From vertex operator algebras to tensor products.

(with bits and pieces from joint work with Robert Allen, Simon Lentner, Christoph Schweigert and David Ridout).

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Overview

1) VOA basics

2) Why tensor products are natural for VOAs

3) Nice cases

4) Less nice (= more interesting) cases

1) VOA basics

Def ($\mathbb{Z}$-graded) Vertex algebra (no operator)

Data:
- $V = \bigoplus_{n\in\mathbb{Z}} V[n]$ vector space (over $\mathbb{C}$)
- $\Omega \in V[0]$, vacuum vector
- $T : V[n] \to V[n+1]$, linear map
- $Y : V \otimes V \to V[[z, z^{-1}]]$, linear map

  $A \otimes B \mapsto Y(A,z)B = \sum_{n\in\mathbb{Z}} A_n B\, z^{-n-h}$

  For $A \in V[h]$, $A_n : V[m] \to V[m-n]$, $h, n, m \in \mathbb{Z}$.

Axiom:
- State-field correspondence.

  $Y(\Omega, z) = \mathrm{id}_V$ $\quad (Y(\Omega,z)A = A,\ \forall A \in V)$

  $Y(A,z)\Omega = A + O(z)$
- Translation

  $T\Omega = 0$

  $[T, Y(A,z)] = \partial_z Y(A,z)$
- locality, $\forall A, B \in V$, $\exists n \in \mathbb{N}$ s.t

  $(z-w)^n [Y(A,z), Y(B,w)] = 0$ in $\mathrm{End}(V)[[z^{\pm 1}, w^{\pm 1}]]$

  "δ-function"

Def Vertex operator algebra

Vertex algebra (V, Ω, T, Y) with

$\omega \in V[2]$ s.t

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad L_{-1} = T, \quad [L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} \frac{c}{12} m(m^2-1)$$

Virasoro algebra

$c \in \mathbb{C}$ central charge.

$L_0|_{V[n]} = n \cdot \mathrm{id}$

Consequences: $\forall A, B, C \in V$

commutative

$$Y(A,z)\,Y(B,w)\,C$$
$$\| $$
$$Y(B,w)\,Y(A,z)\,C$$
$$\|$$
$$Y(Y(A,z-w)B,w)\,C$$

Expansions of same element in $V[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$

$A(BC)$ $\quad (AB)C$ "associative"

Operator product expansion = OPE

This looks like a commutative associative algebra.

Leads to commutation relations via contour integrals

$$[A_m, B_n] = \oint_0 \oint_w Y(Y(A,z-w)B,w)\, z^{m+h_A-1} w^{n+h_B-1}\, dz\,dw$$

Looks like a Lie algebra

Three facets of a VOA

fields	modes	vector space
$Y(A,z)Y(B,w)$ $= Y(B,w)Y(A,z)$ $= Y(Y(A,z-w)B,w)$	$[A_m, B_n]$ $= \oint_0 \oint_w Y(Y(A,z-w)B,w)\, z^{m+h_A-1} w^{n+h_B-1}\, dz\, dw$	$Y(A,z)B$ or $A_n B$
commutative ring	Lie algebra / universal enveloping algebra	module $\downarrow$ many more modules beyond V

Example: The Heisenberg algebra (free boson)

Lie algebra: $\hat{\mathfrak{h}} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C} a_n \oplus \mathbb{C}\mathbb{1}$, $[a_m, a_n] = m\,\delta_{m,-n}\mathbb{1}$, $[\mathbb{1}, -] = 0$

Module (Fock space): $F_\lambda = \mathbb{C}[a_m : m \leq -1]|\lambda\rangle$, $\lambda \in \mathbb{C}$.

a_m, $m \leq -1$ free

$a_0 |\lambda\rangle = \lambda |\lambda\rangle$

$a_m |\lambda\rangle = 0$, $m \geq 1$.

