Towards derived TFT's and eventually CFT's

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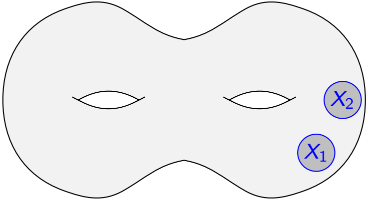
Recall

Theorem (Reshetikhin-Turaev)

Every semisimple modular tensor category $\mathcal{C}$ produces a topological field theory (constructed via surgery along links)

- To a compact oriented surface of genus g with n boundaries, decorated by objects $X_1, \ldots, X_n \in \mathcal{C}$, it assigns a vector space



$$\Sigma_{g,n}^{X_1,\ldots,X_n} \longmapsto \mathcal{Z}(\Sigma_{g,n}^{X_1,\ldots,X_n})$$

- To every 3-manifold M, cobordism between two surface, and links ending in the X_i, it assigns a linear map $\mathcal{Z}({}_{\Sigma_1}\mathrm{M}_{\Sigma_2})$
- such that several axioms are fulfilled, in particular glueing

What about non-semisimple categories?

- Need proper notion of non-semisimple modular tensor category (3 equivalent definitions).
- We do have $\mathcal{Z}(\mathrm{M})$ for special cobordisms, most importantly a proj. action of the **mapping class group** $\Gamma_{g,n}$ on $\mathcal{Z}(\Sigma_{g,n})$ such as an action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{Z}(\Sigma_{1,0})$.

  We now discuss this construction by Lyubaschenko (1995).

Then we discuss our work establishing an action of $\Gamma_{g,n}$ on a derived version $\mathcal{Z}^\bullet(\Sigma_{g,n})$, examples and current work.

Main Reference: L., Mierach, Schweigert, Sommerhäuser (2019): Hochschild Cohomology, Modular Tensor Categories, and Mapping Class Groups arXiv:2003.06527, to appear in "Springer Briefs in Mathematical Physics"

Mapping Class Groups

Take $\Sigma_{g,n}$. On each boundary circle $\rho_1, \ldots, \rho_n$ we fix a marked point.

Definition

The **mapping class group** $\Gamma_{g,n}$ is the group of o-preserving diffeomorphisms of $\Sigma_{g,n}$ that send marked points to marked points, up to homotopies that send marked points to marked points.

The **pure mapping class group** $\mathrm{P}\Gamma_{g,n}$ is the group of o-preserving diffeomorphism of $\Sigma_{g,n}$ that fix all boundary circles pointwise, up to homotopies that fix all boundary circles pointwise.

Lemma

$$1 \to \mathrm{P}\Gamma_{g,n} \to \Gamma_{g,n} \to \mathbb{S}_n \to 1$$

Note the difference between boundary circles and punctures:
A 360° rotation of the boundary circle becomes a trivial element.

Mapping Class Groups

Definition

For a subset $S \subset \Sigma_{g,n}$ define $\Gamma_{g,n}(S)$ as diffeomorphisms fixing S, up to such homotopies. Typical examples are $\Gamma_{g,n}(x)$ and $\Gamma_{g,n}(\rho_n)$.

Lemma (Cap Sequence)

$$\mathbb{Z} \longrightarrow \Gamma_{g,n+1}(\rho_{n+1}) \longrightarrow \Gamma_{g,n}(x) \longrightarrow 1$$

The first map (rotations around ρ_{n+1}) is injective except $g = n = 0$.

Theorem (Birman sequence)

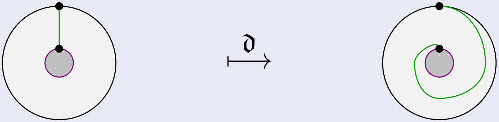
$$\pi_1(\Sigma_{g,n}, x) \longrightarrow \Gamma_{g,n}(x) \longrightarrow \Gamma_{g,n} \longrightarrow 1$$

The first map is called push map, discussed and used later.
The push map is injective, if the Euler characteristic is negative.

Mapping Class Groups

Definition (Dehn twist)

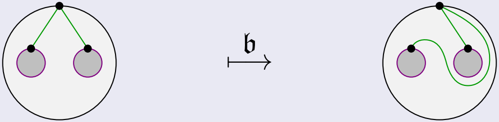
On the annulus $\Sigma_{0,2} = S^1 \times [0,1]$ we define $(\phi, t) \mapsto (\phi + 2\pi \mathrm{i} t, t)$



On any $\Sigma_{g,n}$ and for any simple curve $\gamma : S^1 \to \Sigma_{g,n}$
we define a diffeomorphism $\mathfrak{d}_\gamma$, using a tubular neighbourhood.

