From non-unital skein theory to modified traces and non-semisimple TQFTs

David Reutter, MPIM Bonn

HIM Seminar Series on Modified Traces

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Progress report on ongoing joint work w. Kevin Walker

Goal:

Understand semisimple & non-semisimple TFTs in a uniform & computationally explicit way.

In other words, we propose an answer to the question:
where did the formulas in Ingo's talk last week "come from"?

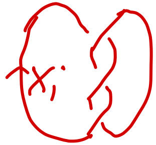
Outline:

i) Semisimple story (where does Reshetikhin–Turaev's invariant "come from"?)

ii) Generalizing to the non-semisimple case.

I. Reshetikhin–Turaev as a ∂-theory of Crane–Yetter

Recall: $\mathcal{M}$ semisimple modular category.

$RT_{\mathcal{M}}$ (M^3 presented as surgery on a link L) $= N \sum \dim(X_i)$ 

$\uparrow$ norm. factor. $\quad$ X_i simples $\quad$ $\uparrow$ Link L evaluated at X_i

$RT_{\mathcal{M}}$ is a ∂-theory of an oriented (fully extended!) 4d theory $CY_{\mathcal{M}}$

$CY_{\mathcal{C}}$ more generally defined for ribbon fusion categories.

In dimensions $(0,1,)\,2,3$ $CY_{\mathcal{C}}$ is skein theory for $\mathcal{C}$:

$\mathrm{Skein}_{\mathcal{C}}(M^3) := K\langle \mathcal{C}\text{-labelled ribbon graphs embedded in } M^3\rangle \big/ \text{isotopies \& local relations from } \mathcal{C}$

$CY_{\mathcal{C}}(M^3) := \mathrm{Skein}_{\mathcal{C}}(M^3)^{\vee}$

E.g: $\mathrm{Skein}_{\mathcal{C}}(S^3) \cong \mathrm{End}_{\mathcal{C}}(I)$

graph $\subseteq S^3$ can be isotoped into a disk and evaluated in $\mathcal{C}$ to an endomorphism of I.

Altogether, skein theory gives:

$$\mathcal{C}\mathcal{Y}_{\mathcal{E}} : \mathrm{Cob}^{\mathrm{or}}_{2,3} \longrightarrow 2\mathrm{Vect}_K$$

Sym. mon. (2,1)-category of
{ closed oriented 2mflds.
compact oriented 3-bordisms
orientation-preserving diffeomorphisms up to isotopy }

Sym. mon. 2-category of
{ finite semisimple linear categories
linear functors
natl trafos. }

[Really: Fully extended TFT into appropriate higher Morita category target, "factorization homology beta" Ayala-Francis, Scheimbauer, Cooke, Morrison-Walker, Johnson-Freyd, ...]

What about 4-manifolds?

(i) I will give an explicit formula to get a feeling & compare with RT

(ii) Then, I will explain that this is forced on us & is the unique extension (up to a $\lambda \in K^\times$) of $\mathcal{C}\mathcal{Y}_{\mathcal{E}}$ to 4-manifolds.

Crash course: Handle decompositions of 4-manifolds.

$$\emptyset \xrightarrow{D^4\text{'s}} \underset{\amalg S^3}{M_0} \xrightarrow[\substack{\text{to embedded}\\ \partial_+(D^1\times D^3)=S^0\times D^3\\ \text{in } M_0}]{\text{glue } D^1\times D^3} M_1 \xrightarrow[\substack{\text{to embedded}\\ \partial_+(D^2\times D^2)=S^1\times D^2\\ \text{in } M_1}]{\text{glue } D^2\times D^2} M_2 \xrightarrow[\substack{\text{to embedded}\\ \partial_+(D^3\times D^1)=S^2\times D^1\\ \text{in } M_2}]{\text{glue } D^3\times D^1} M_3 \xrightarrow[\substack{\text{to embedded}\\ \partial_+(D^4)=S^3\\ \text{in } M_3}]{\text{glue } D^4\text{'s}} M$$

0-handles 1-handles 2-handles 3-handles 4-handles

Simplifying assumption: $\emptyset \xrightarrow{W} M$ built from one 0-handle and 2-handles.

$\Rightarrow$ Handle decomposition encoded in attaching maps of 2-handles, ie. embedded $S^1\times D^2$ in $\partial(\text{0-handle}) = S^3 \Leftrightarrow$ framed link $\subseteq S^3$

$\Rightarrow$ $M = \partial W$ is obtained by surgery on the link L.

