RAMSEY PROPERTIES OF RANDOMLY PERTURBED HYPERGRAPHS

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Abstract. We study Ramsey properties of randomly perturbed 3-uniform hypergraphs. For $t \geqslant 2$, write $\widetilde{K}_t^{(3)}$ to denote the 3-uniform *expanded* clique hypergraph obtained from the complete graph K_t by expanding each of the edges of the latter with a new additional vertex. For an even integer $t \geqslant 4$, let M denote the asymmetric maximal density of the pair $(\widetilde{K}_t^{(3)}, \widetilde{K}_{t/2}^{(3)})$. We prove that adding a set F of random hyperedges satisfying $|F| \gg n^{3-1/M}$ to a given n-vertex 3-uniform hypergraph H with non-vanishing edge density asymptotically almost surely results in a perturbed hypergraph enjoying the Ramsey property for $\widetilde{K}_t^{(3)}$ and two colours. We conjecture that this result is asymptotically best possible with respect to the size of F whenever $t \geqslant 6$ is even. The key tools of our proof are a new variant of the hypergraph regularity lemma accompanied with a *tuple lemma* providing appropriate control over joint link graphs. Our variant combines the so called strong and the weak hypergraph regularity lemmata.

§1 Introduction

1.1. **Ramsey properties of random hypergraphs.** Given a distribution $\mathcal{R}$ over n-vertex hypergraphs, as well as an n-vertex hypergraph H, referred to as the *seed* hypergraph, unions of the form $H \cup R$ with $R \sim \mathcal{R}$ define a distribution over the super-hypergraphs of H, denoted by $H \cup \mathcal{R}$. The hypergraphs $H \cup \mathcal{R}$ are referred to as *random perturbations* of H. The study of the properties of such hypergraph distributions has its origins in the seminal work of Spielman and Teng [52, 53] who coined the term *Smoothed Analysis* whilst investigating the performance of algorithms on randomly perturbed inputs.

Recently, the paradigm of Smoothed Analysis, originating from Theoretical Computer Science, has captured the attention of numerous researchers in Combinatorics. In the latter avenue, two dominant strands of results have emerged. One strand pertains to the study of the thresholds for the emergence of various spanning and nearly-spanning configurations within such structures (see, e.g., [3–6, 10–13, 21, 28, 34, 35, 41]). The second strand pertains to the extremal and Ramsey-type properties (see, e.g., [1, 2, 6, 8, 18–20, 36, 46]) of such hypergraphs. Our result lies in the latter vein. We recall the arrow notation $G \longrightarrow (H_1, H_2)$, signifying the validity of the asymmetric Ramsey statement that every 2-colouring of the edges of G yields a monochromatic copy of H_1 in the first colour or a monochromatic copy of H_2 in the second colour. Moreover, in the symmetric case when $H_1 = H_2 = H$ we simply write $G \longrightarrow (H)$.

Key words and phrases. Ramsey Theory, Smoothed Analysis, Random Hypergraphs.

Ramsey properties of randomly perturbed graphs were first investigated by Krivelevich, Sudakov, and Tetali [36]. In that work it was shown that $n^{-2/(t-1)}$ is the threshold for the asymmetric Ramsey property $G \cup \mathbb{G}(n,p) \longrightarrow (K_3, K_t)$, whenever G is an n-vertex graph of edge density $d \in (0, 1/2)$ independent of n. The general problem, put forth by Krivelevich et al., of determining the threshold for the property $G \cup \mathbb{G}(n,p) \longrightarrow (K_s, K_t)$, whenever G is dense and $s, t \geqslant 4$, was recently (essentially) resolved by Das and Treglown [20]. Those authors showed that $n^{-1/m_2(K_t, K_{\lceil s/2 \rceil})}$ is the threshold for the property $G \cup \mathbb{G}(n,p) \longrightarrow (K_s, K_t)$, when G is a dense n-vertex graph and $t \geqslant s \geqslant 5$, where $m_2(H_1, H_2)$ denotes the asymmetric maximal 2-density of two graphs H_1 and H_2 (see equation (1.2) for the definition). For other values of t and s we also refer to the work of Das and Treglown [20, Theorem 1.7 and Theorem 5(ii)] and for the special case $s = t = 4$ in addition to the work of Powierski [46, Theorem 1.8].

The aforementioned Ramsey-type results for randomly perturbed dense graphs are formulated for 2-colourings only. This restriction is well-justified. Indeed, suppose that more than two colours are available. The colouring in which the seed is coloured using one colour and the random perturbation is coloured using all the remaining colours, reduces the problem to that of studying the Ramsey property at hand for truly random hypergraphs.

The earlier results [20, 36, 46], as well as our result, stated in Theorem 1.1 below, are affected by and closely related to research on Ramsey properties in random graphs and hypergraphs (see, e.g., [17, 26, 27, 29–31, 38–40, 42, 44, 45, 47–50]). For random graphs, the thresholds for *symmetric* Ramsey properties are well-understood due to work of Rödl and Ruciński [47, 49]. Minor exceptions for F being a star forest aside, this work asserts that $n^{-1/m_2(F)}$ is the threshold for the property $\mathbb{G}(n,p) \longrightarrow (F)$, where $m_2(F)$ denotes the maximal 2-density of the given graph F (see equation (1.1) below). The 1-statement of the threshold was extended to random k-uniform hypergraphs by Conlon and Gowers [17] and by Friedgut, Rödl, and Schacht [26]. However, a complete characterisation of the exceptional cases is not yet available and for the progress towards the 0-statement we refer to the work of Nenadov et al. [44] and Gugelmann et al. [27].

The thresholds of asymmetric Ramsey properties in random graphs are the subject of the *Kohayakawa–Kreuter conjecture* [30]. The 1-statement stipulated by this conjecture has been fairly recently verified by Mousset, Nenadov, and Samotij [42] and progress has been made with respect to the corresponding 0-statement by several researchers [27, 29, 39, 40]. Following some progress [14, 37, 42], the conjecture was finally fully resolved by Christoph, Martinsson, Steiner, and Wigderson [15].

1.2. **Main result.** We study Ramsey properties of randomly perturbed hypergraphs; stating our results requires preparation. A hypergraph H is said to be *linear* if $|e \cap f| \leqslant 1$ holds whenever e, $f \in E(H)$ are distinct. Amongst the linear hypergraphs, *expanded cliques* are of special interest. Given $t \geqslant 2$ and $k \geqslant 2$, the *k-uniformly expanded clique of order t*, denoted by $\widetilde{K}_t^{(k)}$, is the k-uniform hypergraph with vertex set of size $t + \binom{t}{2}(k-2)$ obtained from the complete graph K_t by expanding every edge of K_t by $k-2$ new vertices; in particular, $\widetilde{K}_t^{(2)} = K_t$ holds. Expanded cliques have attracted some attention in the literature and related extremal and Ramsy-type questions were addressed by Mubayi [43] and by Conlon, Fox, and Rödl [16].

