Relative Serre functors, Frobenius algebras and some applications to conformal field theory

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based on work with Jürgen Fuchs, Gregor Schaumann and Yang Yang

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Overview

1. Eilenberg-Watts calculus and relative Serre functors
    - A brief recap of Radford's S^4 theorem
    - Eilenberg-Watts equivalences and Nakayama functors
    - Radford's S^4-theorem for bimodules
    - Relative Serre functors and pivotal module categories
2. Modular tensor categories and two-dimensional local conformal field theories
    - Reminder about two-dimensional conformal field theories
    - Frobenius bulk algebras from pivotal module categories
    - Outlook

Chapter 1

Eilenberg-Watts calculus and relative Serre functors

Hopf algebras – conventions and recap

Conventions for this talk:
k is an algebraically closed field.
All vector spaces, algebras, Hopf algebras, modules ... are finite-dimensional k-vector spaces

Definition

A bialgebra $(H, \cdot, 1, \Delta, \epsilon)$ is ...
A Hopf algebra $(H, \cdot, 1, \Delta, \epsilon, S)$ with antipode S is ....

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Facts

- The category $H\text{-mod}$ of left modules over a bialgebra is monoidal.
- The category $H\text{-mod}$ of finite-dimensional left modules over a Hopf algebra with invertible antipode has left and right duals; action on V^* by

$$\rho(h)^\vee = \rho(Sh)^t \qquad {}^\vee\rho(h) = \rho(S^{-1}h)^t$$

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Remark

Order of antipode is related to homological algebra of $H\text{-mod}$.
E.g. $S^2 = \mathrm{id}_H$ and $\mathrm{char}(k) \not| \dim H \Rightarrow H$ and H^* are semisimple.

Radford's S^4 theorem, classical statement

- Since $\dim H < \infty$, space of left / right integrals is one-dimensional, e.g.

$$\dim \mathcal{I}_l = \dim\{t \in H \mid ht = \epsilon(h)t\} = 1$$

- Corollary: $t \in \mathcal{I}_l$ and $h \in H$, then $th \in \mathcal{I}_l$, thus $th = \alpha(h)t$ with $\alpha : H \to k$ a morphism of algebras, i.e. a grouplike element in H^*, i.e. a one-dimensional H-module.
- Dually, there are cointegrals and a grouplike element $a \in H$.
- Action of H^* on H: $\alpha \rightharpoonup h := h_{(1)}\langle\alpha, h_{(2)}\rangle$
  with Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

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Theorem (Radford, 1976)

For a finite-dimensional Hopf algebra H, the following holds:

$$S^4(h) = a(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)a^{-1} = \alpha^{-1} \rightharpoonup (aha^{-1}) \leftharpoonup \alpha \qquad \forall h \in H$$

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Consequences:

- The order of the antipode S is finite.
- If H and H^* are unimodular, i.e. $a = 1_H$ and $\alpha = 1_{H^*}$, then $S^4 = \mathrm{id}_H$.

Finite tensor categories

Let k be a field.

Definition (Finite category)

A k-linear abelian category $\mathcal{C}$ is finite, if

1. $\mathcal{C}$ has finite-dimensional k-vector spaces of morphisms.
2. Every object of $\mathcal{C}$ has finite length.
3. $\mathcal{C}$ has enough projectives.
4. There are finitely many isomorphism classes of simple objects.

Remark

A linear category is finite, if and only if it is equivalent to the category A-mod of finite-dimensional A-modules over a finite-dimensional k-algebra.

Definition (Finite tensor category)

A finite tensor category is a finite rigid monoidal linear category.

In particular, the tensor product is exact in each argument; any left exact functor has a left adjoint.

Eilenberg-Watts calculus

Classical result about finite categories:

Proposition

Let A-mod and B-mod finite categories. Let

$$G : A\text{-mod} \to B\text{-mod}$$

be a right exact functor. Then $G \cong G({}_AA_A) \otimes_A -$.
The B-A-bimodule $G({}_AA_A)$ is a right A-module via the image of right multiplication $r_A : A \to A$ under $\mathrm{End}_A(A) \xrightarrow{G} \mathrm{End}_B(G(A))$.
A similar statement allows to express left exact functors in terms of bimodules.