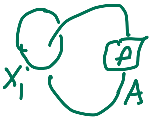
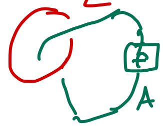
"A framed link $L \subseteq S^3$ is really a presentation of a 4-manifold W together with a bounding 3-manifold"

Let $\phi \xrightarrow{W} M$ be represented by a link $L \subseteq S^3$.

Then:

$$\mathrm{Skein}_{\mathcal{C}}(\phi)^\vee = K \xrightarrow{CY_{\mathcal{C}}(W)} \mathrm{Skein}_{\mathcal{C}}(M)^\vee$$

$$\rotatebox[origin=c]{90}{$\in$}$$

$\lambda^{\#2\text{-handles}+1} \sum_{X_i} \dim(X_i)$  $\longleftrightarrow$ 

handle diagram of $\phi \xrightarrow{D^3\times S^1} S^2\times S^1$

Skein diagram in $S^2\times S^1$

Analogous formulas for general $M_{in} \xrightarrow{W} M_{out}$.

Observe:

$$RT_{\mathcal{C}}\left(\begin{matrix} M^3 \text{ presented by} \\ \text{surgery along } L \subseteq S^3 \end{matrix}\right) = K \xrightarrow{CY_{\mathcal{C}}(W_L)} CY_{\mathcal{C}}(M^3) = \mathrm{Skein}_{\mathcal{C}}(M)^\vee \xrightarrow{(\text{empty skein})^\vee} K$$

[if we choose a normalization s.t. $\lambda^2 = \dim(\mathcal{C})$]

$\Longrightarrow$ $RT_{\mathcal{C}}$ is a boundary theory of $CY_{\mathcal{C}}$: $CY_{\mathcal{C}}(M^3) \xrightarrow{RT_{\mathcal{C}}(M^3) = \text{empty skein}} K$

If $\mathcal{C}$ modular $\Rightarrow$ $CY_{\mathcal{C}}(M^3)$ is one-dimensional.

If $\mathcal{C}$ moreover anomaly-free $\Rightarrow$ $CY_{\mathcal{C}}(W) = CY_{\mathcal{C}}(W')$ for all $W, W' : \phi \to M^3$.

II. Extending field theories

Question: How much choice did we have in extending skein theory to 4-manifolds?

Thm [R.-Walker]: Let $\mathbb{T}$ be a sym. mon. bicategory. Then extensions of a "$(n-1,n,n+\varepsilon)$-TFT"

$$\begin{tikzcd} \mathrm{Cob}_{n-1,n} \arrow[r, "Z"] \arrow[d] & \mathbb{T} \\ \mathrm{Cob}_{n-1,n,n+1} \arrow[ur, dashed] & \end{tikzcd}$$

or

or

correspond to a choice of 1-morphism "D^{n+1}"$: I \longrightarrow Z(S^n)$ [data]

[property]

fulfilling a sequence of n non-degeneracy conditions:

$i=0$: The 1-morphism $I \xrightarrow{D^{n+1}} Z(S^n) \simeq Z(D^n) \circ_{Z(S^{n-1})} Z(D^n)$ is the unit of an adjunction in $\mathbb{T}$.

$i=k$: The 1-morphism $I \xrightarrow{S^k \times D^{n+1-k}} Z(S^k \times S^{n-k}) \simeq Z(S^k \times D^{n-k}) \circ_{Z(S^k \times S^{n-k-1})} Z(S^k \times D^{n-k})$ build from the counit of the adjunction in step $i=k-1$ is the unit of an adjunction.

[cf. the induction step in Lurie's proof sketch of the cobordism hypothesis.]

For skein theory:

$\mathrm{Skein}_{\mathcal{C}}(S^3) \cong \mathrm{End}_{\mathcal{C}}(\mathbb{I})$

Therefore:

extensions of $\mathrm{Skein}_{\mathcal{C}}(-)^{\vee}$ to 4-manifolds $\Longleftrightarrow$ non-degenerate $tr : \mathrm{End}_{\mathcal{C}}(\mathbb{I}) \to k$ $\overset{\mathcal{C}\text{ semisimple ribbon}}{\Longleftrightarrow}$ torsor over $k^{\times}$.