Two natural measures of density, arising in the context of random hypergraphs, are the *maximum density* of a k-uniform $H = (V, E)$, denoted $m(H)$, and its *maximum k-density*, denoted $m_k(H)$. The former is given by

$$m(H) = \max\left\{\frac{e(F)}{v(F)} \colon F \subseteq H \text{ and } v(F) \geqslant 1\right\}$$

and the latter is defined by

$$m_k(H) = \max\left\{d_k(F) \colon F \subseteq H\right\}, \text{ where } d_k(F) = \begin{cases} 0, & \text{if } e(F) = 0, \\ \frac{1}{k}, & \text{if } e(F) = 1,\, v(F) = k, \\ \frac{e(H)-1}{v(H)-k}, & \text{otherwise.} \end{cases} \tag{1.1}$$

It is well known that $n^{-1/m(H)}$ is the threshold for the appearance of H as a subhypergraph in the binomial random k-uniform hypergraph $\mathbb{H}^{(k)}(n,p)$. For $\mathbb{H}^{(k)}(n,p)$ to satisfy the Ramsey property for H a.a.s. it is reasonable to expect that many intermingled copies of H are required; this as to create colour restrictions forcing the Ramsey property for H. Indeed, for (hypergraph) cliques it is necessary that many cliques sharing a single hyperedge would appear a.a.s. in $\mathbb{H}^{(k)}(n,p)$. This results in the higher threshold $n^{-1/m_k(H)}$ being encountered for Ramsey properties.

For asymmetric Ramsey properties, another notion of hypergraph density arises. This notion traces back to the work of Kohayakawa and Kreuter [30]. Given two k-uniform hypergraphs H_1 and H_2, each with at least one edge and satisfying $m_k(H_1) \geqslant m_k(H_2)$, the *asymmetric maximal k-density* of H_1 and H_2 is given by

$$m_k(H_1, H_2) = m_k(H_2, H_1) = \max\left\{\frac{e(F)}{v(F) - k + 1/m_k(H_2)} \colon F \subseteq H_1 \text{ and } e(F) \geqslant 1\right\}, \tag{1.2}$$

where here we do not mean that $m_k(\cdot,\cdot)$ is symmetric only that in our notation we do not keep track over the location in which H_1, the hypergraph with the potentially higher $m_k(\cdot)$-density is higher, is placed. The equality $m_k(H,H) = m_k(H)$ is easy to verify.

With the above notation in place, our main contribution can be stated; this can be viewed as a hypergraph extension of the aforementioned results of Das and Treglown [20]. Below we always tacitly assume that H_n and $\mathbb{H}^{(3)}(n,p)$ share the same vertex set.

Theorem 1.1 (Main result)**.** *For every $d > 0$ and every even integer $t \geqslant 4$, there exists a constant $C > 0$ such that for every sequence of 3-uniform n-vertex hypergraphs $(H_n)_{n\in\mathbb{N}}$ with*

$e(H_n) \geqslant dn^3$ for every $n \in \mathbb{N}$ we have

$$\lim_{n\longrightarrow\infty} \mathbb{P}\big(H_n \cup \mathbb{H}^{(3)}(n,p) \longrightarrow (\widetilde{K}_t^{(3)})\big) = 1,$$

whenever $p = p(n) \geqslant Cn^{-\frac{1}{M}}$ for $M = m_3(\widetilde{K}_t^{(3)}, \widetilde{K}_{t/2}^{(3)})$.

Our proof of Theorem 1.1 relies on two main technical results, which are related to the regularity method for hypergraphs. We present these results in Section 1.3-1.4 below.

The proof of Theorem 1.1 presented here can be adapted for k-uniform hypergraphs and the asymmetric Ramsey properties $H_n \cup \mathbb{H}^{(k)}(n,p) \longrightarrow (\widetilde{K}_s^{(k)}, \widetilde{K}_t^{(k)})$ with $t \geqslant s$. For the sake of brevity, we restrict ourselves to 3-uniform hypergraphs and the symmetric case for even t. In particular, from here on, unless stated otherwise, we use the term hypergraph to mean a 3-uniform hypergraph. We conjecture that Theorem 1.1 uncovers the threshold for the Ramsey property in question.

Conjecture 1.2. For every even integer $t \geqslant 6$ there exist constants d, $c > 0$, and there exists a sequence of 3-uniform n-vertex hypergraphs $(H_n)_{n\in\mathbb{N}}$ with $e(H_n) \geqslant dn^3$ for every $n \in \mathbb{N}$ such that

$$\lim_{n\longrightarrow\infty} \mathbb{P}\big(H_n \cup \mathbb{H}^{(3)}(n,p) \longrightarrow (\widetilde{K}_t^{(3)})\big) = 0\,,$$

whenever $p \leqslant cn^{-1/M}$ for $M = m_3\big(\widetilde{K}_t^{(3)}, \widetilde{K}_{t/2}^{(3)}\big)$.

Conjecture 1.2 may hold for $t = 4$ as well. However, this value is excluded due to the distinct behaviour seen in the graph case [20, 46]. The proof of Theorem 1.1 presented here extends for the asymmetric Ramsey property $H \longrightarrow (\widetilde{K}_t^{(3)}, \widetilde{K}_s^{(3)})$ for sufficiently large integers $t \geqslant s$ and M replaced by $m_3\big(\widetilde{K}_t^{(3)}, \widetilde{K}_{\lceil s/2\rceil}^{(3)}\big)$. It seems plausible that the corresponding generalisation of Conjecture 1.2 may also hold.

1.3. **A tuple lemma for link graphs.**

A key feature of the regularity method of graphs is the control over joint neighbourhoods in the regular environment provided by Szemerédi's regularity lemma (see, e.g., Lemma 2.4 below). For the proof of Theorem 1.1, we establish a similar lemma in the context of the regularity method for hypergraphs.

For a vertex v in a hypergraph $H = (V, E)$, define the *link graph* $L_H(v)$ of v to have vertex set $V \smallsetminus \{v\}$ and edge set comprised of those pairs of vertices which together with v form a hyperedge in H, i.e., $E(L_H(v)) = \{uw : uvw \in E\}$. In particular, $e(L_H(v))$ is the vertex degree of v in H and is also denoted by $\deg_H(v)$. Given a graph G with vertex set $V(G) = V$ we define the *link graph of v supported on G* by

$$L_H(v, G) = E\big(L_H(v)\big) \cap E(G)\,.$$

Link graphs are a natural hypergraph extension of vertex neighbourhoods in the context of graphs. A tuple lemma for hypergraphs would have to control the sizes of the intersections of link graphs. In that, given a set of vertices $U \subseteq V$, we seek to control the sizes of the *joint link graph* and the *joint link graph supported by* G given by

$$L_H(U) = \bigcap_{u \in U} L_H(u) \quad \text{and} \quad L_H(U,G) = \bigcap_{u \in U} L_H(u,G)\,,$$

respectively. For a random hypergraph $H = (V, E)$ with edge density d, one would expect $|L_H(U)| \sim d^{|U|}\binom{|V|}{2}$ to hold with high probability. Our tuple lemma asserts that in the regular environment for hypergraphs this random intuition can be transferred to the deterministic situation. (We defer the definitions concerning regular hypergraphs to Section 2.)

Proposition 1.3 (Tuple lemma for joint links)**.** *For every $t \geqslant 2$ and ε, $d_3 > 0$, there exists a $\delta_3 > 0$ such that for every $d_2 > 0$ there exist $\delta_2 > 0$ and $r \geqslant 1$ such that the following holds.*

Let $H = (X \cup Y \cup Z, E_H)$ be a tripartite hypergraph which is (δ_3, d_3, r)-regular with respect to a (δ_2, d_2)-triad $P = (X \cup Y \cup Z, E_P)$. Then, all but at most $2\varepsilon|X|^t$ of the t-tuples of vertices $X' = \{x_1, \dots, x_t\} \subseteq X$ satisfy

$$\big|L_H(X', P) - d_3^t d_2^{2t+1}|Y||Z|\big| \leqslant \varepsilon d_2^{2t+1}|Y||Z|\,. \tag{1.3}$$

Due to space limitations, our proof of Proposition 1.3 is omitted and can be found in [7] - the full version of this extended abstract - the former extends to all hypergraph uniformities. Alternatives to Proposition 1.3 exerting some control over the sizes of joint link graphs of vertex tuples whilst relying on weaker versions of the hypergraph regularity do exist. Such alternatives are established in the extended account [7].