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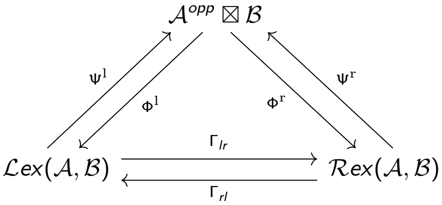
Morita-invariant formulation: triangle of explicit adjoint equivalences:



Ends and coends

Based on the Deligne product and (co)ends.

Remarks

- Examples of coends and ends: trace and natural transformations

$$\int^{v \in \mathrm{vect}_k} v \otimes v^* = k \qquad \text{and} \qquad \mathrm{Nat}(F, G) = \int_{c \in \mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F(c), G(c))$$

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- (Co-)Yoneda lemma: $G:\mathcal{D}\to\mathcal{C}$ linear, then

$$\int^{Y\in\mathcal{D}} G(y)\otimes \mathrm{Hom}_{\mathcal{D}}(y,-) \cong G(-)$$

and

$$\int_{Y\in\mathcal{D}} G(y)\otimes \mathrm{Hom}_{\mathcal{D}}(-,y)^* \cong G(-)$$

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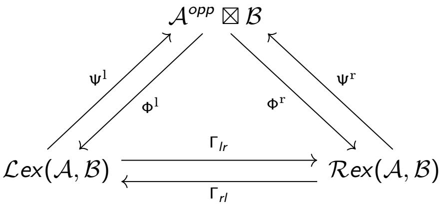
Theorem (Fuchs, Schaumann, CS)

Peter-Weyl theorem: as A-bimodules

$$\int_{m\in A\text{-mod}} m\otimes_k m^* = A \qquad \textit{and} \qquad \int^{m\in A\text{-mod}} m\otimes_k m^* = A^*$$

Eilenberg-Watts calculus



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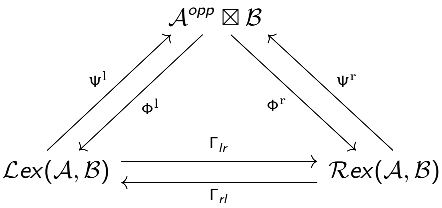

$$\Phi^{\mathrm{l}} \equiv \Phi^{\mathrm{l}}_{\mathcal{A},\mathcal{B}} : \quad \begin{aligned} &\mathcal{A}^{opp} \boxtimes \mathcal{B} \xrightarrow{\simeq} \mathcal{L}ex(\mathcal{A},\mathcal{B})\,, \\ &\overline{a} \boxtimes b \longmapsto \mathrm{Hom}_{\mathcal{A}}(a,-) \otimes b\,, \end{aligned}$$

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In particular, $\mathrm{id}_{\mathcal{A}} \in \mathcal{L}ex(\mathcal{A},\mathcal{A})$ is mapped to the right exact functor

$$N^r_{\mathcal{A}} := \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(-,a)^* \otimes a\,.$$

Nakayama functors

$$N^r_{\mathcal{A}} := \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(-,a)^* \otimes a \quad \text{and} \quad N^l_{\mathcal{A}} := \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,-) \otimes a$$

Lemma

For $\mathcal{A} = A\text{-mod}$*:*

$$N^r_{\mathcal{A}} = A^* \otimes_A - \cong \mathrm{Hom}_A(-,A)^* \quad \textit{and} \quad N^l_{\mathcal{A}} = \mathrm{Hom}_A(A^*,-) \ .$$

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Proof:
Suppose $\mathcal{A} \cong A\text{-mod}$.

- Since $N^r_{\mathcal{A}}$ is right exact, the Eilenberg-Watts theorem implies

$$N^r_{\mathcal{A}} \cong N^r({}_AA_A) \otimes_A -$$

- Thus compute the bimodule $N^r({}_AA_A)$:

$$N^r_{\mathcal{A}}({}_AA_A) = \int^{y\in\mathcal{A}} \mathrm{Hom}_A(A,y)^* \otimes y \cong \int^{y\in\mathcal{A}} y^\vee \otimes y \cong ({}_AA_A)^*$$

where in the last step, we used Peter-Weyl.

