$

No RT inv. for ssi C can distinguish all $L_{1,p}$ as ribbon twist has finite order.

c) 3d TQFT

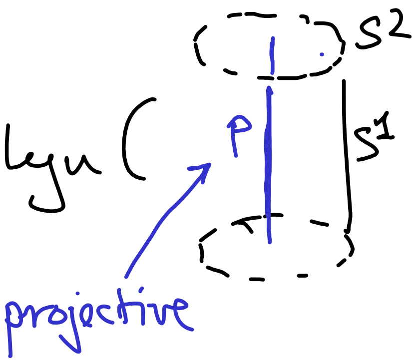
all transparent objects are $\oplus \mathbb{1}$

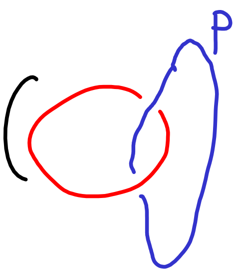
$\mathcal{C}$ non-ssi modular tens. cat.

$$\mathrm{Lyu}(S^2 \times S^1) = \mathcal{D}^{-2} F_1(\bigcirc)$$

$$= \mathcal{D}^{-2}\, \varepsilon \circ 1$$

$$= 0$$

Lyu(

) $= \mathcal{D}^{-2} F_1($ 

)

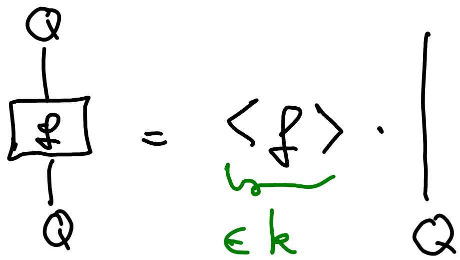
$$= 0$$

Modified traces

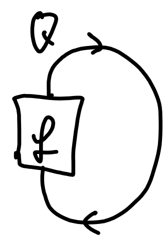
finite ribbon, non-ssi

Idea : Let $Q \in \mathcal{C}$ be simple projective

Then for any $f : Q \to Q$



e.g. $\langle id_Q \rangle = 1$

but $tr_Q f =$ 

$= 0$

So why not take $\langle f \rangle$ instead ?

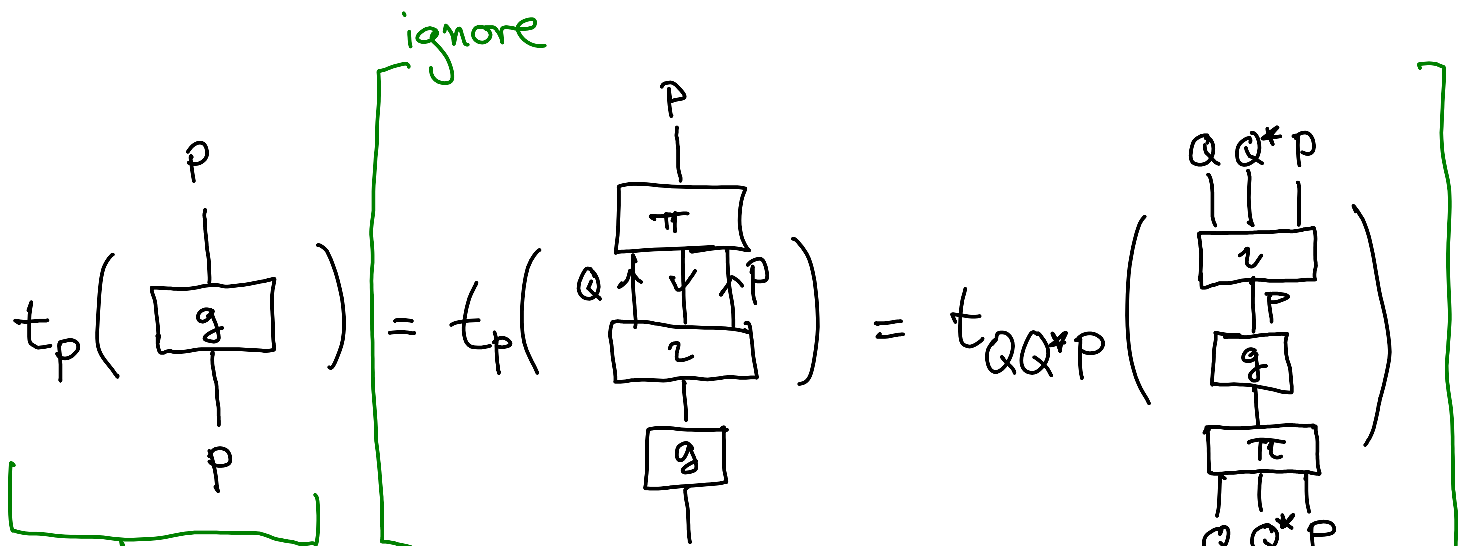
Get $\{ t_P : End(P) \to k \}_{P \in Proj\, \mathcal{C}}$

