CANONICAL COLOURINGS IN RANDOM GRAPHS

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Abstract. Rödl and Ruciński [*Threshold functions for Ramsey properties*, J. Amer. Math. Soc. **8** (1995)] established Ramsey's theorem for random graphs. In particular, for fixed integers r and $\ell \geqslant 2$ they showed that $\hat{p}_{K_\ell,r}(n) = n^{-\frac{2}{\ell+1}}$ is a threshold for the Ramsey property that every r-colouring of the edges of the binomial random graph $G(n,p)$ yields a monochromatic copy of K_ℓ. We investigate how this result extends to arbitrary colourings of $G(n,p)$ with an unbounded number of colours. In this situation, Erdős and Rado [*A combinatorial theorem*, J. London Math. Soc. **25** (1950)] showed that *canonically coloured* copies of K_ℓ can be ensured in the deterministic setting. We transfer the Erdős–Rado theorem to the random environment and show that both thresholds coincide when $\ell \geqslant 4$. As a consequence, the proof yields $K_{\ell+1}$-free graphs G for which every edge colouring contains a canonically coloured K_ℓ.

The 0-statement of the threshold is a direct consequence of the corresponding statement of the Rödl–Ruciński theorem and the main contribution is the 1-statement. The proof of the 1-statement employs the transference principle of Conlon and Gowers [*Combinatorial theorems in sparse random sets*, Ann. of Math. (2) **184** (2016)].

§1. Introduction

In the last three decades, extremal and Ramsey-type properties of random graphs were considered, which led to several general approaches (see, e.g. [2,3,11,13,22–24] and the references therein). We consider Ramsey-type questions for the binomial random graph $G(n,p)$. For graphs G and H and an integer $r \geqslant 2$ we write

$$G \longrightarrow (H)_r$$

to signify the statement that every r-colouring of the edges of G yields a monochromatic copy of H. Ramsey's theorem [18] tells us that for fixed H and r the family of graphs G with $G \longrightarrow (H)_r$ is non-empty. Obviously, this family is monotone and, hence, there is a threshold function $\hat{p}_{H,r} \colon \mathbb{N} \longrightarrow [0,1]$ such that

$$\lim_{n\longrightarrow\infty} \mathbb{P}\big(G(n,p) \longrightarrow (H)_r\big) = \begin{cases} 0\,, & \text{if } p \ll \hat{p}_{H,r}, \\ 1\,, & \text{if } p \gg \hat{p}_{H,r}. \end{cases} \tag{1.1}$$

As usual we shall refer to any such function as *the* threshold of that property, even though it is not unique.

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Rödl and Ruciński [21, 22] determined the threshold $\hat{p}_{H,r}$ for every graph H and every fixed number of colours r. We restrict ourselves to the situation when H is a clique K_ℓ and state their result for that case only.

Theorem 1.1 (Rödl & Ruciński)**.** *For every $r \geqslant 2$ and $\ell \geqslant 3$ we have $\hat{p}_{K_\ell,r}(n) = n^{-\frac{2}{\ell+1}}$.* □

In fact, Rödl and Ruciński established a *semi-sharp* threshold, i.e., the 0-statement in (1.1) holds as long as $p(n) \leqslant c_{\ell,r} n^{-\frac{2}{\ell+1}}$ for some sufficiently small constant $c_{\ell,r} > 0$ and, similarly, the 1-statement becomes true already if $p(n) \geqslant C_{\ell,r} n^{-\frac{2}{\ell+1}}$ for some $C_{\ell,r}$. This was sharpened recently in [8], where the gap between $c_{\ell,r}$ and $C_{\ell,r}$ was closed. Perhaps surprisingly, the asymptotic growth of the threshold function $\hat{p}_{K_\ell,r}(n)$ in Theorem 1.1 is independent of the number of colours r.