1.4. **A variant of the hypergraph regularity lemma.** The second main technical lemma is a new variant of the hypergraph regularity lemma established in [51]. The necessary definitions are deferred to Section 2.

Proposition 1.4 (Variant of the regularity lemma for hypergraphs)**.** *For every $\delta_3 > 0$ and functions $\delta_2 \colon \mathbb{N} \longrightarrow (0,1]$, $r \colon \mathbb{N}^2 \longrightarrow \mathbb{N}$, and constants ℓ_0, t_0, and $s \in \mathbb{N}$, there exist n_0 and $T \in \mathbb{N}$ such that for every $n \geqslant n_0$ and every family $(H_1, \dots, H_s)$ of n-vertex hypergraphs satisfying $V = V(H_1) = \dots = V(H_s)$, there are integers t and ℓ satisfying $t \geqslant t_0$ and $\ell \geqslant \ell_0$, a vertex partition $\mathcal{V}$ with $V_1 \cup \dots \cup V_t = V$ and an ℓ-equitable partition $\mathcal{B}$ with respect to $\mathcal{V}$ such that the following properties hold.*

(R.1): $|V_1| \leqslant |V_2| \leqslant \dots \leqslant |V_t| \leqslant |V_1| + 1$,
(R.2): *for all $1 \leqslant i < j \leqslant t$ and $\alpha \in [\ell]$, the bipartite 2-graph B_α^{ij} is $(\delta_2(\ell), 1/\ell)$-regular,*
(R.3): *H_i is $\delta_2(\ell)$-weakly regular with respect to $\mathcal{V}$ for every $i \in [s]$, and*
(R.4): *H_i is $(\delta_3, r(t, \ell))$-regular with respect to $\mathcal{B}$ for every $i \in [s]$.*

Due to space limitations, our proof of Proposition 1.4 is omitted and can be found in [7] - the full version of this extended abstract. In Proposition 1.4 there is a combination of the environment of the hypergraph regularity lemma [51] (see Lemma 2.6) and the so-called weak hypergraph regularity lemma (see Lemma 2.5 below), which is the straightforward extension of Szemerédi's regularity for graphs. A lemma of similar spirit can be found in the work of Allen, Parczyk, and Pfenninger [9].

In the sequel, these hypergraph regularity lemmata are distinguished by referring to these as the *Strong Lemma* and *Weak Lemma*, respectively. The difference between the Strong Lemma and Proposition 1.4 is Property (R.3). The former, when applied to dense hypergraphs, provides access to triads P set over a vertex set, say, $X \cupdot Y \cupdot Z$ with respect to which the regularised hypergraphs is (δ_3, d, r)-regular. This, in turn, provides ζ-weak regularity control for $\zeta = \delta_3^{1/3}$, by which we mean the ability to control the hyperedge distribution of the hypergraphs along sets $X' \subseteq X$, $Y' \subseteq Y$, and $Z' \subseteq Z$ satisfying $|X'| \geqslant \zeta|X|$, $|Y'| \geqslant \zeta|Y|$, and $|Z'| \geqslant \zeta|Z|$.

The added Property (R.3), however, provides weak regularity control over vertex sets with much smaller density. In fact, there the control δ_2 is allowed to be a function of ℓ and the quantification of the Strong Lemma leads to $\delta_3 \gg \ell^{-1}$.

Organisation. Theorem 1.1 is proved in Section 3. Various required preliminaries are collected in Section 2. As mentioned above, the proofs of Propositions 1.3 and 1.4 are omitted from this account due to space limitations and can be seen in [7] - the full version of this account.

Notational remark. Throughout, we often write the enumeration of a result in the subscripts of the constants that it presides over. For instance, the constant t_0 in Proposition 1.4 becomes $t_{1.4}$ and the constant δ_3 in the same lemma is written $\delta_{1.4}^{(3)}$ and so on. This aids in keeping track of the various constants encountered throughout the proofs.

§2 Preliminaries

Let V be a finite set. A partition $\mathcal{U}$ of V given by $V = U_1 \cupdot \cdots \cupdot U_r$ is said to be *equitable* if $|U_1| \leqslant |U_2| \leqslant \cdots \leqslant |U_r| \leqslant |U_1| + 1$. Given an additional partition of V, namely $\mathcal{V}$, of the form $V = V_1 \cupdot \cdots \cupdot V_\ell$, we say that $\mathcal{V}$ *refines* $\mathcal{U}$, and write $\mathcal{V} \prec \mathcal{U}$, if for every $i \in [\ell]$ there exists some $j \in [r]$ such that $V_i \subseteq U_j$ holds. For $k \geqslant 2$, write $K^{(k)}(\mathcal{U})$ to denote the complete $|\mathcal{U}|$-partite k-uniform hypergraph whose vertex set is V and whose edge set is given by all sets of $V^{(k)} = \{K \subseteq V : |K| = k\}$ meeting every member of $\mathcal{U}$ (termed *cluster* hereafter) in at most one vertex. If $\mathcal{U} = \{U, U'\}$ consists of only two clusters, then we abbreviate $K^{(2)}(\mathcal{U})$ to $K^{(2)}(U, U')$. We write $K^{(2)}(V)$ to denote the complete graph whose vertex set is V.

2.1. **Graph regularity.** Let $d, \delta > 0$ be given. A bipartite 2-graph $G = (X \cupdot Y, E)$ is said to be (δ, d)-*regular* if

$$e_G(X', Y') = d|X'||Y'| \pm \delta|X||Y|$$

holds[1] for every $X' \subseteq X$ and $Y' \subseteq Y$. If d coincides with the edge density of G, i.e. $d = \frac{e(G)}{|X||Y|}$, then we abbreviate (δ, d)-regular to δ-regular. It follows directly from the definition that G is a (δ, d)-regular bipartite graph if, and only if, its (bipartite) complement is $(\delta, 1-d)$-regular.

A tripartite 2-graph P with vertex set $V(P) = X \cup Y \cup Z$ is said to be a (δ, d)-*triad*, if $P[X, Y]$, $P[Y, Z]$, and $P[X, Z]$ are all (δ, d)-regular. For a 2-graph G, let $\mathcal{K}_3(G)$ denote the family of members of $V(G)^{(3)}$ spanning a triangle in G. We shall employ the well known triangle countling lemma (see, e.g., [25, Fact A]).

Lemma 2.1 (Triangle counting lemma). *Let $d > 0$, let $0 < \delta < d/2$, and let P be a (δ, d)-triad with vertex set $V(P) = X \cup Y \cup Z$. Then,*

$$(1 - 2\delta)(d - \delta)^3|X||Y||Z| \leqslant |\mathcal{K}_3(P)| \leqslant ((d + \delta)^3 + 2\delta)|X||Y||Z|.$$

In particular, if $d \leqslant 1/2$, then

$$|\mathcal{K}_3(P)| = (d^3 \pm 4\delta)|X||Y||Z| \tag{2.1}$$

holds. □

We shall also use the variant of the triangle counting lemma with only two of the bipartite graphs being regular and its proof is included for completeness.