Definition (Braiding)

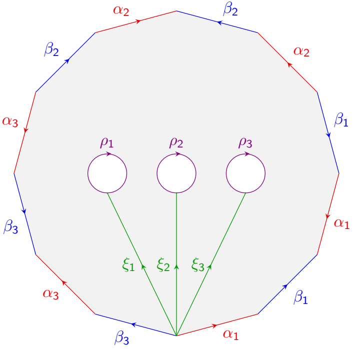
On the three-punctured sphere $\Sigma_{0,3}$ we define the diffeomorphism

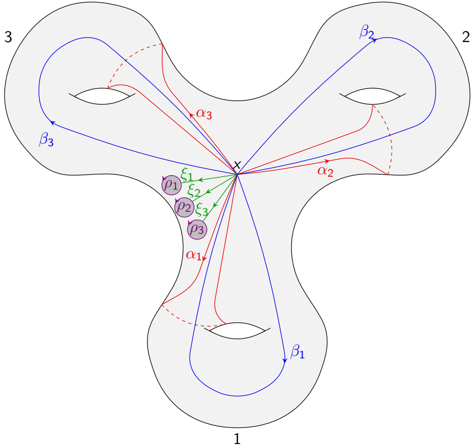


On any $\Sigma_{g,n}$ define a diffeomorphism $\mathfrak{b}_{i,j}$ for any $1 \leq i < j \leq n$.

Mapping Class Groups

For explicit calculations we use the polygon model of $\Sigma_{g,n}$:





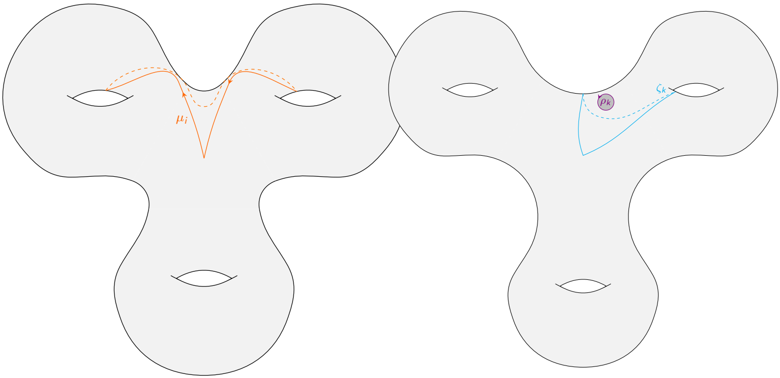
$\pi_1(\Sigma_{g,n}, x)$ has the relation $\prod_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \prod_{k=1}^{n} \xi_k \rho_k \xi_k^{-1} = 1$

Mapping Class Groups

Theorem (Dehn-Lickorish)

The following diffeomorphism classes generate $\Gamma_{g,n}$ *as a group:*

$$\mathfrak{t}_i := \mathfrak{d}_{\alpha_i}, \quad \mathfrak{r}_i := \mathfrak{d}_{\beta_i}, \quad \mathfrak{d}_k := \mathfrak{d}_{\rho_k}, \quad \mathfrak{b}_{k,k+1}, \quad \mathfrak{n}_i := \mathfrak{d}_{\mu_i}, \quad \mathfrak{z}_k := \mathfrak{d}_{\zeta_k}$$



We further define the diffeomorphism class $\mathfrak{s}_i := \mathfrak{t}_i^{-1}\mathfrak{r}_i^{-1}\mathfrak{t}_i^{-1}$.

Mapping Class Groups

Fact

The group $\Gamma_{g,n}(x)$ acts on $\pi_1(\Sigma_{g,n}, x)$ by group automorphisms.

The group $\Gamma_{g,n}$ acts on $\pi_1(\Sigma_{g,n}, x)$ by outer automorphism classes.

Recall that the abelianization of $\pi_1(\Sigma_{g,n})$ is $\mathrm{H}_1(\Sigma_{g,n}, \mathbb{Z}) = \mathbb{Z}^{2g+n}$.

Fact

The action of $\Gamma_{g,n}(x)$ on $\mathrm{H}_1(\Sigma_{g,n}, \mathbb{Z})$ factors over $\Gamma_{g,n}$. Explicitly

$$\mathfrak{t}_i = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \mathfrak{r}_i = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\ \mathfrak{s}_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \mathfrak{n}_i = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathfrak{d}_k = (1), \quad \mathfrak{b}_{k,k+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathfrak{z}_k = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a representation of $\Gamma_{g,n}$ on $\mathbb{Z}^{2g}$ factoring over $\mathrm{Sp}_{2g}(\mathbb{Z})$, where the symplectic form on $\mathbb{Z}^{2g}$ is the intersection form on H_1

Mapping Class Groups

For the torus the previous action of $\Gamma_{g,n}$ on $\mathbb{Z}^{2g}$ is faithful:

Example

On the torus $\Sigma_{1,0}$ the mapping class group is $\mathrm{SL}_2(\mathbb{Z})$, which is generated by $\mathfrak{s}, \mathfrak{t}$ with relations $\mathfrak{s}^4 = 1$, $\mathfrak{s}\mathfrak{t}\mathfrak{s} = \mathfrak{t}^{-1}\mathfrak{s}\mathfrak{t}^{-1}$.