We are interested in arbitrary edge colourings of $G(n,p)$, i.e., colourings which are not restricted to a fixed number of colours. However, if the number of colours is unrestricted, then this allows injective edge colourings and, consequently, monochromatic (nontrivial) subgraphs might be prevented. Erdős and Rado [6], however, showed that certain *canonical* patterns are unavoidable in edge colourings of sufficiently large cliques. Obviously, the monochromatic and the injective pattern must be canonical. Another type of canonical patterns arises by ordering the vertices of K_n and colouring every edge uv by $\min\{u,v\}$ or colouring every edge by its maximal vertex. More generally, for finite graphs G and H with ordered vertex sets, we write

$$G \xrightarrow{\ *\ } (H)$$

if for every edge colouring $\varphi \colon E(G) \longrightarrow \mathbb{N}$ there exists an order-preserving graph embedding $\varsigma \colon H \longrightarrow G$ such that one of the following holds:

(*a*) the copy $\varsigma(H)$ of H is monochromatic under φ,
(*b*) or φ restricted to $E(\varsigma(H))$ is injective,
(*c*) or for all edges e, $e' \in E(\varsigma(H))$ we have $\varphi(e) = \varphi(e') \iff \min(e) = \min(e')$,
(*d*) or for all edges e, $e' \in E(\varsigma(H))$ we have $\varphi(e) = \varphi(e') \iff \max(e) = \max(e')$.

We call an ordered copy of H in G *canonical* if it displays one of the four patterns described in (*a*)–(*d*).

Note that for the patterns described in (*a*) and (*b*) the orderings of the vertex sets have no bearing. Moreover, we shall refer to copies enjoying an injective colouring as *rainbow* copies of H (even if $|E(H)| \neq 7$). Similarly, we refer to the patterns appearing in (*c*) and (*d*) as *min-coloured* and *max-coloured*, respectively. In case only the backward implications in (*c*) or (*d*) are enforced, then we refer to those colourings as *non-strict*, e.g., if $\min(e) = \min(e')$ yields $\varphi(e) = \varphi(e')$ for all edges e, $e' \in E(\varsigma(H))$, then $\varsigma(H)$ is a non-strictly min-coloured copy of H. Obviously, monochromatic copies are also non-strictly min- and max-coloured.

From now on, the vertex sets of all graphs considered are ordered. In particular, for cliques and random graphs we simply assume

$$V(K_n) = [n] \qquad \text{and} \qquad V(G(n,p)) = [n]\,.$$

With this notation at hand, the aforementioned canonical Ramsey theorem of Erdős and Rado [6] restricted to the graph case asserts that canonical copies are unavoidable.

Theorem 1.2 (Erdős & Rado)**.** *For every $\ell \geqslant 3$, there exists n such that $K_n \xrightarrow{\ *\ } (K_\ell)$.* □

We are interested in a common generalisation of Theorems 1.1 and 1.2. Owing to Theorem 1.2, for any ordered graph H, the monotone family $\{G\colon G \xrightarrow{\ *\ } (H)\}$ is non-empty and it raises the problem of estimating the threshold $\hat{p}_H\colon \mathbb{N} \longrightarrow [0,1]$ such that

$$\lim_{n\longrightarrow\infty} \mathbb{P}\big(G(n,p) \xrightarrow{\ *\ } (H)\big) = \begin{cases} 0\,, & \text{if } p \ll \hat{p}_H\,, \\ 1\,, & \text{if } p \gg \hat{p}_H\,. \end{cases} \tag{1.2}$$

It follows from the definition, that for every graph H not admitting a vertex cover of size at most two, the only canonical colourings of H using at most two colours are monochromatic. Consequently,

$$\hat{p}_H \geqslant \hat{p}_{H,2}$$

for every such graph H. In particular, for cliques on at least four vertices this may suggest that the asymptotics of the thresholds for those Ramsey properties coincide and our main result verifies this.

Theorem 1.3. *For every $\ell \geqslant 4$, there exists $C > 0$ such that for $p = p(n) \geqslant Cn^{-\frac{2}{\ell+1}}$ we have*

$$\lim_{n\longrightarrow\infty} \mathbb{P}\big(G(n,p) \xrightarrow{\ *\ } (K_\ell)\big) = 1\,.$$

Combining Theorem 1.3 with the corresponding lower bound on $\hat{p}_{K_\ell,2}$ shows that the threshold for the canonical Ramsey property is semi-sharp for $\ell \geqslant 4$. For $\ell = 3$ we recall that the canonical Ramsey threshold is indeed smaller than the Ramsey threshold $n^{-1/2}$. In fact, one can check that every edge colouring of K_4 yields a canonical copy of the triangle and, hence, $\hat{p}_{K_3} \leqslant n^{-2/3}$.