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For this reason, we call $N^r_{\mathcal{A}}$ and $N^l_{\mathcal{A}}$ Nakayama functors.

Proposition

1. *The Nakayama functors are adjoints, $N^l_{\mathcal{A}} \dashv N^r_{\mathcal{A}}$.*
2. *$N^l_{\mathcal{A}}$ equivalence $\Leftrightarrow$ $N^r_{\mathcal{A}}$ equivalence. $\Leftrightarrow$ $\mathcal{A}$ is selfinjective.*

Radford's S^4-theorem

For linear functors, we have

Theorem (Fuchs, Schaumann, CS)

Let $\mathcal{A}, \mathcal{B}$ be finite categories. Let $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{B})$ such that F^{la} is left exact so that F^{lla} exists. Assume that F^{lla} is left exact as well. Then there is a natural isomorphism

$$\varphi^l_F : \quad N^l_{\mathcal{B}} \circ F \cong F^{lla} \circ N^l_{\mathcal{A}}$$

that is coherent with respect to composition of functors.

Apply this to bimodule categories over finite tensor categories:
Tensor ideals are bimodule categories. We will only consider (bi)module categories over finite tensor categories that are finite categories and thus in particular abelian. The ideal of projectives is not abelian.

Radford's S^4-theorem

Apply this to bimodule categories over finite tensor categories:

Definition (Module categories)

Let $\mathcal{A}$ and $\mathcal{B}$ be linear monoidal categories.

1. A left $\mathcal{A}$-module category is a linear category $\mathcal{M}$ with a bilinear functor $\otimes : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ and natural isomorphisms
$$\alpha : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{M}}) \xrightarrow{\sim} \otimes \circ (\mathrm{id}_{\mathcal{A}} \times \otimes) \qquad \lambda : \otimes \circ (\mathrm{id}_{\mathcal{A}} \times -) \xrightarrow{\sim} \mathrm{id}_{\mathcal{M}}$$
satisfying obvious pentagon and triangle axioms. We write $a.m := a \otimes m$.
2. Right module categories are defined analogously.
3. An $\mathcal{A}$-$\mathcal{B}$ bimodule category is a linear category $\mathcal{D}$, with the structure of a left $\mathcal{A}$ and right $\mathcal{D}$-module category and a natural associator isomorphism $(a.d).b \cong c.(d.b)$.
4. Module functors, module natural transformations defined in obvious way.

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Apply this to bimodule categories over finite tensor categories:

Theorem (Fuchs, Schaumann, CS)

Let $\mathcal{A}, \mathcal{B}$ be finite tensor categories and $\mathcal{M}$ an $\mathcal{A}$-$\mathcal{B}$ bimodule. Then the Nakayama functor has the structure of a twisted bimodule functor:

$$N^l_{\mathcal{M}}(a.m.b) \cong a^{\vee\vee}.N^l_{\mathcal{M}}(m).{}^{\vee\vee}b$$

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Recovering Radford's S^4-theorem

$$N^l_{\mathcal{M}}(a.m.b) \cong a^{\vee\vee}.N^l_{\mathcal{M}}(m).{}^{\vee\vee}b$$

Observe

- The finite tensor category $\mathcal{A}$ is a bimodule over itself.
- $$N^l_{\mathcal{A}}(1) = \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,1)\otimes a = D_{\mathcal{A}}$$
  is the canonical invertible object of $\mathcal{A}$.
- Compute
  $$N^l_{\mathcal{A}}(a) = N^l_{\mathcal{A}}(a\otimes 1) = a^{\vee\vee}\otimes N^l_{\mathcal{A}}(1) = a^{\vee\vee}\otimes D_{\mathcal{A}}$$
  and
  $$N^l_{\mathcal{A}}(a) = N^l_{\mathcal{A}}(1\otimes a) = N^l_{\mathcal{A}}(1)\otimes {}^{\vee\vee}a = D_{\mathcal{A}}\otimes {}^{\vee\vee}a$$
- We recover Radford's S^4-theorem in its categorical form $D_{\mathcal{A}}\otimes a\otimes D_{\mathcal{A}}^{-1}\cong a^{\vee\vee\vee\vee}$ [ENO, 2004]