Lemma 2.2. *Let $P = (X \cup Y \cup Z, E_P)$ be a tripartite 2-graph such that $P[X, Y]$ and $P[X, Z]$ are both (δ, d)-regular. In addition, let $X' \subseteq X$ be a set of size $|X'| \geqslant \delta|X|$. Then,*

$$(d - \delta)d|X'|e(P[Y, Z]) - 2\delta|X||Y||Z| \leqslant |\mathcal{K}_3(P, X')| \leqslant (d + \delta)d|X'|e(P[Y, Z]) + 2\delta|X||Y||Z|$$

holds, where $\mathcal{K}_3(P, X')$ denotes the set of triangles of P meeting X'.

Proof. Let $Y' \subseteq Y$ consist of all vertices $y \in Y$ satisfying $\deg_P(y, X') \geqslant (d - \delta)|X'|$; note that $|Y'| \geqslant (1 - \delta)|Y|$ holds by Lemma 2.4. We may then write

$$\begin{aligned}|\mathcal{K}_3(P, X')| &\geqslant \sum_{y \in Y'} \Big(d(d - \delta)|X'| \deg_P(y, Z) - \delta|X||Z|\Big) \\ &= d(d - \delta)|X'| \left(\sum_{y \in Y} \deg_P(y, Z) - \sum_{y \in Y \smallsetminus Y'} \deg_P(y, Z)\right) - \sum_{y \in Y'} \delta|X||Z| \\ &\geqslant d(d - \delta)|X'|e(P[Y, Z]) - d(d - \delta)\delta|X||Y||Z| - \delta|X||Y||Z| \\ &\geqslant d(d - \delta)|X'|e(P[Y, Z]) - 2\delta|X||Y||Z|.\end{aligned}$$

Next, we prove the upper bound. Let $Y'' \subseteq Y$ consist of all vertices $y \in Y$ satisfying $\deg_P(y, X') \leqslant (d + \delta)|X'|$; note that $|Y''| \geqslant (1 - \delta)|Y|$ holds by Lemma 2.4. We may then write

$$|\mathcal{K}_3(P, X')| \leqslant \sum_{y \in Y''} \Big(d(d + \delta)|X'| \deg_P(y, Z) + \delta|X||Z|\Big) + \sum_{y \in Y \smallsetminus Y''} |X'||Z|$$

[1] Given $x, y, z \in \mathbb{R}$, we write $x = y \pm z$ if $y - z \leqslant x \leqslant y + z$.

$$\leqslant d(d+\delta)|X'|\left(\sum_{y\in Y}\deg_P(y,Z)-\sum_{y\in Y\smallsetminus Y''}\deg_P(y,Z)\right)$$
$$+\sum_{y\in Y''}\delta|X||Z|+\sum_{y\in Y\smallsetminus Y''}|X||Z|$$
$$\leqslant d(d+\delta)|X'|e(P[Y,Z])+2\delta|X||Y||Z|\,. \qquad \square$$

The next lemma is commonly referred to as the *Slicing Lemma* (see, e.g., [33, Fact 1.5]).

Lemma 2.3 (Slicing lemma). *Let $d = d_{2.3}$, let $\delta = \delta_{2.3} > 0$, and let $G = (A \cup B, E)$ be a (δ, d)-regular bipartite graph. Let $\delta \leqslant \alpha = \alpha_{2.3} \leqslant 1$, and let $A' \subseteq A$ and $B' \subseteq B$ be sets of sizes $|A'| \geqslant \alpha|A|$ and $|B'| \geqslant \alpha|B|$. Then, $G[A', B']$ is (δ', d')-regular where $\delta' = \max\{\delta/\alpha, 2\delta\}$ and $d' = d \pm \delta$.* $\square$

The *tuple property* of dense regular bipartite graphs, also referred to as the *intersection property*, reads as follows (see [33, Fact 1.4]).

Lemma 2.4 (Tuple lemma for graphs). *Let $G = (X \cup Y, E)$ be a δ-regular bipartite graph of edge density $d > 0$. Then, all but at most $2\delta\ell|X|^\ell$ of the tuples $\{x_1, \ldots, x_\ell\} \subseteq X$ satisfy*

$$|N_G(x_1,\ldots,x_\ell,Y')| = |\{y \in Y' \colon x_i y \in E(G) \text{ for all } i \in [\ell]\}| = (d \pm \delta)^\ell |Y'|, \tag{2.2}$$

whenever $Y' \subseteq Y$ satisfies $(d-\delta)^{\ell-1}|Y'| \geqslant \delta|Y|$. $\square$

2.2. **Hypergraph regularity.** A direct generalisation of the notion of δ-regularity, defined in the previous section for 2-graphs, reads as follows. Let $d, \delta > 0$. A tripartite hypergraph $H = (X \cup Y \cup Z, E)$ is said to be (δ, d)-*weakly regular* if

$$e_H(X',Y',Z') = d|X'||Y'||Z'| \pm \delta|X||Y||Z|$$

holds whenever $X' \subseteq X$, $Y' \subseteq Y$, and $Z' \subseteq Z$. If $d = \frac{e(H)}{|X||Y||Z|}$, then we abbreviate (δ, d)-weakly regular to δ-weakly regular.

Given a partition $\mathcal{V}$ of a finite set V defined by $V = V_1 \cup \cdots \cup V_t$, a hypergraph H with $V(H) = V$ is said to be δ-*weakly regular with respect to* $\mathcal{V}$ if $H[X, Y, Z]$[2] is δ-weakly regular with respect to all but at most $\delta\binom{t}{3}$ triples $\{X, Y, Z\} \in \mathcal{V}^{(3)}$. We state the straightforward adaptation of Szemerédi's graph regularity lemma [32, 33, 54].

Lemma 2.5 (Weak hypergraph regularity lemma). *For every $\delta = \delta_{2.5} > 0$ and positive integers $s = s_{2.5}$, $t = t_{2.5}$, and $h = h_{2.5}$ satisfying $t \geqslant h$, there exist positive integers n_0 and $T = T_{2.5}$ such that the following holds whenever $n \geqslant n_0$. Let $(H_1, \ldots, H_s)$ be a sequence of n-vertex hypergraphs, all on the same vertex set, namely V, and let $\mathcal{U} = \mathcal{U}_{2.5}$ be a vertex partition of V given by $V = U_1 \cup \ldots \cup U_h$. Then, there exists an equitable vertex partition $\mathcal{V}$, given by $V = V_1 \cup V_2 \cup \cdots \cup V_{t'}$, where $t \leqslant t' \leqslant T$, such that $\mathcal{V} \prec \mathcal{U}$ and, moreover, H_i is δ-weakly regular with respect to $\mathcal{V}$ for every $i \in [s]$.* $\square$

[2] $H[X, Y, Z]$ is the subgraph of H over $X \cup Y \cup Y$ whose edge set is $\{\{x, y, z\} \in E(H) \colon x \in X, y \in Y, z \in Z\}$.