Moreover, we note that for $p = O(n^{-\frac{2}{\ell+1}})$ the random graph $G(n,p)$ is likely to contain only $o(pn^2)$ cliques $K_{\ell+1}$. In the proof of Theorem 1.3 we can delete an edge from each such clique. Consequently, we obtain the following statement in structural Ramsey theory, which can be viewed as a Folkman-type extension of the Erdős–Rado theorem for graphs.

Corollary 1.4. *For every $\ell \geqslant 4$ there exists a $K_{\ell+1}$-free graph G such that $G \xrightarrow{\ *\ } (K_\ell)$. Moreover, G contains no two distinct copies of K_ℓ that share at least three vertices.*

In the context of Ramsey's theorem, the existence of such a graph G was asked for by Erdős and Hajnal [5]; for two colours this was established by Folkman [7], and for any fixed number of colours by Nešetřil and Rödl [16]. The graph G in Corollary 1.4 will be obtained by modifying the random graph and, hence, the proof is non-constructive. Reiher and Rödl [19] pointed out that the first part of Corollary 1.4 can also be proved in a constructive manner by means of the *partite construction method* of Nešetřil and Rödl [17]. While this approach falls short to exclude K_ℓ's intersecting in triangles, it has the advantage that it readily extends to k-uniform hypergraphs for every $k \geqslant 3$.

We conclude this introduction with a short overview of the main ideas of the proof of Theorem 1.3. Roughly speaking, the proof is inspired by the proof of the canonical graph Ramsey theorem laid out by Lefmann and Rödl [12] and Alon et al. [1]. This approach pivots on a case distinction of the edge colouring of the underlying graph K_n. The first case, when many different colours appear everywhere, which is captured by assuming that every vertex is incident to only $o(n)$ edges of the same colour, leads to rainbow copies of K_ℓ. In the other case, there is a vertex with a monochromatic neighbourhood of size $\Omega(n)$, which by iterated applications, as in the standard proof of Ramsey's theorem, leads to a non-strictly min- or max-coloured $K_{(\ell-2)^2+2}$. Such a non-strictly min/max-coloured clique contains a canonical K_ℓ by a straightforward application of Dirichlet's box principle.

Transferring such an approach from K_n to $G(n,p)$ for $p = O(n^{-\frac{2}{\ell+1}})$ faces several challenges. Firstly, we shall not use a $K_{(\ell-2)^2+2}$ in the coloured host graph, as such large cliques are not very likely to appear in $G(n,p)$ for that edge probability. Moreover, in the more challenging second case, when the colouring is unbounded, it is certainly not sufficient to consider one vertex with a large monochromatic neighbourhood (of size $\Omega(pn)$), as again, this neighbourhood is too sparse to contain any useful structure in $G(n,p)$. Thus we resort to a robust version of the above-mentioned argument, building a large non-strictly min- or max-coloured subgraph which contains $\Omega(n^2p)$ edges.

The bounded case, with at most λ edges of every colour incident to any given vertex of $G(n,p)$, is a problem of independent interest. For example, $\lambda = 1$ corresponds to studying proper edge colourings of $G(n,p)$ and anti-Ramsey properties (see, e.g. [9, 10, 14] and the references therein). In fact, for $\ell \geqslant 5$, there are proper colourings of $G(n,p)$ with $p = cn^{-\frac{2}{\ell+1}}$ which do not contain a rainbow copy of K_ℓ (see [10]), which is an alternative argument for $\hat{p}_{K_\ell} \geqslant cn^{-\frac{2}{\ell+1}}$ and another obstruction for Theorem 1.3. For the proof of Theorem 1.3 presented here, we will need to guarantee rainbow copies of H under the weaker assumption that $\lambda = o(pn)$. This can be viewed as a partial extension of the work of Kohayakawa, Konstadinidis, and Mota [9]. In both cases (bounded and unbounded colourings) the transference principle for random discrete structures developed by Conlon and Gowers [3] is an integral part of the proof.