VOA: $V = F_0$, $\Omega = |0\rangle$

$$Y(a_{-1}\Omega, z) = a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad Y(a_{-n}\Omega, z) = \frac{\partial^{n-1}}{(n-1)!} a(z), \quad n \in \mathbb{N}.$$

$$Y(a_{-n_1} \cdots a_{-n_k}\Omega, z) = :\frac{\partial^{(n_1-1)}}{(n_1-1)!} a(z) \cdots \frac{\partial^{n_k-1}}{(n_k-1)!} a(z):, \quad k, n_1, \ldots, n_k \in \mathbb{N},$$

$\omega = \frac{1}{2} a_{-1}^2 \Omega \rightarrow c = 1$

2) Why tensor products are natural for VOAs

Correlation functions and tensor products

In quantum field theory a correlation function is probability amplitude encoding the probability of a quantum system being in a certain state at a given point of space time.

$$\langle Y(v, z)\, Y(m_1, x_1)\, Y(m_2, x_2) \cdots Y(m_k, x_k) \rangle$$

Desired property from CFT: Keep locality and OPE

$$= \langle Y(Y(v, z-x_1)m_1, x_1)\, Y(m_2, x_2) \cdots Y(m_k, x_k) \rangle$$

$$= \langle Y(m_1, x_1)\, Y(Y(v, z-x_2)m_2, x_2) \cdots Y(m_k, x_k) \rangle$$

$$\vdots$$

$$= \langle Y(m_1, x_1) \cdots Y(m_{k-1}, x_{k-1})\, Y(Y(v, z-x_k)m_k, x_k) \rangle$$

$z, x_i \in$ Space time (usually $\mathbb{P}$)

$v \in V$, $m_i \in M_i$ module

Recall: Tensor products over a ring/field R

Let V, W be R-modules.

The <u>tensor product</u> of V, W is a pair $(V \otimes_R W, \kappa)$, where

$V \otimes_R W$, R-module, $\kappa : V \times W \to V \otimes_R W$ bilinear, satisfying:

For every bilinear $\alpha : V \times W \to U$, $\exists!\, \varphi_\alpha : V \otimes_R W \to U$ such that

$$\begin{tikzcd} V \times W \arrow[r, "\kappa"] \arrow[dr, "\alpha"'] & V \otimes_R W \arrow[d, dashed, "\varphi_\alpha"] \\ & U \end{tikzcd}$$

$\alpha = \varphi_\alpha \circ \kappa$.

Note: This universal property does not "give" you $V \otimes_R W$ or κ, but it can be constructed via set theory: $V \otimes_R W = \dfrac{\mathrm{span}\{V \times W\}}{\text{relations}}$.

Intertwining operators and VOA tensor products

Def Intertwining operator of type $\binom{R}{M,\ N}$, M, N, R modules

$$\mathcal{Y} : M \otimes N \longrightarrow R\{x\}[\log x]$$

$\otimes$ tensor ← power series with arbitrary exponents

$$m \otimes n \longmapsto \mathcal{Y}(m,x)n$$

$\partial \log x = \frac{1}{x}$

+ more

such that

$\forall v \in V,\ \forall m \in M,\ \forall n \in N$

hiding regularisation trickery

$$Y(v,z)\,\mathcal{Y}(m,x)n \approx \mathcal{Y}(Y(v,z-x)m,x)n \approx \mathcal{Y}(m,x)Y(v,z)n.$$

Interpretation: $|z|>|x|$ $\qquad |x|>|z-x| \qquad |x|>|z|$

Expand as power series and compare coefficients to get "coproduct" for vn.

VOA-ring facet: bilinear map

VOA-universal enveloping algebra facet: coproduct

Example: The Heisenberg algebra (again)

Auxilliary operator $e^{\lambda\hat{a}} : F_\mu \to F_{\mu+\lambda}$, $\lambda, \mu \in \mathbb{C}$,

characterised by $e^{\lambda\hat{a}}|\mu\rangle = |\mu+\lambda\rangle$

$$[a_n, e^{\lambda\hat{a}}] = \lambda\,\delta_{n,0}, \quad n \in \mathbb{Z}.$$

Prop $\mathcal{Y} : F_\lambda \otimes F_\mu \to F_{\lambda+\mu}[[x^{\pm 1}]]\,x^{\lambda\mu}$, $\lambda, \mu \in \mathbb{C}$, $p \in \mathbb{C}[a_{-n} : n \geq 1]$

$$\mathcal{Y}(p|\lambda\rangle, x)\,q|\mu\rangle = e^{\lambda\hat{a}} x^{\lambda\mu} \prod_{m\geq 1} \exp\left(\lambda \frac{a_{-m}}{m} x^m\right) Y(p,x) \prod_{m\geq 1} \exp\left(-\lambda \frac{a_m}{m} x^{-m}\right) q|\mu\rangle$$

In fact: Here $\mathcal{Y}$ is unique up to scaling and

$$\dim \binom{F_\nu}{F_\lambda, F_\mu} = \delta_{\nu,\lambda+\mu}$$

Tensor categories from VOAs [Huang-Lepowsky-Zhang]

For a "sufficiently nice" VOA module category $\mathcal{C}$ we get a braided tensor category.