On the punctured torus $\Sigma_{1,1}$ we have a central element $\mathfrak{s}^4 = \mathfrak{d}_1^{-1}$.

Example

On the punctured sphere we have a group homomorphism

$$\mathbb{Z}^n \ltimes \mathbb{B}_n \longrightarrow \Gamma_{0,n}$$

using Dehn twists $\mathfrak{d}_k$ and braidings $\mathfrak{b}_{i,j}$, with $\mathbb{B}_n$ the braid group. The map is not injective, but factors to an isomorphism, for $n > 1$

$$\mathbb{Z}^{n-1} \ltimes \mathbb{B}_{n-1} \xrightarrow{\sim} \Gamma_{0,n}$$

Modular Tensor Categories

Let $(\mathcal{C}, 1, \otimes,)$ be a finite tensor category over a field $\mathbb{K}$.

Definition

Recall: The coend $L = \int^X F(X, X)$ of a bifunctor $F : \mathcal{C}^{op} \otimes \mathcal{C} \to \mathcal{D}$ is the universal object L having a dinatural trafo $\iota_X : F(X, X) \to L$

Theorem

The coend L of the bifunctor $X^* \otimes X$ is a Hopf algebra inside $\mathcal{C}$. (product from $\iota_{X\otimes Y}$, unit from ι_1, coproduct from coeval_{X,X^*} etc.)

Example

If $\mathcal{C}$ is semisimple, with simple objects X_i, then $L = \bigoplus_i X_i^* \otimes X_i$

Example

If $\mathcal{C}$ is the category of representations of a Hopf algebra H, then L is the coadjoint representation H^*_{coad} (and transmuted algebra)

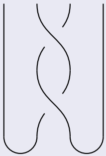
Modular Tensor Categories

Let $(\mathcal{C}, 1, \otimes, c_{X,Y}, \theta_X)$ be a finite ribbon category.

Definition (Modular Tensor Category)

Call $\mathcal{C}$ modular, if one of the following equivalent conditions holds

- The only objects X with $c_{Y,X}c_{X,Y} = \mathrm{id}$ for all objects Y, called transparent objects, are trivial $X = 1 \oplus \cdots \oplus 1$.
- The map sending an object X to $X, c_{X,Y}$ and $X, c^{-1}_{Y,X}$
$$\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \to \mathrm{DrinfeldCenter}(\mathcal{C})$$
is an equivalence of braided tensor categories.
- The pairing $\omega : L \otimes L \to 1$ defined by dinat. maps
$$(\mathrm{eval}_{X^*,X} \otimes \mathrm{eval}_{Y^*,Y}) \circ (\mathrm{id}_{X^*} \otimes c_{Y^*,X}c_{X,Y^*} \otimes \mathrm{id}_Y)$$
is non-degenerate. It represents the open Hopf link, and generalizes the matrix S_{ij} for semisimple $\mathcal{C}$.



Definition in Lyubaschenko (1996), equivalence see Müger, Shimizu.

Modular Tensor Categories

Example

Semisimple modular tensor categories, such as Vect_A^Q for a (finite) abelian group A and a nondegenerate quadratic form $Q : A \to \mathbb{K}^\times$.

Example

Yetter-Drinfeld modules ${}^G_G\mathcal{YD}$ of a finite group G over any field $\mathbb{K}$.

Simple/indecomposable/projective objects $\mathcal{O}_{[g]}^\chi$ for any conjugacy class $[g]$ and simple/indecomposable/projective rep χ of $\mathrm{Cent}(g)$.

Example

$\mathrm{Rep}(H)$ for a finite-dimensional factorizable ribbon Hopf algebra H, for example the small (quasi-)quantum group $u_q(\mathfrak{g})$.

Lyubaschenko's Modular Functor

Let $\mathcal{C}$ be a modular tensor category and $\Sigma_{g,n}^{X_1,\dots,X_n}$ a decorated surface.

Definition (Block space)

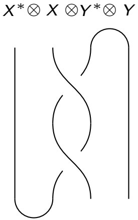
$$\mathcal{Z}(\Sigma_{g,n}^{X_1,\dots,X_n}) := \mathrm{Hom}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, L^{\otimes g})$$

Theorem

$\mathrm{P}\Gamma_{g,n}$ *acts projectively on* $\mathcal{Z}(\Sigma_{g,n}^{X_1,\dots,X_n})$*, and* $\Gamma_{g,n}$ *on a resp. sum.*

For example, $\mathfrak{d}_k$ acts via θ_{X_k}, the braiding $\mathfrak{b}_{k,k+1}$ acts via $c_{X_k,X_{k+1}}$, $\mathfrak{t}_i$ acts via θ_X on any $X^* \otimes X$ dinaturally, and thereby on the i-th L, $\mathfrak{s}_i$ acts again by a variant of the Hopf link on the i-th L, explicitly

$$L \xrightarrow[\text{integral, Kirby color}]{\mathrm{id}\otimes\Lambda_L} L \otimes L \xrightarrow[\text{dinatural}]{\mathrm{eval}_{X^*,X}\, c_{Y^*,X}\, c_{X,Y^*}} L$$



Towards Derived Topological Field Theories

Theorem (L., Mierach, Schweigert, Sommerhäuser 2018)

$\mathrm{SL}_2(\mathbb{Z})$ *acts on the the Hochschild cohomology* $\mathrm{HH}^\bullet(H, \mathbb{K})$ *of a finite-dimensional factorizable ribbon Hopf algebra.*

The twisted class functions reappear as $\mathrm{HH}^0(H, \mathbb{K})$.