In the next section, we present the two main lemmata rendering the case distinction sketched above, and deduce Theorem 1.3. The proofs of these lemmata are deferred to the full version of the manuscript. We conclude with a discussion of possible generalisations of this work from cliques K_ℓ to general graphs H and of related open problems in Section 3.

§2. Proof of the main result

2.1. **Proof of the canonical Ramsey theorem for graphs.** The proof of Theorem 1.3 adopts some ideas of the canonical Ramsey theorem for graphs from the work of Lefmann and Rödl [12] and Alon et al. [1] and below we recall their argument. For $\ell \geqslant 3$ we fix

$$\delta = \frac{1}{4\ell^3} \quad \text{and} \quad n \geqslant 2^{6\ell^2(\log_2(\ell)+1)} \tag{2.1}$$

and first we consider bounded colourings $\varphi\colon E(K_n) \longrightarrow \mathbb{N}$. We say such a colouring is *δ-bounded* if for every colour $c \in \mathbb{N}$ and every vertex $v \in V(K_n)$ we have

$$d_c(v) = \big|N_c(v)\big| = \big|\{w \in V(K_n)\colon \varphi(vw) = c\}\big| \leqslant \delta n\,.$$

Roughly speaking, bounded colourings have the property that many different colours are "present everywhere" and this yields rainbow copies of K_ℓ. In fact, a simple counting argument shows for δ-bounded colourings that at most $\delta n^3/2$ triples contain two edges of the same colour and at most $\delta n^4/8$ quadruples contain two disjoint edges of the same colour. Consequently, selecting every vertex of K_n independently with probability $2\ell/n$ and removing a vertex from every such triple and every such quadruple, establishes the existence of ℓ vertices inducing a rainbow K_ℓ.

The second part of the proof resembles the standard proof of Ramsey's theorem for graphs and iterates along large monochromatic neighbourhoods. Given the observation above for bounded colourings, we may assume that the edge colouring φ is unbounded in a hereditary way and this requires the exponential lower bound on n above.

More precisely, assuming that φ fails to induce a rainbow copy of K_ℓ gives rise to a vertex $v \in V(K_n)$, a colour c, and a comparability sign $\diamond \in \{<, >\}$ such that

$$d_c^\diamond(v) = \big|N_c^\diamond(v)\big| = \big|\{w \in V(K_n)\colon \varphi(vw) = c \text{ and } v \diamond w\}\big| > \frac{\delta n}{2}\,.$$

Restricting our attention to the colouring φ on the edges contained in $N_c^\diamond(v)$ and iterating this argument $L = 2(\ell-2)^2 + 2$ times leads to a sequence $(v_i, c_i, \diamond_i)_{i\in[L]}$ such that for every $i \in [L]$ we have

$$\Big|\bigcap_{j=1}^{i} N_{c_j}^{\diamond_j}(v_j)\Big| > \Big(\frac{\delta}{2}\Big)^i n\,. \tag{2.2}$$

In fact, owing to the choices in (2.1) we can iterate this step L times.

Furthermore, we may assume that there are indices $1 \leqslant i_0 < \dots < i_{(\ell-2)^2} < L$ such that $\diamond_{i_j}$ is $<$ for all j. Consequently, the correspondingly indexed vertices $v_{i_0}, \dots, v_{i_{(\ell-2)^2}}$ together with v_L induce a non-strictly min-coloured clique on $(\ell-2)^2 + 2$ vertices. Finally, if one of the colours appears $\ell - 1$ times among $c_{i_0}, \dots, c_{i_{(\ell-2)^2}}$, then this yields a monochromatic K_ℓ among $v_{i_0}, \dots, v_{i_{(\ell-2)^2}}$, and v_L. Otherwise, at least $\ell - 1$ distinct colours appear and we are guaranteed to find a min-coloured K_ℓ instead.