We proceed to the statement of the Strong hypergraph Regularity Lemma for hypergraphs following the formulation seen in [51]. Given a 2-graph G, the *relative density* of a hypergraph H with vertex set $V(H) = V(G)$, with respect to G is given by

$$d(H|G) = \frac{|E(H) \cap \mathcal{K}_3(G)|}{|\mathcal{K}_3(G)|}. \tag{2.3}$$

For $\delta, d > 0$ and a positive integer r, a tripartite hypergraph $H = (X \cupdot Y \cupdot Z, E_H)$ is said to be (δ, d, r)-*regular* with respect to a tripartite 2-graph $P = (X \cupdot Y \cupdot Z, E_P)$ if

$$\left|\left|\bigcup_{i=1}^{r}(E_H \cap \mathcal{K}_3(Q_i))\right| - d\left|\bigcup_{i=1}^{r}\mathcal{K}_3(Q_i)\right|\right| \leqslant \delta\big|\mathcal{K}_3(P)\big| \tag{2.4}$$

holds for every family of, not necessarily disjoint, subgraphs $Q_1, \ldots, Q_r \subseteq P$ satisfying

$$\left|\bigcup_{i=1}^{r}\mathcal{K}_3(Q_i)\right| \geqslant \delta\big|\mathcal{K}_3(P)\big| > 0\,.$$

Let V be a finite set and let $\mathcal{V}$ be a partition $V_1 \cupdot \ldots \cupdot V_h$ of V, where h is some positive integer. Given an integer $\ell \geqslant 1$, a partition $\mathcal{B}$ of $K^{(2)}(\mathcal{V})$ is said to be ℓ-*equitable with respect to* $\mathcal{V}$ if it satisfies the following conditions:

(B.1): every $B \in \mathcal{B}$ satisfies $B \subseteq K^{(2)}(V_i, V_j)$ for some distinct $i, j \in [h]$; and

(B.2): for any distinct $i, j \in [h]$, precisely ℓ members of $\mathcal{B}$ partition $K^{(2)}(V_i, V_j)$.

We view partitions of $K^{(2)}(\mathcal{V})$ as partitions of $V^{(2)}$ under the *agreement*[3] that the set $\{K^{(2)}(V_i)\colon i \in [h]\}$ of complete graphs is added to the former; such an addition of cliques does not hinder the equitability notion defined in (B.2); it does violate (B.1), but this will not harm our arguments. Moreover, it is under this agreement that we say that a partition of $V^{(2)}$ refines a partition of $K^{(2)}(\mathcal{V})$.

For distinct indices $i, j \in [h]$, the partition of $K^{(2)}(V_i, V_j)$ induced by $\mathcal{B}$ is denoted by

$$\mathcal{B}^{ij} = \{B^{ij}_\alpha = (V_i \cupdot V_j, E^{ij}_\alpha)\colon \alpha \in [\ell]\}\,.$$

The *triads* of $\mathcal{B}$ are the tripartite 2-graphs having the form

$$B^{ijk}_{\alpha\beta\gamma} = (V_i \cupdot V_j \cupdot V_k, E^{ij}_\alpha \cupdot E^{ik}_\beta \cupdot E^{jk}_\gamma),$$

where $i, j, k \in [h]$ are distinct and $\alpha, \beta, \gamma \in [\ell]$. Recall that a triad is called a (δ, d)-*triad* if each of the three bipartite graphs comprising it is (δ, d)-regular. A hypergraph H with vertex set $V(H) = V$ is said to be (δ, r)-*regular* with respect to $\mathcal{B}$ if

$$\left|\left\{\bigcup_{\substack{1\leqslant i<j<k\leqslant h\\ \alpha,\beta,\gamma\in[\ell]}} \mathcal{K}_3(B^{ijk}_{\alpha\beta\gamma})\colon H_{ijk} \text{ is not } \big(\delta, d(H|B^{ijk}_{\alpha\beta\gamma}), r\big)\text{-regular w.r.t. } B^{ijk}_{\alpha\beta\gamma}\right\}\right| \leqslant \delta|V|^3\,,$$

[3]We appeal to this agreement in our proof of Proposition 1.4 omitted from this account and which can be found in [7].

where $H_{ijk} = H[V_i \cup V_j \cup V_k]$. A formulation of the Strong Lemma [51, Theorem 17] for hypergraphs, reads as follows.

Lemma 2.6 (Strong hypergraph regularity lemma)**.** *For all $0 < \delta_3 \in \mathbb{R}$, $\delta_2 \colon \mathbb{N} \longrightarrow (0,1]$, $r \colon \mathbb{N}^2 \longrightarrow \mathbb{N}$, and $s, t, \ell \in \mathbb{N}$, there exist n_0, $T \in \mathbb{N}$ such that for every $n \geqslant n_0$ and every sequence of n-vertex hypergraphs $(H_1, \dots, H_s)$, satisfying $V = V(H_1) = \cdots = V(H_s)$, there are $t', \ell' \in \mathbb{N}$ satisfying $t \leqslant t' \leqslant T$ and $\ell \leqslant \ell' \leqslant T$, a vertex partition $V = V_1 \cup \cdots \cup V_{t'}$, namely $\mathcal{V}$, and an ℓ'-equitable partition $\mathcal{B}$ with respect to $\mathcal{V}$ such that the following properties hold.*

(S.1): $|V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_{t'}| \leqslant |V_1| + 1$;
(S.2): *for all $1 \leqslant i < j \leqslant t'$ and $\alpha \in [\ell']$, the bipartite 2-graph B_α^{ij} is $(\delta_2(\ell'), 1/\ell')$-regular; and*
(S.3): *H_i is $(\delta_3, r(t', \ell'))$-regular with respect to $\mathcal{B}$ for every $i \in [s]$.* □

§3 Monochromatic expanded cliques

In this section, we prove Theorem 1.1. The required Ramsey properties of $\mathbb{H}^{(3)}(n,p)$ are collected in Section 3.1; a proof of Theorem 1.1 can be found in Section 3.2. For an integer $t \geqslant 3$, the t vertices of $\widetilde{K}_t^{(k)}$ having their 1-degree strictly larger than one are called the *branch-vertices* of $\widetilde{K}_t^{(k)}$. Set

$$v(t) = v(\widetilde{K}_t^{(3)}) \quad \text{and} \quad e(t) = e(\widetilde{K}_t^{(3)}).$$

3.1. **Properties of random hypergraphs.** The main goal of this section is to state Proposition 3.1 which is an adaptation of [20, Theorem 2.10]. This proposition collects the Ramsey properties of $\mathbb{H}^{(3)}(n,p)$ that will be utilised throughout our proof of Theorem 1.1.

A k-graph H is said to be *balanced* if $m_k(H) = d_k(H)$ holds; if all proper subgraphs F of H satisfy $m_k(F) < m_k(H)$, then H is said to be *strictly balanced*. It is not hard to verify that expanded cliques are strictly balanced. In particular,

$$m_k\left(\widetilde{K}_t^{(k)}\right) = \frac{\binom{t}{2} - 1}{t + (k-2)\binom{t}{2} - k}$$

holds for any $k \geqslant 2$ and $t \geqslant 3$. In the special case $k = 3$ we obtain

$$m_3\big(\widetilde{K}_t^{(k)}\big) = \frac{t^2 - t - 2}{t^2 + t - 6} = 1 - \frac{2t - 4}{t^2 + t - 6} < 1, \tag{3.1}$$

that is, 3-uniformly expanded cliques are *sparse*. Note that this is in contrast to graph cliques (on at least 3 vertices) whose 2-density is larger than one. For a simpler notation we set an integer $t \geqslant 2$

$$m(t) = m(\widetilde{K}_t^{(3)}) \quad \text{and} \quad M_t = m_3(\widetilde{K}_t^{(3)})\,.$$

Similarly for integers t_1, $t_2 \geqslant 2$ we set

$$M_{t_1,t_2} = M_{t_2,t_1} = m_3\big(\widetilde{K}_{t_1}^{(3)}, \widetilde{K}_{t_2}^{(3)}\big)\,.$$

Let H_1 and H_2 be two k-graphs, each with at least one edge and such that $m_k(H_1) \geqslant m_k(H_2)$. If $m_k(H_1) = m_k(H_2)$, then $m_k(H_1, H_2) = m_k(H_1)$; otherwise $m_k(H_2) < m_k(H_1, H_2) < m_k(H_1)$ holds. The k-graph H_1 is said to be *strictly balanced with respect to* $m_k(\cdot, H_2)$ if no proper subgraph $F \subsetneq H_1$ maximises (1.2). For instance, it is not hard to verify that $\widetilde{K}_t^{(3)}$ is strictly balanced with respect to $m_3(\cdot, \widetilde{K}_{t/2}^{(3)})$, assuming $t \geqslant 4$ is even.