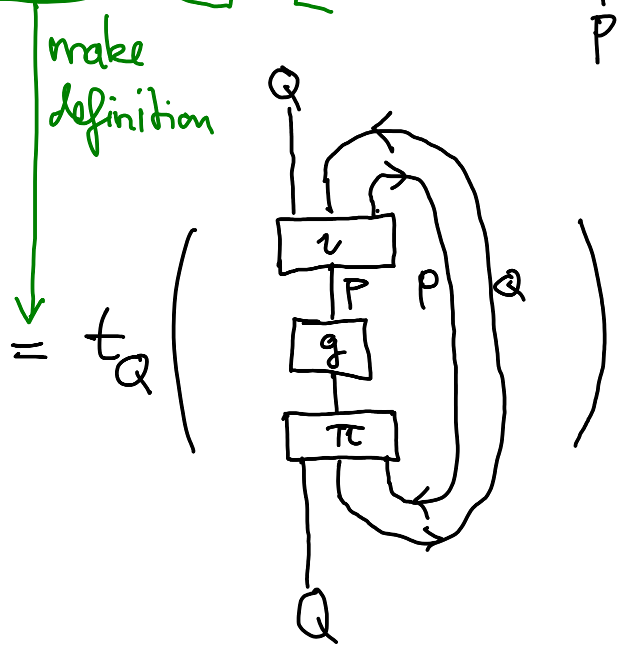
via

1) $t_Q(f) = \langle f \rangle$

2)
$$\begin{tikzcd} P \arrow[r, hook, "\iota"] \arrow[rr, bend right, "id"'] & QQ^* P \arrow[r, two heads, "\pi"] & P \end{tikzcd}$$

and cycl. + partial trace





$\mathcal{C}$ modular tens. cat $/\mathbb{C}$ (not nec. ssi)

<u>Thm</u> (Gainutdinov, IR '17)

$\mathcal{C}$ contains a simple projective object.

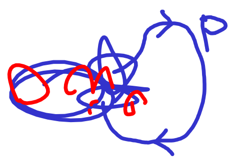
<u>Thm</u> (Geer, Kujawa, Patureau-Mirand '13)

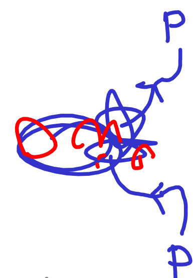
The above defines a modified trace $\{t_P\}$ on $\mathrm{Proj}\,\mathcal{C}$.

Renormalised Lyubashenko invariant

b) Surgery invariant

Γ : closed bichrome graph, at least one projective edge

$F_{1}'\Big($ 

$\Big) := t_P\Big($ 

$\Big)$

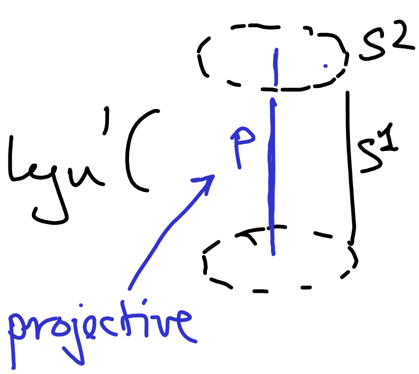
C finite ribbon, unimod, twist-nondeg, anomaly-free

$$\mathrm{Lyu}'\Big(M + \substack{\text{rib w.}\\ \text{proj}}\Big) \overset{\Downarrow}{=} \mathcal{D}^{-\ell-1}\, F_{1}'\big(\text{surg. link} + \text{rib.}\big)$$

<u>Thm</u> (De Renzi, Geer, Patureau-Mirand '17
... & Gainutdinov, IR '19)

Lyu' is an invariant.

E.g. $\cancel{\mathrm{Lyu}'(S^2 \times S^1)}$

$\mathrm{Lyu}'\Big($ 

$\Big) = \dim C(P, \mathbb{1})$

c) 3d TQFT

$\mathcal{C}$: modular tens. cat, not nec. ssi
(here: assume $\mathcal{C}$ anomaly free)

$$Z \;:\; \mathrm{Bord}'_3(\mathcal{C}) \longrightarrow \mathrm{vect}$$

same objects as $\mathrm{Bord}_3(\mathcal{C})$
<u>but</u>
every connected comp. of bordism which does not intersect in-going bnd must contain a projective edge.