2.2. **Bounded and unbounded colourings in random graphs.** For the proof of Theorem 1.3 we derive appropriate random versions of the facts above that analyse bounded and unbounded colourings of $G(n,p)$ (see Lemmata 2.1 and 2.2 below). We begin by defining a notion of boundedness central to our proof. Roughly speaking, an edge colouring of $G(n,p)$ is bounded if at most $o(pn)$ edges of the same colour are incident to any given vertex. However, similar to the proof in the deterministic setting, it will be useful to define this property for large subsets of vertices, which is made precise as follows.

Given a graph $G = (V,E)$ with an edge colouring $\varphi\colon E \longrightarrow \mathbb{N}$, a subset $U \subseteq V$, and reals $\delta > 0$, $p \in (0,1]$ we say φ is *(δ,p)-bounded on U* if for every colour $c \in \mathbb{N}$ and every vertex $u \in U$ we have

$$d_c(u,U) = \big|N_c(u,U)\big| = \big|\{w \in U\colon \varphi(uw) = c\}\big| \leqslant \delta p |U|\,.$$

The first lemma asserts that bounded edge colourings of $G(n,p)$ for $p \gg n^{-\frac{2}{\ell+1}}$ yield rainbow copies of K_ℓ asymptotically almost surely, i.e., with probability tending to 1 as $n \longrightarrow \infty$.

In view of Corollary 1.4, we define the *ℓ-clean* subgraph G_ℓ of a given graph G on $[n]$ as follows: Consider all edges of G in lexicographic order and remove an edge e in the current subgraph of G, if the edge e is contained in two distinct K_ℓ's intersecting in at least three vertices. Actually, the precise definition is not relevant for our argument, but it will be convenient that this way the ℓ-clean subgraph $G_\ell \subseteq G$ is unique. Note that G_ℓ contains no copy of $K_{\ell+1}$, since this would yield two K_ℓ's intersecting in $\ell-1$ vertices.

Lemma 2.1. *For all integers $\ell \geqslant 4$ and every $\nu > 0$ there is some constant $C > 0$ such that for $p = p(n) \geqslant Cn^{-\frac{2}{\ell+1}}$ asymptotically almost surely the following holds for $G \in G(n,p)$.*

If $\varphi\colon E(G) \longrightarrow \mathbb{N}$ is $(\ell^{-5}/4, p)$-bounded on some $U \subseteq V(G)$ with $|U| \geqslant \nu n$, then U induces a rainbow copy of K_ℓ in G.

Moreover, if in addition we have $p(n) \leqslant n^{-\frac{2\ell-2}{\ell^2+\ell-4}}/\omega(n)$ for some arbitrary function ω tending to infinity as $n \longrightarrow \infty$, then the ℓ-clean subgraph $G_\ell \subseteq G$ also contains a rainbow copy of K_ℓ.

Lemma 2.1 strengthens a result of Kohayakawa, Konstadinidis, and Mota [9], where a more restrictive boundedness assumption is required. The proof of Lemma 2.1 makes use of the transference principle of Conlon and Gowers [3], which allows us to transfer the bounded case in the deterministic setting to the random environment. The second lemma yields canonical copies in unbounded colourings.

Lemma 2.2. *For all integers $\ell \geqslant 3$ and every $\delta > 0$ there is some constant $C > 0$ such that for $p = p(n) \geqslant Cn^{-\frac{2}{\ell+1}}$ asymptotically almost surely the following holds for $G \in G(n,p)$.*

If $\varphi\colon E(G) \longrightarrow \mathbb{N}$ has the property that every $U \subseteq V(G)$ with size $|U| \geqslant \delta^{5\ell^2} n$ satisfies

$$\big|\{u \in U\colon d_c(u,U) \geqslant 8\delta p|U| \text{ for some colour } c\}\big| \geqslant \frac{|U|}{2}\,, \tag{2.3}$$

then G contains a canonical copy of K_ℓ.

Moreover, if in addition we have $p(n) \leqslant n^{-\frac{2\ell-2}{\ell^2+\ell-4}}/\omega(n)$ for some arbitrary function ω tending to infinity as $n \longrightarrow \infty$, then the ℓ-clean subgraph $G_\ell \subseteq G$ also contains a rainbow copy of K_ℓ.