Fixing tr s.t. $tr(id_{\mathbb{I}}) = \lambda \in k^{\times}$ $\rightsquigarrow$ get CY formula from before

III. Non-unital skein theory & modified traces

Let $\mathcal{C}$ be a (non-semisimple) finite ribbon category.

Observation: TQFTs really only see $\mathrm{Proj}(\mathcal{C})$. ← Karoubi complete k-lin. cat. with finite-dim. Hom spaces & finitely many indecomposables.

$\Rightarrow$ Let's do skein theory for $\mathrm{Proj}(\mathcal{C})$!

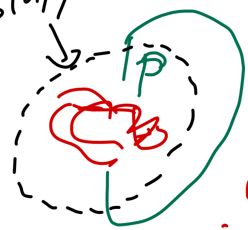
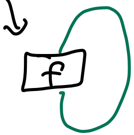
But: $\mathrm{Proj}(\mathcal{C})$ doesn't have a tensor unit.

$\Rightarrow$ Non-unital skein theory!

$$\mathrm{Skein}_{\mathrm{Proj}(\mathcal{C})}(M^3) = K\left\langle \begin{array}{l} \mathcal{C}\text{-labelled ribbon graphs in } M^3 \\ \text{s.t. every component of } M^3 \text{ contains} \\ \text{at least one projective label} \end{array} \right\rangle \Big/ \begin{array}{l} \text{isotopies \&} \\ \text{local relations in} \\ \text{disks with at least one} \\ \text{projective on the } \partial. \end{array}$$

$\exists$ a more general expression which works for general "non-unital ribbon categories $\mathcal{A}$".
For $\mathcal{A} = \mathrm{Proj}(\mathcal{C}) \rightsquigarrow$ simplifies by allowing labels from $\mathcal{C}$.

Example:

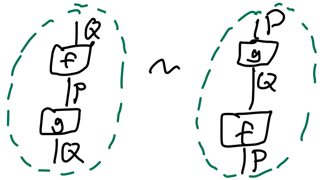
$\mathrm{Skein}_{\mathrm{Proj}(\mathcal{C})}(S^3) =$ ribbon graphs in S^3 with at least one projective label P $=$  $=$ 

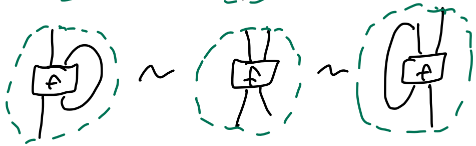
more P string away from rest

evaluate in disk

can't be simplified further!

$$\cong \bigoplus_{\substack{\text{projectives} \\ P}} \mathrm{End}(P) \Big/$$

i)  $\forall\, P \underset{g}{\overset{f}{\rightleftarrows}} Q$

ii) 

Thm [R.-Walker]: $\mathrm{Skein}_{\mathrm{Proj}(\mathcal{C})}(S^3)^\vee \cong$ space of modified traces on $\mathrm{Proj}(\mathcal{C})$

IV Combining everything

80%-Thm [R.-Walker]

Let $\mathcal{C}$ be a modular, unimodular finite ribbon category.
Then, extensions of $\mathrm{Skein}_{\mathrm{Proj}(\mathcal{C})}(-)^\vee$ to an oriented 4d TQFT correspond to choices of non-degenerate modified traces on $\mathrm{Proj}(\mathcal{C})$.

Moreover, if $\mathcal{C}$ is anomaly-free and if S is a $\mathcal{C}$-labeled ribbon graph in a closed oriented M^3, then

In contrast to RT, "empty skein" is not an allowed vector.

$$k \xrightarrow{Z_{\mathcal{C}}(W^4)} Z_{\mathcal{C}}(M^3) = \mathrm{Skein}_{\mathrm{Proj}(\mathcal{C})}(M)^\vee \xrightarrow{S^\vee} k$$

is independent of the choice of compact oriented W^4 with $\partial W^4 = M^3$ and agrees with the renormalized Lyubashenko invariant of [De Renzi — Gainutdinov — Geer — Patureau-Mirand — Runkel]

Further Comments: More generally, I expect

- the non-degeneracy conditions to hold for any non-degenerate modified trace on a unimodular finite ribbon category whose symmetric center is semisimple. Moreover, the resulting 4D TFT should be invertible iff $\mathcal{C}$ is modular.

- the RGGPMR 3D TFT to be a not-everywhere defined boundary TQFT of this (everywhere-defined!) 4D theory, analogous to the RT - CY situation.

Thanks for listening!