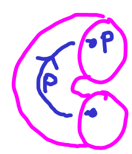
E.g.:

$\Sigma \sqcup -\Sigma \longrightarrow \emptyset$ ✓

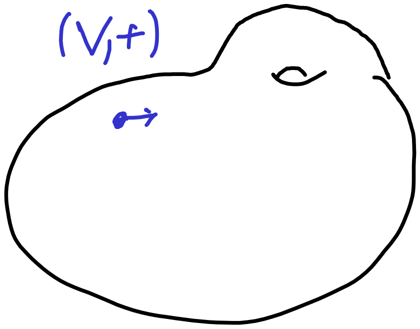
$\emptyset \longrightarrow \Sigma \sqcup -\Sigma$ ✗

but for P proj:

$\Sigma \sqcup -\Sigma \longrightarrow \emptyset$ ✓

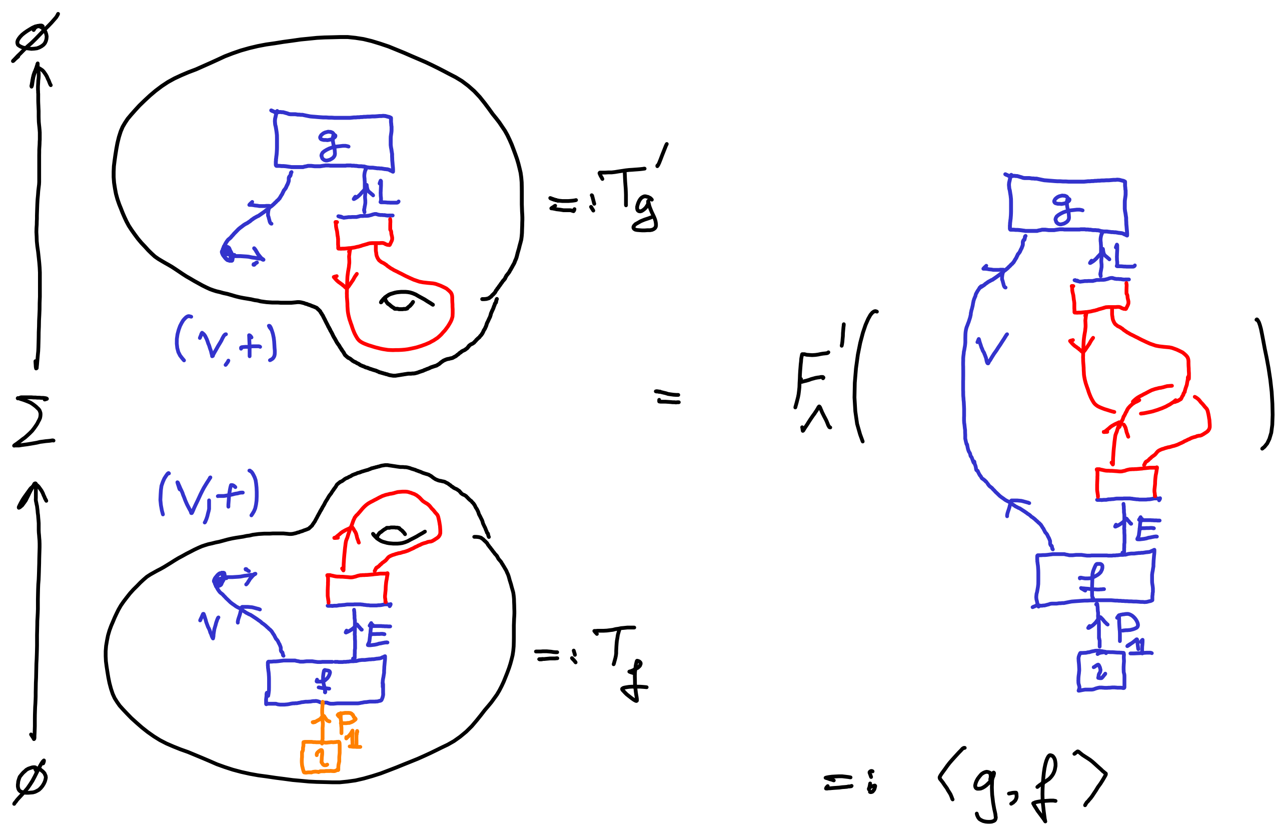


$\emptyset \longrightarrow \Sigma \sqcup -\Sigma$ ✓



$\longmapsto \quad C(P_{\mathbb{1}}, VE)/\sim$

Pairing with $C(VL, \mathbb{1})$ via Heegard splitting of S^3



State space is

$$C(P_{\mathbb{1}}, VE)/\mathrm{rad}_\ell\langle\, , \,\rangle \simeq C(VL, \mathbb{1})^*$$

For $\Sigma \xrightarrow{M} \Sigma'$ define $Z(M)$ via

$$\langle g, Z(M) f \rangle := \langle T'_g \circ M \circ T_f \rangle$$

<u>Thm</u> (De Renzi, Geer, Patureau-Mirand '17 ... & Gainutdinov, IR '19)

This defines a sym. mon. fun.

$$Z : \mathrm{Bord}'_3(\mathcal{C}) \longrightarrow \mathrm{vect}$$

References:

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* Gainutdinov, Runkel,
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  tensor categories
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* Berger, Gainutdinov, Runkel,
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* De Renzi, Geer, Patureau-Mirand,
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* De Renzi, Gainutdinov, Geer, Patureau-Mirand, Runkel,
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  on its centre
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