Theorem (L., Mierach, Schweigert, Sommerhäuser 2020)

There is an action of the mapping class group $\mathrm{P}\Gamma_{g,n}$ *on the spaces*

$$\mathcal{Z}^\bullet(\Sigma_{g,n}^{X_1,\dots,X_n}) := \mathrm{Ext}^\bullet_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, L^g)$$

The Lyubaschenko modular functor reappears as degree zero part.

Theorem (Schweigert, Woike 2019, 2020)

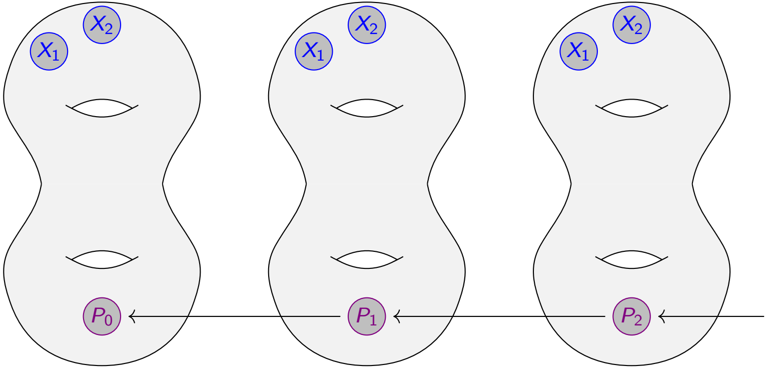
There is a homotopy coherent action on $\mathrm{P}\Gamma_{g,n}$ *on a suitable Hochschild complex, in the resp. homotopy theoretic setting.*
$\Rightarrow$ *A modular functor with values in chain complexes.*

Construction and Proof

Take a projective resolution of the tensor unit

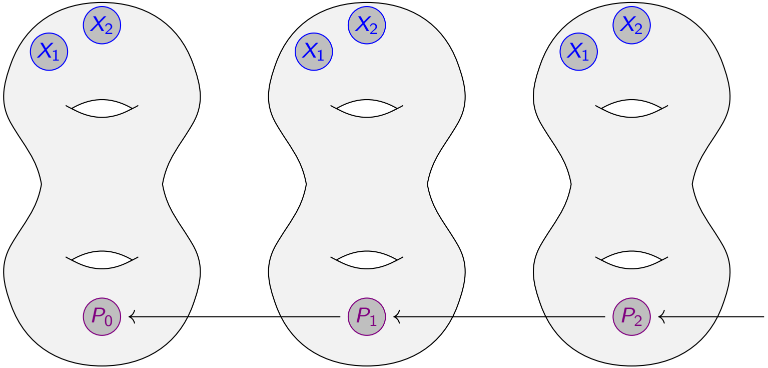
$$1 \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots$$

Functoriality of Lyubaschenko's $\mathcal{Z}$ gives a chain complex



$$\mathrm{Hom}_{\mathcal{C}}(X_1 \cdots X_n \otimes P_0, L^g) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X_1 \cdots X_n \otimes P_1, L^g) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X_1 \cdots X_n \otimes P_2, L^g) \longrightarrow$$

Towards Derived Topological Field Theories: Proof



$$\mathrm{Hom}_{\mathcal{C}}(X_1 \cdots X_n \otimes P_0, L^g) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X_1 \cdots X_n \otimes P_1, L^g) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X_1 \cdots X_n \otimes P_2, L^g) \longrightarrow$$

The mapping class group $\mathrm{P}\Gamma_{g,n+1}$ acts strictly (but projectively) on this chain complex by chain maps.

Does this factor to an action of $\mathrm{P}\Gamma_{g,n}$ up to chain homotopy? **YES...**

Towards Derived Topological Field Theories: Examples

Example (Sphere)

$$\mathcal{Z}^\bullet(\Sigma_{0,0}) = \mathrm{Ext}^\bullet_{\mathcal{C}}(1, 1)$$

This is an **algebra** via the cup-product. It acts on any $\mathcal{Z}^\bullet(\Sigma_{g,n})$, commuting with $\Gamma_{g,n}$-action. It plays the role of a new ground ring.