Relative Serre functors

Definition (Fuchs, Schaumann, CS 2016)

Let $\mathcal{M}$ be a $\mathcal{C}$-module. A right/left relative Serre functor is an endofunctor $\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$ / $\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}$ of $\mathcal{M}$ together with a family

$$\begin{aligned} \underline{\mathrm{Hom}}(m,n)^{\vee} &\xrightarrow{\cong} \underline{\mathrm{Hom}}(n,\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)) \\ {}^{\vee}\underline{\mathrm{Hom}}(m,n) &\xrightarrow{\cong} \underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(n),m) \end{aligned}$$

of isomorphisms natural in $m, n \in \mathcal{M}$.

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of isomorphisms natural in $m,n \in \mathcal{M}$.

- Relative Serre functors exist, iff $\mathcal{M}$ is an exact module category (i.e. $p.m$ projective, if $p \in \mathcal{C}$ projective).
- Serre functors are equivalences of categories.
- Serre functors are twisted module functors:

$$\phi_{c,m} : \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(c.m) \longrightarrow c^{\vee\vee}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \quad \text{and} \quad \tilde{\phi}_{c,m} : \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(c.m) \longrightarrow {}^{\vee\vee}c.\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(m)$$

Theorem

Let $\mathcal{M}$ be an exact $\mathcal{A}$-module. Then

$$N^{l}_{\mathcal{M}} \cong D_{\mathcal{A}}.\mathrm{S}^{\mathrm{l}}_{\mathcal{M}} \quad \textit{and} \quad N^{r}_{\mathcal{M}} \cong D_{\mathcal{A}}^{-1}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$$

Pivotal module categories

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Definition (Schaumann 2015, Shimizu 2019)

A pivotal structure on an exact module category $\mathcal{M}$ over a pivotal finite tensor category $(\mathcal{C}, \pi)$ is an isomorphism of functors $\tilde{\pi}: \mathrm{id}_{\mathcal{M}} \to \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$ such that the following diagram commutes for all $c \in \mathcal{C}$ and $m \in \mathcal{M}$:

$$\begin{array}{ccc} c.m & \xrightarrow{\;\pi_c.\tilde{\pi}_m\;} & c^{\vee\vee}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \\ & {\scriptstyle \tilde{\pi}_{c.m}} \searrow \qquad \nearrow {\scriptstyle \phi_{c,m}} & \\ & \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(c.m) & \end{array}$$

Pivotal module categories

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$$\begin{array}{ccc} c.m & \xrightarrow{\;\pi_c.\tilde{\pi}_m\;} & c^{\vee\vee}.\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) \\ & {\scriptstyle \tilde{\pi}_{c.m}}\searrow \qquad \nearrow{\scriptstyle \phi_{c,m}} & \\ & \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(c.m) & \end{array}$$

- For indecomposable exact module categories, the pivotal structure is unique up to scalar.
- The algebras $\underline{\mathrm{Hom}}(m,m) \in \mathcal{C}$ for m in a pivotal module category have the structure of symmetric Frobenius algebras.

Frobenius algebras and traces

Proposition

The algebras $\underline{\mathrm{Hom}}(m,m) \in \mathcal{C}$ for m in a pivotal module category $\mathcal{M}$ have the structure of symmetric Frobenius algebras.