Let F and F' be k-graphs and let $\mu = \mu(n)$ be given. An n-vertex k-graph H is said to be (F, μ)-*Ramsey* if $H[U] \longrightarrow (F)_2$ holds for every $U \subseteq V(H)$ is of size $|U| \geqslant \mu n$. Similarly, H is said to be (F, F', μ)-*Ramsey* if $H[U] \longrightarrow (F, F')$ holds for every $U \subseteq V(H)$ of size $|U| \geqslant \mu n$. Given $\mathcal{F} \subseteq \binom{[n]}{v(F)}$ and $\mathcal{F}' \subseteq \binom{[n]}{v(F')}$, we say that H is (F, F')-*Ramsey with respect to* $(\mathcal{F}, \mathcal{F}')$ if any 2-colouring of $E(H)$ yields a monochromatic copy K of F (in the first colour) with $V(K) \notin \mathcal{F}$ or a monochromatic copy K' of F' (in the second colour) with $V(K') \notin \mathcal{F}'$.

Proposition 3.1. *Let $t \geqslant 4$ be an even integer. The binomial random hypergraph $H \sim \mathbb{H}^{(3)}(n, p)$ a.a.s. satisfies the following properties.*

- (P.1) *There are constants $\gamma_{3.1} = \gamma_{3.1}(t)$ and $C_{3.1}^{(1)} = C_{3.1}^{(1)}(t)$ such that if $\mathcal{F}_1 \subseteq \binom{[n]}{v(t)}$ and $\mathcal{F}_2 \subseteq \binom{[n]}{v(t/2)}$ satisfy $|\mathcal{F}_1| \leqslant \gamma_{3.1} n^{v(t)}$ and $|\mathcal{F}_2| \leqslant \gamma_{3.1} n^{v(t/2)}$, then H is $(\widetilde{K}_t^{(3)}, \widetilde{K}_{t/2}^{(3)})$-Ramsey with respect to $(\mathcal{F}_1, \mathcal{F}_2)$, whenever $p = p(n) \geqslant C_{3.1}^{(1)} n^{-1/M_{t,t/2}}$.*
- (P.2) *For every fixed $\mu > 0$, there exists a constant $C_{3.1}^{(2)} = C_{3.1}^{(2)}(\mu, t)$ such that H is $(\widetilde{K}_{t-1}^{(3)}, \mu)$-Ramsey, whenever $p = p(n) \geqslant C_{3.1}^{(2)} n^{-1/M_{t-1}}$.*
- (P.3) *For every fixed $\mu > 0$, there exists a constant $C_{3.1}^{(3)} = C_{3.1}^{(3)}(\mu, t)$ such that H is $(\widetilde{K}_t^{(3)}, \widetilde{K}_{t/2}^{(3)}, \mu)$-Ramsey, whenever $p = p(n) \geqslant C_{3.1}^{(3)} n^{-1/M_{t,t/2}}$.*

Remark 3.2. *A straightforward albeit somewhat tedious calculation shows that $M_{t,t/2} \geqslant M_{t-1}$ holds for every even integer $t \geqslant 4$. It thus follows that Properties (P.1) and (P.3) are the most stringent in terms of the bound these impose on p. Hence, if $p = p(n) \geqslant \max\left\{C_{3.1}^{(1)}, C_{3.1}^{(3)}\right\} \cdot n^{-1/M_{t,t/2}}$, then a.a.s. H satisfies Properties (P.1), (P.2), and (P.3) simultaneously.*

Property (P.1) is modelled after [20, Theorem 2.10(i)]; Properties (P.2) and (P.3) are both specific instantiations of [20, Theorem 2.10(ii)]. The aforementioned results of [20] handle 2-graphs only. Nevertheless, proofs of Properties (P.1-3) can be attained by straightforwardly adjusting the proofs of their aforementioned counterparts in [20, Theorem 2.10] so as to accommodate the transition from 2-graphs to hypergraphs. Theorem 2.10 in [20] requires that the maximal 2-densities of the two (fixed) configurations would both be at least one; this can be omitted in our setting. Indeed, this condition is imposed in [20, Theorem 2.10] in order to handle setting (a) in that theorem where the maximal 2-densities of the two configurations *coincide*; by (3.1), this is not an issue in our case. The fact that $\widetilde{K}_t^{(3)}$ is strictly balanced with respect to $m_3(\cdot, \widetilde{K}_{t/2}^{(3)})$ is required by setting (b) appearing in [20, Theorem 2.10].

3.2. **Proof of Theorem 1.1.** We commence our proof of Theorem 1.1 with a few observations facilitating our arguments; proofs of these observations are included for completeness.

Observation 3.3. *Let $d \in (0,1]$, let $G = (A \cupdot B, E)$ be a bipartite graph satisfying $e(G) \geqslant d|A||B|$, and let $k \leqslant d|B|/2$ be a positive integer. Then, $|\{v \in A : \deg_G(v) \geqslant k\}| \geqslant d|A|/2$.*

Proof. Let $A_k = \{v \in A : \deg_G(v) \geqslant k\}$ and suppose for a contradiction that $|A_k| < d|A|/2$. Then,

$$e(G) < k|A| + |A_k||B| < d|A||B|/2 + d|A||B|/2 \leqslant e(G)$$

which is clearly a contradiction. □

The next lemma captures the phenomenon of *supersaturation* (first [4] recorded in [22–24]) for bipartite graphs; to facilitate future references, we phrase this lemma with the host graph being bipartite as well.

Lemma 3.4. *For every bipartite graph K and every $d \in (0,1)$, there exists a constant $\zeta = \zeta_{3.4} > 0$ and a positive integer n_0 such that every n-vertex bipartite graph $G = (A \cupdot B, E)$ satisfying $n \geqslant n_0$, $|A| \leqslant |B| \leqslant |A| + 1$, and $e(G) \geqslant d|A||B|$ contains at least $\zeta n^{v(K)}$ distinct copies of K.*

Observation 3.5. *For every graph K and every $d \in (0,1)$, there exists a constant $\xi = \xi_{3.5} > 0$ and an integer n_0 such that the following holds whenever $n \geqslant n_0$. If an n-vertex graph G contains $dn^{v(K)}$ distinct copies of K, then it contains at least ξn pairwise vertex-disjoint copies of K.*

Proof. Any given copy of K meets $O\left(n^{v(K)-1}\right)$ copies of K. □

Proof of Theorem 1.1. Given d, t, and H as in the premise of Theorem 1.1, set

$$0 < d_3 \ll d \text{ and } 0 < \varepsilon \ll \min\left\{d_3^{v(t/2)}, \gamma_{3.1}(t)\right\}. \tag{3.2}$$