Example (Punctured Sphere)

$$\mathcal{Z}^\bullet(\Sigma_{0,n}^{X_1,\ldots,X_n}) = \mathrm{Ext}^\bullet_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, 1)$$

This has an action of the (pure) braid group on n strands via c_{X_i,X_j}. This action factorizes over the mapping class group $\Gamma_{0,n}$, because $\theta_{X_1 \otimes \cdots \otimes X_n}$ acts trivial up to homotopy, although $\theta_{P_i} \neq \mathrm{id}$

Towards Derived Topological Field Theories: Examples

Example (Genus 1, Torus without punctures)

$$\mathcal{Z}^\bullet(\Sigma_{1,0}) = \mathrm{Ext}^\bullet_{\mathcal{C}}(1, L)$$

this has an action of the modular group $\mathrm{SL}_2(\mathbb{Z})$.

It comes from an action of $\Gamma_{1,1}$ on L by morphisms in $\mathcal{C}$, where

$$\langle \mathfrak{d} \rangle \to \Gamma_{1,1} \to \mathrm{SL}_2(\mathbb{Z}),$$

is a central extension with $\mathfrak{s}^4 = \mathfrak{d}^{-1}$. The element $\mathfrak{d}$ acts by θ_L, so it acts trivially on $\mathrm{Hom}_{\mathcal{C}}(1, L)$ and all $\mathrm{Ext}^\bullet_{\mathcal{C}}(1, L)$.

For $\mathcal{C} = \mathrm{Rep}(H)$ we recover our previous result (1707.04032):

$\Gamma_{1,1}$ acts on the coadjoint representation $L = H^*_{coad}$, the quotient $\mathrm{SL}_2(\mathbb{Z})$ acts on the Hochschild cohomology $\mathrm{Ext}^\bullet_{\mathcal{C}}(1, L) \cong \mathrm{HH}^\bullet(H, H)$, compatible with cup product by the algebra $\mathrm{Ext}^\bullet_{\mathcal{C}}(1, 1) \cong \mathrm{HH}^\bullet(H, \mathbb{K})$.

Towards Derived Topological Field Theories: Examples

Example (Commutative Case)

Suppose that $\mathcal{C}$ has the property that $L = 1 \oplus \cdots \oplus 1$ as object. (for example, representations of a commutative Hopf algebra)

Then $\Gamma_{1,1}$ and also $\Gamma_{1,0}$ act on $\mathbb{K}^n = \mathrm{Hom}_{\mathcal{C}}(1, L)$. $\mathrm{P}\Gamma_{g,n+1}$ acts on

$$\begin{aligned} &\mathrm{Hom}(X_1 \otimes \cdots \otimes X_n \otimes P_i, L^{\otimes g}) \\ &= \mathrm{Hom}(X_1 \otimes \cdots \otimes X_n \otimes P_i, 1) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(1, L^{\otimes g}) \end{aligned}$$

where the decomposition is preserved by $\mathfrak{d}, \mathfrak{b}$ and $\mathfrak{t}, \mathfrak{s}, \mathfrak{n}$, not $\mathfrak{z}$.

This action factorizes to an action of $\mathrm{P}\Gamma_{g,n}$ on

$$\mathcal{Z}^\bullet(\Sigma_{g,n}) = \mathrm{Ext}^\bullet(X_1 \otimes \cdots \otimes X_n, 1) \otimes_{\mathbb{K}} (\mathbb{K}^n)^g$$

In particular $\mathcal{Z}^\bullet(\Sigma_{g,0})$ is a free module of the Ext-algebra $\mathcal{Z}^\bullet(\Sigma_{0,0})$ generated by Lyubaschenko's part in degree zero $\mathcal{Z}(\Sigma_{g,0})$.

Towards Derived Topological Field Theories: Groups

We now treat a class of nonsemisimple examples more elaborately. Let G be a finite group, $\mathbb{K}$ of arbitrary characteristic, recall:

Definition (Yetter-Drinfeld modules ${}^G_G\mathcal{YD}$)

- Objects: G-graded G-representations V with $g.(V_h) = V_{ghg^{-1}}$
- The simple, indecomposable, or projective objects are $\mathcal{O}_{[h]}^V$, parametrized by a conjugacy class $[h]$ of G and a simple, indecomposable, or projective representations V of $\mathrm{Cent}(h)$
- Semisimple iff $\mathrm{Rep}(G)$ is semisimple, i.e. $\mathrm{char}(\mathbb{K}) \nmid |G|$.
- Braiding $v_g \otimes v_h \mapsto g.v_h \otimes v_g$.

For example, the symmetric group $\mathbb{S}_3$ over $\mathbb{K} = \mathbb{C}$ has simples

$$\mathcal{O}_e^1,\ \mathcal{O}_e^{\mathtt{sgn}},\ \mathcal{O}_e^{\mathtt{std}},\ \mathcal{O}_{[(12)]}^{\pm 1},\ \mathcal{O}_{[(123)]}^{\zeta_3^k}$$

In characteristic 2 or 3 the category is nonsemisimple .

Towards Derived Topological Field Theories: Groups

More generally, for every tensor category $\mathcal{C}$ we can define a modular tensor category called Drinfeld center $\mathcal{D}(\mathcal{C})$.