As in the unbounded case in the deterministic setting, the proof of Lemma 2.2 yields either a monochromatic, or a min-coloured, or a max-coloured copy of K_ℓ. We conclude this section with the short proof of Theorem 1.3 and Corollary 1.4 based on Lemmata 2.1 and 2.2.

Proof of Theorem 1.3 and Corollary 1.4. Given $\ell \geqslant 4$ we set $\delta = \ell^{-5}/64$ and $\nu = \delta^{5\ell^2}/2$ and let C be sufficiently large so that we can appeal to Lemma 2.1 with ℓ and ν and to Lemma 2.2 with ℓ and δ. Owing to the monotonicity of the canonical Ramsey property, for the proof of Theorem 1.3 we may assume $p = p(n) = Cn^{-\frac{2}{\ell+1}}$. Since $\ell \geqslant 4$ this implies $p(n) \leqslant n^{-\frac{2\ell-2}{\ell^2+\ell-4}}/\omega(n)$ for some function ω tending to infinity with n.

Let $G \in G(n,p)$ satisfy the conclusion of both lemmata and consider an arbitrary edge colouring $\varphi\colon E(G) \longrightarrow \mathbb{N}$ of G.

For every $U \subseteq V(G)$ we consider its subset of unbounded vertices in U

$$B(U) = \{w \in U\colon d_c(w,U) \geqslant 8\delta p|U| \text{ for some colour } c\}\,.$$

Owing to Lemma 2.2 we may assume that there is a set $U \subseteq V(G)$ satisfying $|U| \geqslant \delta^{5\ell^2} n$ and $|B(U)| < |U|/2$. Removing the unbounded vertices from U we arrive at a set

$$U' = U \smallsetminus B(U) \quad \text{with} \quad |U'| > \frac{|U|}{2} \geqslant \nu n\,.$$

For every vertex $u \in U'$ and every colour c we have

$$d_c(u, U') \leqslant d_c(u, U) < 8\delta p|U| < 16\delta p|U'|\,.$$

In other words, φ is $(16\delta, p)$-bounded on U' and Lemma 2.1 yields asymptotically almost surely a rainbow copy of K_ℓ in the ℓ-clean subgraph $G_\ell \subseteq G$. □

§3. Concluding remarks

3.1. **Thresholds for canonical Ramsey properties for general graphs.** Recall that for an ordered graph H, we defined $\hat{p}_H$ as the threshold for the property $G(n,p) \xrightarrow{*} (H)$ and Theorem 1.3 establishes $\hat{p}_{K_\ell} = n^{-\frac{2}{\ell+1}}$. The problem of determining the threshold $\hat{p}_H$ for ordered graphs H which are not complete is still open, but there are some partial results.

Firstly, Alvarado, Kohayakawa, Morris, and Mota [15] studied a closely related problem for even cycles $C_{2\ell}$. Their result implies that for $p = Cn^{-1/m_2(C_{2\ell})} \log n$, any colouring of $G(n,p)$ contains a canonical copy of the cycle $C_{2\ell}$. However, in their work the ordering of the random graph $G(n,p)$ is determined after the colouring.

Secondly, for a strictly balanced graph H, our proof guarantees for $p \gg n^{-1/m_2(H)}$ a canonical copy of H, but one cannot require a specific vertex ordering of H.

3.2. **Canonical colourings in random hypergraphs.** Furthermore, it would be interesting to investigate extensions of Theorem 1.3 to k-uniform hypergraphs for $k \geqslant 3$. Namely, in their original work Erdős and Rado [6] established a canonical Ramsey theorem for k-uniform hypergraphs. However, their proof for k-uniform hypergraphs used Ramsey's theorem for $2k$-uniform hypergraphs and this seems to be an obstacle for transferring it to random hypergraphs at the right threshold. Hence, for transferring their result to the random setting, it seems necessary to start with a proof which avoids the use of hypergraphs with larger uniformity. Such proofs can be found in [20, 25].

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