- For an exact module category $\mathcal{M}$, use the Serre functor to define a trace on internal Homs, twisted by the Serre functor:

$$\underline{\mathrm{tr}}: \quad \underline{\mathrm{Hom}}(m, \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)) \cong \underline{\mathrm{Hom}}(m,m)^{\vee} \xrightarrow{\mathrm{coev}^{\vee}} 1$$

- Now suppose that $\mathcal{M}$ is pivotal. Then we get a trace on internal Ends:

$$\epsilon_m: \quad \underline{\mathrm{Hom}}(m,m) \xrightarrow{(\pi^{\mathcal{M}}_m)_*} \underline{\mathrm{Hom}}(m, \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)) \xrightarrow{\underline{\mathrm{tr}}} 1$$

  which endows $\underline{\mathrm{Hom}}(m,m)$ with the structure of a symmetric Frobenius algebra (Shimizu, 2019).

- In particular, given an endomorphism $m \xrightarrow{f} m$ in $\mathcal{M}$, find

$$1 \to \underline{\mathrm{Hom}}(m,m) \xrightarrow{f_*} \underline{\mathrm{Hom}}(m,m) \xrightarrow{\epsilon_m} 1 \ .$$

The Drinfeld center

For CFT, we need symmetric Frobenius algebras in the Drinfeld center $\mathcal{Z}(\mathcal{C})$.

Definition (Half-braiding, Drinfeld center)

Let $\mathcal{A}$ be a monoidal category.
A half-braiding for $V \in \mathcal{A}$ is a natural isomorphism

$$\sigma_V : V \otimes - \to - \otimes V$$

such that $\sigma_V(X \otimes Y) = (\mathrm{id}_X \otimes \sigma_V(Y)) \circ (\sigma_V(X) \otimes \mathrm{id}_Y)$ for all $X, Y \in \mathcal{C}$.
The Drinfeld center $\mathcal{Z}(\mathcal{A})$ has pairs (V, σ_V) as objects.

The Drinfeld center

For CFT, we need symmetric Frobenius algebras in the Drinfeld center $\mathcal{Z}(\mathcal{C})$.

Definition (Half-braiding, Drinfeld center)

Let $\mathcal{A}$ be a monoidal category.
A half-braiding for $V \in \mathcal{A}$ is a natural isomorphism

$$\sigma_V : V \otimes - \to - \otimes V$$

such that $\sigma_V(X \otimes Y) = (\mathrm{id}_X \otimes \sigma_V(Y)) \circ (\sigma_V(X) \otimes \mathrm{id}_Y)$ for all $X, Y \in \mathcal{C}$.
The Drinfeld center $\mathcal{Z}(\mathcal{A})$ has pairs (V, σ_V) as objects.

Remarks

1. $\mathcal{Z}(\mathcal{A})$ is a braided monoidal category.
2. The forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ is exact.
   Left adjoint $L : c \mapsto \int^{x \in \mathcal{C}} x \otimes c \otimes {}^{\vee}x$
   Right adjoint $R : c \mapsto \int_{x \in \mathcal{C}} {}^{\vee}x \otimes c \otimes c$
3. $\mathcal{C}$ unimodular $\Leftrightarrow L \cong R$
   $\Leftrightarrow R(1) \in \mathcal{Z}(\mathcal{A})$ is a (commutative) Frobenius algebra (Shimizu 2017)

Symmetric Frobenius algebras in the Drinfeld center

For CFT, we need symmetric Frobenius algebras in $\mathcal{Z}(\mathcal{C})$.
Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{C}$-modules.
The functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is a module category over $\mathcal{Z}(\mathcal{C})$:

$$(z.F)(m) := z.F(m)$$

with module functor structure given by half braiding:

$$(z.F)(c.m) = z.F(c.m) \cong (z \otimes c).F(m) \cong (c \otimes z).F(m) \cong c.(z.F)(m)$$

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Theorem (Fuchs, CS 2020)

$\mathcal{C}$ a pivotal finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ exact $\mathcal{C}$-modules.