The Reshetikhin-Turaev-TFT of $\mathcal{D}(\mathcal{C})$ is the Turaev-Viro TFT of $\mathcal{C}$ as a state-sum model (also extended in [FSS]). Recall the example

Example (Dijkgraaf-Witten theory $\mathcal{C} = {}^G_G\mathcal{YD}$)

$$\begin{aligned}\mathcal{Z}(\Sigma_{g,0}) &= \mathrm{Hom}_{\mathcal{C}}(1, (DG)^{\otimes g}) \\ &= \mathrm{span}_{\mathbb{K}} \left\{ (a_1, b_1, \ldots, a_g, b_g) \in G^{2g} \mid \prod [a_i, b_i] = 1) \right\}^{\mathrm{ad}_G} \\ &= \mathrm{span}_{\mathbb{K}} \left\{ \mathrm{Hom}_{group}\left(\pi_1(\Sigma_{g,0}), G\right) / \mathrm{ad}_G \right\}\end{aligned}$$

$\mathcal{Z}(\Sigma_{g,n}^{\mathcal{O}_{[g_1]}^{\chi_1} \cdots \mathcal{O}_{[g_n]}^{\chi_n}})$ is roughly the span of G-bundels with prescribed monodromy g_i around ρ_i; taking a resp. ad_G-isotypical component.

For example for $G = \mathbb{Z}_N$ we get $\mathcal{Z}(\Sigma_{g,0}) = \mathrm{H}_1(\Sigma_{g,0}, \mathbb{Z}) = \mathbb{Z}_N^{2g}$
The mapping class group $\Gamma_{g,0}$ acts via its quotient $\mathrm{Sp}_{2g}(\mathbb{Z}_N)$.

Towards Derived Topological Field Theories: Groups

Let G be a finite group and $\mathcal{C} = {}^G_G\mathcal{YD}$. Define the span

$$M_g := \mathbb{K}\,\mathrm{Hom}_{Group}(\pi_1(\Sigma_{g,0}), G)$$

It has commuting actions of G-module via conjugation on G, and of $\Gamma_{g,1}$ via the action of $\Gamma_{g,0}(x)$ on $\pi_1(\Sigma_{g,0}, x)$.

Theorem (L., Mierach, Schweigert, Sommerhäuser, to appear soon)

$$\mathcal{Z}^\bullet(\Sigma_{g,0}) = \mathrm{H}^\bullet_{Group}(G, M_g)$$

and similarly for $\mathcal{Z}^\bullet(\Sigma_{g,n})$ with boundaries decorated by $\mathcal{O}^\chi_{[h]}$.

We recover our main result: The action of $\Gamma_{g,1}$ on $\pi_1(\Sigma_{g,0})$ does **not** factor to an action of $\Gamma_{g,0}$ but it does on cohomology. E.g.

$$\mathrm{H}^0_{Group}(G, M) = M^G = \mathbb{K}\mathrm{Bun}_G(\Sigma_g)$$

Towards Derived Topological Field Theories: Groups

Example (some Γ_g-representaitons factoring over $\mathrm{Sp}_{2g}(\mathbb{Z})$)

We have $\mathrm{Hom}(\pi_1(\Sigma_g, x), \mathbb{Z}_N) = \mathbb{Z}_N^{2g}$, with diagonal action by $\mathbb{Z}_N^\times$, define $\Omega_{\mathbb{Z}_N}^g$ as all vectors with coefficient gcd 1.

If $\mathbb{K}$ contains all N-th roots of unity, then we further decompose the span according to Dirichlet characters $\chi : \mathbb{Z}_N \to \mathbb{K}^\times$ as follows

$$\mathbb{K}\Omega_{\mathbb{Z}_N}^g = \bigoplus_\chi \mathbb{K}_\chi[\mathbb{Z}_N\mathbb{P}^{2g-1}]$$

interpreted as sections in line bundles on projective space $\mathbb{Z}_N\mathbb{P}^{2g-1}$.

The group $\mathrm{Sp}_{2g}(\mathbb{Z}_N)$ acts on $\mathbb{K}_\chi[\mathbb{Z}_N\mathbb{P}^{2g-1}]$, diagonals acting by χ.

For $g = 1$ the stabilizers of vectors in $\mathbb{K}[\Omega_{\mathbb{Z}_N}^1]$ and $\mathbb{K}_1[\mathbb{Z}_N\mathbb{P}^1]$ give a short exact sequence of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$

$$\mathbb{Z}_N^\times \to \Gamma_0(N) \to \Gamma_1(N)$$

This hints at modular forms, part of a vector valued modular form.