Tensor product: Let $M, N \in \mathcal{C}$, then $(M \boxtimes N, \mathcal{Y}_{M,N})$ are a tensor product, if $\forall \alpha \in \binom{X}{M,N}$, $\exists! \, \varphi_\alpha$ such that

suppressed variables

$$\begin{tikzcd} M \otimes N \arrow[r, "\mathcal{Y}"] \arrow[dr, "\alpha"'] & M \boxtimes N \arrow[d, dashed, "\varphi_\alpha"] \\ & X \end{tikzcd} \qquad \alpha = \varphi_\alpha \circ \mathcal{Y}$$

Complications:
- $\mathcal{Y} \notin \mathrm{Hom}_{\mathcal{C}}$ (as with usual $\otimes$)
- $\mathrm{im}(\mathcal{Y}) \notin \mathrm{Obj}(\mathcal{C})$
- Set theoretic construction of $M \boxtimes N$ horrendous
- $\mathcal{Y}$ hard to find (e.g. solution of KZ-eq).

Tensor structures

Consider $M \otimes N \longrightarrow N \boxtimes M\{z\}[\log z]$ Intertwining operator of

$m \otimes n \longmapsto e^{zL_{-1}} \mathcal{Y}_{N,M}(n,-z)m.$ type $\binom{N \boxtimes M}{M\ ,\ N}$.

$\mathcal{Y}_{N,M} \in \binom{N \boxtimes N}{N, M}$

Units: V unit object

$\ell_M : V \boxtimes M \to M$

$\ell_M(\mathcal{Y}_{V,M}(v,z)m) = \underbrace{Y(v,z)m}_{\text{module action of } V \text{ on } M}$

$r_M : M \boxtimes V \to M$

$r_M(\mathcal{Y}_{M,V}(m,z)v) = e^{zL_{-1}} Y(v,-z)m$

Braiding: $M, N \in \mathcal{C}$, $(M \boxtimes N, \mathcal{Y}_{M\boxtimes N})$, $(N \boxtimes M, \mathcal{Y}_{N \boxtimes M})$ tensor products.

Let $c_{M,N} : M \boxtimes N \to N \boxtimes M$ be map characterized by

$$c_{M,N}(\mathcal{Y}_{M,N}(m,z)n) = e^{zL_{-1}} \mathcal{Y}_{N,M}(n,-z)m$$

Tensor structures continued

Associativity M, N, R modules,

tensor products $\left(M\boxtimes(N\boxtimes R),\ \mathcal{Y}_{M,(N,R)}(-,x_1)\,\mathcal{Y}_{N,R}(-,x_2)-\right)$

$\left((M\boxtimes N)\boxtimes R,\ \mathcal{Y}_{(M,N),R}\left(\mathcal{Y}_{M,N}(-,x_1-x_2)-\right)-\right)$

$$A_{M,N,R} : M\boxtimes(N\boxtimes R) \longrightarrow (M\boxtimes N)\boxtimes R$$

characterised by $\forall m\in M, \forall n\in N, \forall r\in R$

$$A_{M,N,R}\left(\mathcal{Y}_{M,(N,R)}(m,x_1)\,\mathcal{Y}_{N,R}(n,x_2)\,r\right) \approx \mathcal{Y}_{(M,N),R}\left(\mathcal{Y}_{M,N}(m,x_1-x_2)n, x_2\right)r$$

$|x_1| > |x_2|$ $\qquad$ $|x_2| > |x_1 - x_2|$

Many convergence details suppressed!

3) Nice cases

i.e rational

Rational VOAs

Thm [Moore, Seiberg, Verlinde, Huang, ...]

Let V be a VOA satisfying:
- C_2-cofinite
- $\mathbb{N}$-gradable modules are semisimple
- $V^* \cong V$
- $\dim V[0] = 1$, $\dim V[n] < \infty$, $\dim V[-n] = 0$, $n \geq 1$.

The category of $\mathbb{N}$-gradable modules is a modular tensor category!

Examples:
- Virasoro minimal models
- Affine VOAs at level $k \in \mathbb{Z}_{\geq 0}$
- Lattice VOAs on even lattices
- Heisenberg VOA is <u>not</u> an example.

The Verlinde formula (or why we like MTCs)

The MTC S-matrix is proportional to the charcter S-matrix and the S-matrix determines the tensor product!

Let $\{M_i\}_{i\in I}$ be simple iso classes, $|I|<\infty$, $M_0 = V$ unit.