The Tuple Property (Theorem 1.3) applied with $t_{1.3} = v(t/2)$, $\varepsilon_{1.3} = \varepsilon$, and $d_{1.3}^{(3)} = d_3$, yields the existence of a constant

$$0 < \delta_3 = \delta_{1.3}^{(3)}(v(t/2), \varepsilon, d_3) \ll d_3 \tag{3.3}$$

as well as the functions

$$\widetilde{\delta}_2(x) = \delta_{1.3}^{(2)}(x, t_{1.3}, \varepsilon, d_3, \delta_3) \text{ and } r(x) = r_{1.3}(x, t_{1.3}, \varepsilon, d_3, \delta_3),$$

where $\widetilde{\delta}_2 : \mathbb{R} \longrightarrow (0,1]$ and $r : \mathbb{N} \longrightarrow \mathbb{N}$. Define $\delta_2 : \mathbb{N} \longrightarrow (0,1]$ such that

$$0 < \delta_2(x) \ll \min\left\{\widetilde{\delta}_2(x), \frac{d_3^{2v(t/2)}}{v(t/2) \cdot x^{6\cdot(2v(t/2)+1)}}\right\} \tag{3.4}$$

[4] Rademacher (1941, unpublished) was first to prove that every n-vertex graph with $\lfloor n^2/4 \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles

holds for every $x \in \mathbb{N}$. Lemma 1.4, applied with

$$H_1 = \cdots = H_s = H,\ \delta^{(3)}_{1.4} = \delta_3,\ \delta^{(2)}_{1.4} = \delta_2,\ r_{1.4} = r^{5},\ \ell_{1.4} \gg d_3^{-1},\ \text{and } t_{1.4} \gg d^{-1}, \tag{3.5}$$

yields the existence of constants $T_{1.4}, \tilde{t}, \ell \in \mathbb{N}$ satisfying $t_{1.4} \leqslant \tilde{t} \leqslant T_{1.4}$ and $\ell_{1.4} \leqslant \ell \leqslant T_{1.4}$, along with partitions $\mathcal{V} = V_1 \cup \cdots \cup V_{\tilde{t}} = V(H)$ and $(\mathcal{P}^{ij})_{1 \leqslant i < j \leqslant \tilde{t}}$ satisfying Properties (R.1-4). Set auxiliary constants

$$d_2 = 1/\ell \quad \text{and} \quad \eta = \frac{d_3^{v(t/2)} d_2^{2v(t/2)+1}}{2} \tag{3.6}$$

and fix

$$0 < \mu \ll \frac{\xi_{3.5}(\zeta_{3.4}(\eta/2)) \cdot d_3^{3+2v(t/2)} \cdot d_2^{10+4v(t/2)}}{v(t/2)^2 \cdot T_{1.4}}. \tag{3.7}$$

We claim that there exist three distinct clusters $X, Y, Z \in \mathcal{V}$ along with a $(\delta_2(\ell), d_2)$-triad $P = P^{ijk}_{\alpha\beta\gamma}$, with $i, j, k, \alpha, \beta, \gamma$ appropriately defined, satisfying $V(P) = X \cup Y \cup Z$ such that $H[X \cup Y \cup Z]$ is $\delta_2(\ell)$-weakly-regular and, moreover, $H[X \cup Y \cup Z]$ is (δ_3, d_3, r)-regular with respect to P. To see this, note first that at most $\tilde{t}\binom{\lceil n/\tilde{t} \rceil}{3} \leqslant \frac{n^3}{\tilde{t}^2} \ll dn^3$ edges of H reside within the members of $\mathcal{V}$, where the last inequality relies on $\tilde{t} \geqslant t_{1.4} \gg d^{-1}$, supported by (3.5). Second, by Property (R.3), the number of edges of H captured within $\delta_2(\ell)$-weakly-irregular triples (V_i, V_j, V_k), where $i, j, k \in [\tilde{t}]$, is at most $\delta_2(\ell) \cdot \tilde{t}^3 \cdot \left(\frac{n}{\tilde{t}} + 1\right)^3 \leqslant 2\delta_2(\ell) n^3 \ll dn^3$, where the last inequality holds by (3.2) and (3.4). Third, by Property (R.4), the number of edges of H residing[6] in $(\delta_3, d(H|P^{ijk}_{\alpha\beta\gamma}), r)$-irregular triads $P^{ijk}_{\alpha\beta\gamma}$ is at most $\delta_3 n^3 \ll dn^3$, where the last inequality holds by (3.2) and (3.3). Fourth and lastly, it follows by the Triangle Counting Lemma (Lemma 2.1) and by (2.3), that the number of edges of H found in $(\delta_2(\ell), d_2)$-triads $P^{ijk}_{\alpha\beta\gamma}$, where $i, j, k \in [\tilde{t}]$ and $\alpha, \beta, \gamma \in [\ell]$, satisfying $d(H|P^{ijk}_{\alpha\beta\gamma}) < d_3$ is at most

$$\tilde{t}^3 \ell^3 d_3 \left(d_2^3 + 4\delta_2(\ell)\right) \left(\frac{n}{\tilde{t}} + 1\right)^3 \leqslant 2d_3 \left(\ell^3 d_2^3 + 4\ell^3 \delta_2(\ell)\right) n^3 \overset{(3.6)}{=} (2 + 8\ell^3\delta_2(\ell)) d_3 n^3 \ll dn^3,$$

where the last inequality holds by (3.2) and (3.4).

It follows that at least $dn^3/2$ edges of H are captured in $(\delta_2(\ell), d_2)$-triads with respect to which H is (δ_3, d_3, r)-regular and such that H is $\delta_2(\ell)$-weakly-regular with respect to the three members of $\mathcal{V}$ defining the vertex-sets of these triads. The existence of $X, Y, Z \in \mathcal{V}$ and P as defined above is then established. Throughout the remainder of the proof, we **identify** H with $H[X \cup Y \cup Z]$.

Let $\mathcal{F} \subseteq \binom{X}{v(t/2)}$ be the family of all sets $\{x_1, \ldots, x_{v(t/2)}\} \subseteq X$ satisfying

$$\left| \bigcap_{j \in [v(t/2)]} L_H(x_j, P) \right| < \left(d_3^{v(t/2)} - \varepsilon\right) d_2^{2v(t/2)+1} |Y||Z|. \tag{3.8}$$

[5]Formally, r is a function of one integer whereas $r_{1.4}$ is a function of two. However, this "loss of information" is a technicality that will not hinder our proof.

[6] Supported by triangles of such triads.

Then,

$$|\mathcal{F}| \leqslant \varepsilon |X|^{v(t/2)} \overset{(3.2)}{\ll} \gamma_{3.1}(t)|X|^{v(t/2)}$$

holds by (1.3). This application of the Tuple Lemma is supported by our choice $\ell_{1.4} \gg d_3^{-1}$, seen in (3.5), ensuring that $d_2 \ll d_3$ holds and thus fitting the quantification of the Tuple Lemma. With foresight (see (C.1) and (C.2) below), let

$$C = \max\left\{C_{3.1}^{(1)}(t), C_{3.1}^{(2)}(\mu, t), C_{3.1}^{(3)}(\mu, t)\right\} \cdot \tilde{t}^{1/M_{t,t/2}}$$

and put

$$p = p(n) = C \max\left\{n^{-1/M_{t,t/2}}, n^{-1/M_{t-1}}\right\} = Cn^{-1/M_{t,t/2}};$$

for the last equality consult Remark 3.2. Proposition 3.1 then asserts that the following properties are all satisfied simultaneously a.a.s. whenever $R \sim \mathbb{H}^{(3)}(n,p)$; in the following list of properties, whenever an asymmetric Ramsey property is stated, the first colour is assumed to be red and the second colour is assumed to be blue.