Towards Derived Topological Field Theories: Groups

We give a complete example for $G = \mathbb{S}_3,\ \mathrm{char}(\mathbb{K}) = 3$:

For the torus we get in this case

$$\mathcal{Z}^0(\Sigma_{1,0}) \cong \mathbb{K} \oplus \mathbb{K}[\mathbb{F}_2\mathbb{P}^1] \oplus \mathbb{K}[\mathbb{F}_3\mathbb{P}^1]$$

$$\mathcal{Z}^i(\Sigma_{1,0}) \cong \begin{cases} \mathbb{K}\mu^{i/2} \\ 0 \\ 0 \\ \mathbb{K}\mu^{(i-1)/2}\nu \end{cases} \oplus \begin{cases} \mathbb{K}[\mathbb{F}_3\mathbb{P}^1]\mu^{i/2} \\ 0 \\ 0 \\ \mathbb{K}[\mathbb{F}_3\mathbb{P}^1]\mu^{(i-1)/2}\nu \end{cases} \oplus \begin{cases} 0, \\ \mathbb{K}_{sgn}[\mathbb{F}_3\mathbb{P}^1]\mu^{(i-1)/2}\nu, \\ \mathbb{K}_{sgn}[\mathbb{F}_3\mathbb{P}^1]\mu^{i/2}, \\ 0, \end{cases}$$

We find

- A large portion is generated from degree zero.
  (conjugacy classes of pairs of commuting elements)
- Not free as $\mathcal{Z}^\bullet(\Sigma_{0,0})$-module, $\mathbb{K}[\mathbb{F}_2\mathbb{P}^1]$ is killed (G-projective).
- $\mathbb{K}_{sgn}[\mathbb{F}_3\mathbb{P}^1]$ new in degree $i \equiv 2, 3$, nontrivial Dirichlet character
  From $\mathcal{Z}(\Sigma_{1,1}^{sgn})$ cup $\mathcal{Z}^\bullet(\Sigma_{0,1}^{sgn})$, as sgn is in the principal block.

Towards Derived Topological Field Theories: Outlook

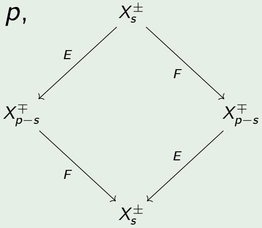
Ongoing work in computing $\mathcal{Z}^\bullet(\Sigma_{g,n})$ for quantum groups:

For every $\mathfrak{g}$ and q a primitive ℓth root of unity,
there exists a small (quasi-)quantum group $u_q(\mathfrak{g})$,
giving a non-semisimple modular tensor category,
related to $\mathfrak{g}$-representations in characteristic ℓ.

Drinfeld Jimbo 1986, Lusztig 1990, Andersen Jantzen Soergel 1996, Kazhdan Lusztig
Creutzig Gainutdinov Runkel 2017, Gainutdinov L. Ohrmann 2018, Negron 2018.

Example

$\tilde{u}_q(\mathfrak{sl}_2) = \langle E, F, K \rangle / (K^{2p} - 1, [E, F] = \frac{K - K^{-1}}{q - q^{-1}})$ at $q^{2p} = 1$ has
simple reps $X_s^\pm$ of dimension s for $1 \leq s \leq p$,
nontrivial $\mathrm{Ext}^1(X_s^\pm, X_{p-s}^\mp) = \mathbb{C}^2$ for $s \neq p$,
and projective covers as follows:



It produces a modular tensor category,
with nontrivial associator from $\mathrm{Vect}_{\mathbb{Z}_{2p}}^Q$.

Towards Derived Topological Field Theories: Outlook

Ongoing work in computing $\mathcal{Z}^\bullet(\Sigma_{g,n})$ for quantum groups.

For $\tilde{u}_q(\mathfrak{sl}_2)$ the following picture holds resp. should hold:

Gainutdinov L. Schweigert, work in progress, drawing from

Feigin Gainutdinov Semikhatov Tipunin (2005), Farsad Gainutdinov Runkel (2017)

The Ext-ring and one important module are

$$\mathrm{Ext}^\bullet(1,1) = \begin{cases} \mathbb{C}^{n+1}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$\mathrm{Ext}^\bullet(1, X^-_{p-1}) = \begin{cases} 0, & n \text{ even} \\ \mathbb{C}^{n+1}, & n \text{ odd} \end{cases}$$

which are simple $\mathfrak{sl}_2$-representations under a categorical action, and the cup product is the respective leading direct summand in

$$\mathbb{C}^{n+1} \otimes \mathbb{C}^{m+1} = \mathbb{C}^{(n+m)+1} \oplus \dots \mathbb{C}^{|n-m|+1}$$

Towards Derived Topological Field Theories: Outlook

Ongoing work in computing $\mathcal{Z}^\bullet(\Sigma_{g,n})$ for quantum groups.