1. *The functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is an exact module category over $\mathcal{Z}(\mathcal{C})$ with relative Serre functor $N^r_{\mathcal{N}} \circ (D.-) \circ N^r_{\mathcal{M}}$.*
2. *If $\mathcal{C}$ is unimodular pivotal and $\mathcal{M}$ and $\mathcal{N}$ are pivotal $\mathcal{C}$-modules, then $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is a pivotal $\mathcal{Z}(\mathcal{C})$-module category.*
3. *In particular, then $\underline{\mathrm{Nat}}(F,F)$ is a symmetric Frobenius algebra in the Drinfeld center $\mathcal{Z}(\mathcal{C})$ and $\underline{\mathrm{Nat}}(\mathrm{id}_{\mathcal{M}},\mathrm{id}_{\mathcal{M}})$ has a natural structure of a commutative symmetric Frobenius algebra.*

Chapter 2

Modular tensor categories and two-dimensional local conformal field theories

Modular tensor categories

Definition (Modular tensor category)

A modular tensor category $\mathcal{C}$ is a finite ribbon category such that the braiding is maximally non-degenerate. Various formulations exist and are equivalent [Shimizu 2016]:

- Braided equivalence $\mathcal{C} \boxtimes \mathcal{C}^{rev} \simeq \mathcal{Z}(\mathcal{C})$
- Coend $L := \int^{\mathcal{C}} U^{\vee} \otimes U$ has non-degenerate Hopf pairing $\omega_{\mathcal{C}}$
- Map $\mathrm{Hom}(1, L) \to \mathrm{Hom}(L, 1)$ induced by $\omega_{\mathcal{C}}$ is isomorphism.
- $\mathcal{C}$ has no transparent objects.

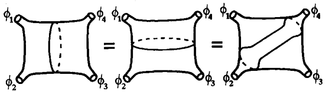
Remarks

- The representation category of suitable vertex algebras or nets of observable algebras has naturally the structure of a modular tensor category:
  The chiral data of a (finite) conformal field theory are described by a modular tensor category.
- From a modular tensor category, one can construct a modular functor (Lyubashenko, $\sim$ 1995)

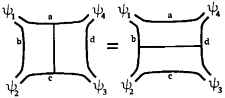
Fields in two-dimensional local conformal field theory

- Fields + OPE $\rightsquigarrow$ (symmetric Frobenius) algebras.
- Frobenius algebras in the appropriate monoidal category

(a)



(c)



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Additional datum to specify local CFT given a modular tensor category:
Suitable module category $\mathcal{M}$ over the modular tensor category $\mathcal{C}$.

Boundary

Boundary condition:	Object of $\mathcal{M}$
Boundary fields from bc m to n	$\underline{\mathrm{Hom}}(m,n) \in \mathcal{C}$
OPE	composition of inner Homs

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- Modular tensor category $\mathcal{C}$ is pivotal.
- Require $\mathcal{M}$ to be a pivotal module category
- Then $\underline{\mathrm{Hom}}(m,m)$ is a symmetric Frobenius algebra for each $m \in \mathcal{M}$.

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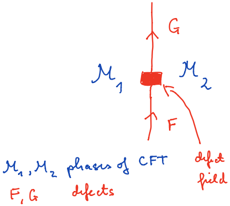
Bulk algebra: commutative algebra in $\mathcal{C} \boxtimes \mathcal{C}^{rev} \simeq \mathcal{Z}(\mathcal{C})$.

Tasks:

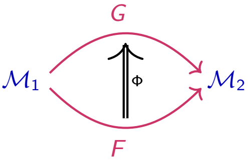
1. Obtain bulk Frobenius algebras from boundary data
2. Describe correlators for any surface from OPE
   (This talk focuses on bulk fields.)

Bulk fields and defect fields for a fixed modular tensor category $\mathcal{C}$

Include defects and defect fields:



Poincaré dual $\leftrightarrow$



Defects are labelled by right exact $\mathcal{C}$-module functors $F, G : \mathcal{M}_1 \to \mathcal{M}_2$.
For defect field, need an object $\mathbb{D}^{F,G}$ in $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C}^{rev} \boxtimes \mathcal{C}$:

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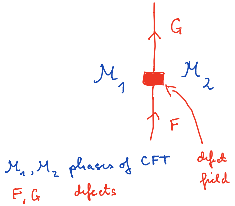

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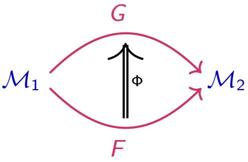