If $M_i \boxtimes M_j \cong \bigoplus_{k\in I} N_{i,j}^k M_k$, $N_{i,j}^k \in \mathbb{Z}_{\geq 0}$

Then

$$N_{i,j}^k = \sum_{\ell\in I} \frac{S_{i,\ell} S_{j,\ell} \overline{S_{k,\ell}}}{S_{0,\ell}}.$$

Character S-matrix: $\chi_M(\tau) = Tr_M\left(e^{2\pi i\tau(L_0 - c/24)}\right)$ $\tau \in \mathbb{H}$

$$\chi_{M_i}(-1/\tau) = \sum_{j\in I} S_{i,j}\, \chi_{M_j}(\tau)$$

Easy to compute.

4) Less nice

(= more interesting) cases

Observations of nice patterns [Ridout, Creutzig, Milas, SW, ...]

For certain VOAs and module categories we have:

0) There is a distinguished class of module called <u>standard</u>.

1) Standard module iso classes and characters are parameterised by a measurable space (M, μ)

2) Simple standard modules are called <u>typical</u>, the rest <u>atypical</u>. Atypicals form a set of measure 0 with respect to μ.

3) Typical modules are injective and projective in their module category.

4) Standard module characters form a (topological) basis for the space of all characters ($\simeq$ Grothendieck group)

5) Standard characters span a representation of $SL(2, \mathbb{Z})$. In the basis of standard characters, S-matrix is symmetric, unitary and squares to conjugation

$$S_{i,j} = S_{j,i}, \quad (S^{-1})_{i,j} = \overline{S_{j,i}}, \quad (S \circ S)_{i,j} = \delta_{j,i^*}$$

6) The naive generalisation of the Verlinde formula gives non-negative "tensor" multiplicities on the Grothendieck group.

If $\chi[R_i \boxtimes R_j] = \int_M N_{i,j}^{k} \chi[R_k]\, d\mu(k) \to$ evals to a finite sum of chars.

standard modules

$$N_{i,j}^{k} = \int_M \frac{S_{i,\ell}\, S_{j,\ell}\, \overline{S_{k,\ell}}}{S_{vac,\ell}}\, d\mu(\ell)$$

In all cases to be listed below

$N_{i,j}^{k}$ is $\mathbb{Z}_{\geq 0}$-linear combination of finitely many δ-functions on M.

Tensor products, where they have been computed, match this formula.

Examples

- Any VOA & module category statisfying Huang's rationality conditions [Huang]
- Heisenberg VOA [Ridout, SW, Tuite, Zuevsky + lots of physics history]

$\mathcal{C} = \left\{ \begin{array}{l} \text{finitely generated modules on which } a_n, n \geq 1 \text{ are locally nilpotent} \\ \text{and } a_0 \text{ is semisimple, } a_0\text{-eigenvalues} \in \mathbb{R}. \end{array} \right\}$

$= \{\text{finite sums of Fock spaces}\}$

Standard modules: Fock spaces. All simple, hence typicall.

- Affine sl_2 at level $k = \frac{u}{v} - 2$, $\gcd(u,v) = 1$, $u,v \geq 2$.

[Ridout, Creutzig]

$\mathcal{C}$ defined in stages

$\mathcal{R} = \left\{ \begin{array}{l} \text{finitely generated modules on which positive modes locally nilpotent} \\ \text{and finite Cartan Subalgebra is semisimple + reality conds.} \end{array} \right\}$

$= \{\text{modules vizible to Zhu's algebra}\}$

$\mathcal{C} = \langle \sigma \mathcal{R}, \sigma \in Q^* \rangle$

coweight lattice

spectral flow twists

Tensor product formulae still conjectural.

Examples continued

- β-γ ghosts [Allen, Ridout, SW]

  $\mathcal{C}$ defined analogously to sl_2

- Rank $n \in \mathbb{N}$ free boson on a lattice of rank $k \leq n$. [Allen, Lentner, Schweigert, SW, In preparation]

- $W_{1,p}$ singlet, $p \geq 2$ [Creutzig, Milas]

  tensor product formulae conjectured

- $W_{p,q}$ singlet, $p,q \geq 2$, $\gcd(p,q) = 1$ [Creutzig, Milas, Ridout, SW]

  tensor product formulae conjectured

- Affine sl_3 at level $k = -\frac{3}{2}$ [Kawasetsu, Ridout, SW, in prep]

  tensor product formulae conjectured

Fin!

Thanks for tuning in.