For $\tilde{u}_q(\mathfrak{sl}_2)$ the following picture holds resp. should hold:

$$L = (\mathbb{C}^{2'} \otimes \mathbb{C}^{p-1})\ 1 \oplus (\mathbb{C}^{2''} \otimes \mathbb{C}^{p-1})\ X^-_{p-1} \oplus \text{projectives} \oplus \text{other blocks}$$

with commuting actions of $\mathfrak{sl}_2$ and $\Gamma_{1,1}$ factorizing to $\Gamma_{1,0}$, acting on $\mathbb{C}^{2'}$ the standard way and $\mathbb{C}^{p-1}$ as in the minimal model $(\hat{\mathfrak{sl}}_2)_{p-2}$.

$$\mathrm{Hom}(1, L) = \qquad \mathbb{C}^{2'} \otimes \mathbb{C}^{p-1} \oplus \mathbb{C}^{p+1}$$

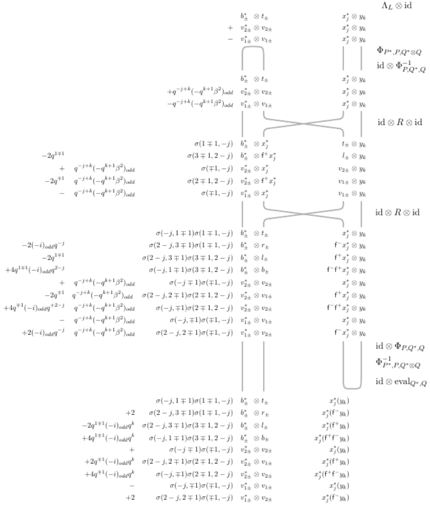
$$\mathrm{Ext}^\bullet(1, L) = \begin{cases} \mathbb{C}^{n+1} \otimes \mathbb{C}^{2'} \otimes \mathbb{C}^{p-1}, & n \text{ even} \\ \mathbb{C}^{n+1} \otimes \mathbb{C}^{2''} \otimes \mathbb{C}^{p-1}, & n \text{ odd} \end{cases}$$

Hence again $\mathrm{Ext}^\bullet(1, L)$ should be generated as $\mathrm{Ext}^\bullet(1, 1)$-module, from Lyubaschenko's degree zero part and a degree one part

$$\mathrm{Hom}(X^-_{p-1}, L) = \qquad \mathbb{C}^{2''} \otimes \mathbb{C}^{p-1}$$

Towards Derived Topological Field Theories: Outlook

Explicit calculation for $p = 2$ suggest:
$\mathbb{C}^{2'}$ has trivial $\mathfrak{sl}_2$ action and standard projective $\mathrm{SL}_2(\mathbb{Z})$-action,
$\mathbb{C}^{2''}$ has standard $\mathfrak{sl}_2$ action and trivial projective $\mathrm{SL}_2(\mathbb{Z})$-action.



Outlook Question

What do these $\mathrm{Ext}^\bullet_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, L^{\otimes g})$ **mean (analytically)** if the modular tensor category $\mathcal{C}$ arises as category $\mathrm{Rep}(\mathcal{V})$ of (suitable) representations of a (suitable) vertex operator algebra $\mathcal{V}$?

And can we **construct elements** in them from $\mathrm{Rep}(\mathcal{V})$-characters?

Recall, very roughly:

- A vertex operator algebra $\mathcal{V}$ is a graded vector space with an action of Virasoro algebra and a "multiplication" map

$$\mathrm{Y} : \mathcal{V} \otimes_{\mathbb{C}} \mathcal{V} \to \mathcal{V}[[z, z^{-1}]]$$

- If $\mathcal{V}$ is C_2-cofinite [SM][HLZ] construct a tensor product of $\mathcal{V}$-modules by the universal property of admitting intertwiner

$$X \otimes_{\mathbb{C}} Y \to (X \boxtimes Y)\{z\}[\log(z)]$$

  and a braiding by continuing this multivalued analytic function with regular singularity $z = 0$ from z counterclockwise to $-z$.

Example: Heisenberg algebra, Lattice algebra, Triplet algebra $\mathcal{W}_p$.

Conformal blocks / Chiral correlators

Elements in $\mathrm{Hom}_{\mathcal{C}}(X_1 \boxtimes \cdots \boxtimes X_n, L^g)$ are functions on the space of complex structures on $\Sigma_{g,n}$ depending on elements $x_k \in X_k$. Lyubaschenko's action of $\Gamma_{g,n}$ (should) match the geometric action.

Examples:

- $\Sigma_{0,n}$ returns matrix elements of composed vertex operators $\mathrm{Y}(x_1, z_1) \cdots \mathrm{Y}(x_n, z_n)$, transforming under the braid group.
- $\Sigma_{1,0} = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ returns functions in $q = e^{2\pi \mathrm{i}\tau}$. They piece together to a vectorvalued modular form under $\Gamma_{1,0} = \mathrm{SL}_2(\mathbb{Z})$.

  Spanned (for semisimple $\mathcal{C}$) by graded characters of $\mathcal{V}$-irreps.

We surely can consider a projective resolution in $\mathrm{Rep}(V)$, but what about the additional insertion? (homotopy?) What about traces?

Question (maybe known to some experts?)

Is $\mathrm{Ext}^{\bullet}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n, L^{\otimes g})$ for $\mathcal{C} = \mathrm{Rep}(\mathcal{V})$ dual to chiral homology in [Beilinson-Drinfeld Chp. 4] associated to the chiral algebra of $\mathcal{V}$?

....some sketches on the chiral homology of Virasoro algebra in respect to the previous discussion, as well as the first chiral homologies in general following [vEH].