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Theorem (Fuchs, CS 2020)

$$\underline{\mathrm{Nat}}(F, G) = \int_{m_1 \in \mathcal{M}_1} \underline{\mathrm{Hom}}(F(m_1), G(m_1)) \ \in \mathcal{Z}(\mathcal{C})$$

Bulk and defect fields II

$$\mathbb{D}^{F,G} = \int_{m_1 \in \mathcal{M}_1} \underline{\mathrm{Hom}}(F(m_1), G(m_1)) \;\in \mathcal{Z}(\mathcal{C})$$

Remarks

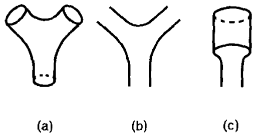
- Recall natural transformations:

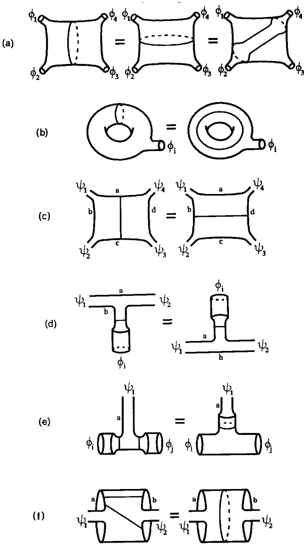
$$\mathrm{Nat}(F,G) \;=\; \int_{m_1 \in \mathcal{M}_1} \mathrm{Hom}(F(m_1), G(m_1)) \;\subset\; \prod_{m_1 \in \mathcal{M}_1} \mathrm{Hom}(F(m_1), G(m_1))$$

  For $\mathcal{C} = \mathcal{M} = A$-mod, get $Z(A) = \mathrm{Nat}(\mathrm{id}, \mathrm{id}) \;=\; \int_{m_1 \in \mathcal{M}_1} \mathrm{Hom}(m_1, m_1)$

- Defect fields = "internalized" natural transformations.
  In particular, bulk algebra $= \int_{m \in \mathcal{M}} \underline{\mathrm{Hom}}(m,m) =$ "internalized center".
- We have horizontal and vertical compositions of relative natural transformations.

Sewing constraints





(Lewellen, 1992)

Structure morphisms:

– Multiplications and comultiplications
– Component maps $\underline{\mathrm{Nat}}(\mathrm{id},\mathrm{id}) \to \underline{\mathrm{Hom}}(m, m)$

Relations:

– (a), (c): bulk and boundary are Frobenius
– (e): component map is morphism of algebras
– (d) dinaturality of the (co)end component morphisms
– (b) and (f)=Cardy relation are genus 1

Outlook

Outlook

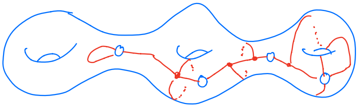
1. $\mathcal{C}$ semisimple: correlators for boundary and defect fields though string nets.
2. Stringnets beyond semisimplicity.
3. Bulk algebras and other fields beyond semisimplicity.
4. Combination with approximation schemes.

Appendix

Correlators for semisimple modular tensor categories via string nets

String net models

Σ oriented smooth surface, possibly with boundary
Γ unoriented graph on Σ.



Coloring: $\mathcal{C}$ a spherical fusion category
Edge: Object $V(e) \in \mathcal{C}$ not necessarily simple Vertex: Morphism $v \in V(\Gamma)$

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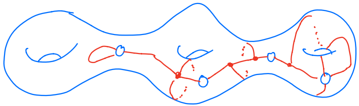

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Edge: Object $V(e) \in \mathcal{C}$ not necessarily simple Vertex: Morphism $v \in V(\Gamma)$
Define: $\mathrm{Graph}(\Sigma, V)$:= Set of all graphs on Σ with boundary value V
$\mathrm{VGraph}(\Sigma, V) := \mathrm{span}_{\mathbb{C}} \mathrm{Graph}(\Sigma, V)$
Impose local relations via graphical calculus on disks.

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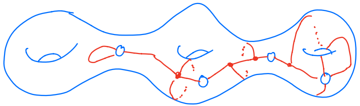

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Definition

The string net space is the quotient

$$H^{string}(\Sigma, V) := \mathrm{VGraph}(\Sigma, V)/N(\Sigma, V)$$

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$$H^{string}(\Sigma, V) := \mathrm{VGraph}(\Sigma, V)/N(\Sigma, V)$$

Remarks

- A colored graph Γ defines a vector $\langle\Gamma\rangle \in H^{string}(\Sigma, V)$.
- $H^{string}(\Sigma, V)$ carries a geometric action of the mapping class group of Σ.

Remark

String nets can be used to define a fully-fledged 3-2-1 topological field theory that is equivalent to the Turaev-Viro-Barrett-Westbury state sum model.

Consistent systems of correlators

Correlators for bulk fields with bulk object F
=vector $v_\Sigma \in \mathrm{tft}_{\mathcal{C}^{rev}\boxtimes\mathcal{C}}(\Sigma)$ for all surfaces Σ
(since $\mathcal{C}^{rev} \boxtimes \mathcal{C} \simeq \mathcal{Z}(\mathcal{C})$)
=specific vector in the string net space $H^{string}(\Sigma)$ for all Σ

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- Boundary value F at each boundary component

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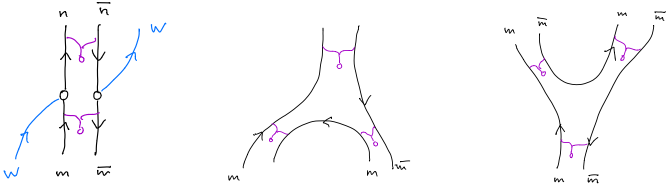
$\mathcal{C}$ semisimple and modular, $\mathcal{M}$ pivotal. Write $\mathcal{M} = \mathrm{mod}_{\mathcal{C}} - A$. Then

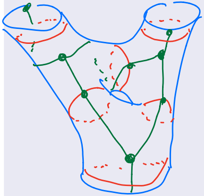
$$F_{\mathcal{M}} = \underline{\mathrm{Nat}}(\mathrm{id},\mathrm{id}) = \int_{m\in\mathcal{M}} \underline{\mathrm{Hom}}(m,m) = \oplus_{\alpha\in I_{\mathcal{M}}} \underline{\mathrm{Hom}}(m_\alpha m_\alpha) = \oplus_\alpha m_\alpha \otimes_A \overline{m_\alpha}$$



Correlators from string nets

Theorem (Fuchs, CS, Yang Yang, 2020)

The vector v_Σ specified by the following string net on Σ is invariant under the mapping class group:

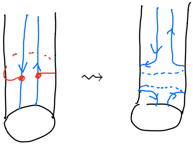


red lines	$\rightsquigarrow$	*canonical color*
green lines	$\rightsquigarrow$	*Bulk Frobenius algebra $F_{\mathcal{M}}$*

Proof of the theorem

Cardy case: $\mathcal{M} = \mathcal{C}$:

Locally, at the boundary



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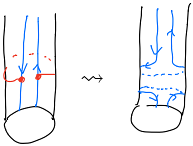
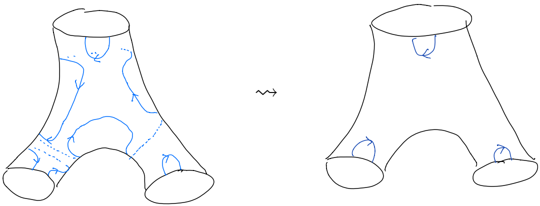


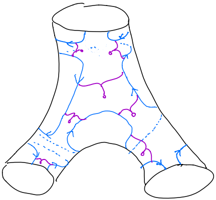
Globally, on a pair of pants



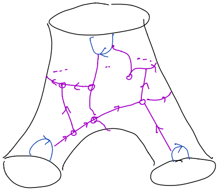
We get essentially empty string nets that are manifestly invariant under the mapping class group.

General pivotal module categories

The string net on the pair of pants reduces to the dual of a triangulation labelled by the Frobenius algebra A which is famously an invariant under the mapping class group.



$\rightsquigarrow$



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