(C.1): $R[X]$ is $(\widetilde{K}_t^{(3)}, \widetilde{K}_{t/2}^{(3)})$-Ramsey with respect to $(\varnothing, \mathcal{F})$;
(C.2): $R[X]$ is $(\widetilde{K}_{t/2}^{(3)}, \widetilde{K}_t^{(3)})$-Ramsey with respect to $(\mathcal{F}, \varnothing)$;
(C.3): R is $(\widetilde{K}_{t-1}^{(3)}, \mu)$-Ramsey;
(C.4): R is $(\widetilde{K}_t^{(3)}, \widetilde{K}_{t/2}^{(3)}, \mu)$-Ramsey;
(C.5): R is $(\widetilde{K}_{t/2}^{(3)}, \widetilde{K}_t^{(3)}, \mu)$-Ramsey.

Fix $R \sim \mathbb{H}^{(3)}(n,p)$ satisfying Properties (C.1-5) and set $\Gamma = H \cup R$.

Let ψ be a red/blue colouring of $E(\Gamma)$ and suppose for a contradiction that ψ does not yield any monochromatic copy of $\widetilde{K}_t^{(3)}$. For every $v \in V(H)$, let $L_H^{(r)}(v)$ denote the *red link graph of v in H under ψ*, that is, $L_H^{(r)}(v)$ is a spanning subgraph of $L_H(v)$ consisting of the edges of $L_H(v)$ that together with v yield a red edge of H under ψ. Similarly, let $L_H^{(b)}(v)$ denote the *blue link graph of v in H under ψ*. Note that, for any fixed vertex v, these two link subgraphs are edge-disjoint.

We say that blue (respectively, red) is *a majority colour* of ψ in H if $|\{e \in E(H) : \psi(e) \text{ is blue}\}| \geqslant |\{e \in E(H) : \psi(e) \text{ is red}\}|$ (respectively, $|\{e \in E(H) : \psi(e) \text{ is red}\}| \geqslant |\{e \in E(H) : \psi(e) \text{ is blue}\}|$).

Claim 3.6. *If blue is a majority colour of ψ in H, then $e\left(L_H^{(r)}(v)\right) \leqslant \frac{\eta}{2v(t/2)} \cdot |Y||Z|$ holds for every $v \in X$.*

Proof. Suppose for a contradiction that there exists a vertex $v \in X$ which violates the assertion of the claim. The Triangle Counting Lemma (Lemma 2.1) coupled with the assumption of H being (δ_3, d_3, r)-regular with respect to the $(\delta_2(\ell), d_2)$-triad P (take $Q_1 = \cdots = Q_r = P$ in (2.4)) collectively yield

$$e(H) \;\geqslant\; (d_3 - \delta_3)|\mathcal{K}_3(P)|$$

$$\overset{(2.1)}{\geqslant} (d_3 - \delta_3)\left(d_2^3 - 4\delta_2(\ell)\right)|X||Y||Z|$$
$$\geqslant \left(d_3 d_2^3 - \delta_3 d_2^3 - 4 d_3 \delta_2(\ell)\right)|X||Y||Z|$$
$$\geqslant \frac{d_3 d_2^3}{2}|X||Y||Z|, \tag{3.9}$$

where the last inequality is owing to $\delta_3 \ll d_3$ and $\delta_2(\ell) \ll d_2^3$ supported by (3.3) and (3.4), respectively. Blue being the majority colour implies that at least $\frac{d_3 d_2^3}{4}|X||Y||Z|$ of the edges of H are blue and thus there exists a vertex $u \in Z$ satisfying $e\left(L_H^{(b)}(u)\right) \geqslant \frac{d_3 d_2^3}{4}|X||Y|$; note that $L_H^{(b)}(u) \subseteq X \times Y$. Set

$$A_v = \left\{z \in Z : \deg_{L_H^{(r)}(v)}(z) \geqslant t\right\} \subseteq Z \quad \text{and} \quad A_u = \left\{x \in X : \deg_{L_H^{(b)}(u)}(x) \geqslant t\right\} \subseteq X.$$

Then,

$$|A_v| \geqslant \frac{\eta}{4v(t/2)}|Z| \overset{(3.4)}{\geqslant} \delta_2(\ell)|Z| \quad \text{and} \quad |A_u| \geqslant \frac{d_3 d_2^3}{8}|X| \overset{(3.4)}{\geqslant} \delta_2(\ell)|X| \tag{3.10}$$

both hold by Observation 3.3. Since H is $\delta_2(\ell)$-weakly-regular, it follows that

$$e_H(A_u, Y, A_v) \overset{(3.9)}{\geqslant} \left(\frac{d_3 d_2^3}{2}\right) \cdot |A_u||Y||A_v| - \delta_2(\ell)|X||Y|Z|$$
$$\overset{(3.10)}{\geqslant} \left(\frac{d_3 d_2^3}{2}\right) \cdot \left(\frac{\eta}{4v(t/2)}\right) \cdot \left(\frac{d_3 d_2^3}{8}\right)|X||Y||Z| - \delta_2(\ell)|X||Y|Z|$$
$$= \left(\frac{d_3^2 d_2^6 \eta}{64v(t/2)} - \delta_2(\ell)\right) \cdot |X||Y||Z|$$
$$\overset{(3.4)}{\geqslant} \left(\frac{d_3^2 d_2^6 \eta}{65v(t/2)}\right) \cdot |X||Y||Z|. \tag{3.11}$$

If red is a majority colour seen along $E_H(A_u, Y, A_v)$, then there exists a vertex $v' \in A_v \subseteq Z$ satisfying

$$\left|E\left(L_H^{(r)}(v')\right) \cap (A_u \times Y)\right| \overset{(3.11)}{\geqslant} \left(\frac{d_3^2 d_2^6 \eta}{130v(t/2)}\right)|X||Y| \geqslant \left(\frac{d_3^2 d_2^6 \eta}{130v(t/2)}\right)|A_u||Y|.$$

Consequently, the set

$$A_{u,v'} = \left\{x \in A_u : \deg_{L_H^{(r)}(v')}(x) \geqslant t\right\} \subseteq A_u \subseteq X$$

satisfies

$$|A_{u,v'}| \geqslant \left(\frac{d_3^2 d_2^6 \eta}{260v(t/2)}\right)|A_u|$$
$$\overset{(3.10)}{\geqslant} \left(\frac{d_3^2 d_2^6 \eta}{260v(t/2)}\right) \cdot \left(\frac{d_3 d_2^3}{8}\right)|X|$$
$$\geqslant \left(\frac{d_3^3 d_2^9 \eta}{2100v(t/2)}\right) \cdot \left\lfloor \frac{n}{\tilde{t}} \right\rfloor$$
$$\overset{(3.7)}{\geqslant} \mu n,$$

where the first inequality holds by Observation 3.3. We may then write that $\Gamma[A_{u,v'}] \longrightarrow (\widetilde{K}^{(3)}_{t-1})_2$ owing to R being $(\widetilde{K}^{(3)}_{t-1}, \mu)$-Ramsey, by Property (C.3). Let K be a copy of $\widetilde{K}^{(3)}_{t-1}$ appearing monochromatically under ψ within $\Gamma[A_{u,v'}]$. Let $x_1, \ldots, x_{t-1}$ denote the branch vertices of K. It follows by the definition of $A_{u,v'}$ that there are distinct vertices $y_1, \ldots, y_{t-1} \in Y$ such that $\{x_i, y_i, v'\}$ is a red edge of H for every $i \in [t-1]$. Similarly, since $A_{u,v'} \subseteq A_u$, there are distinct vertices $y'_1, \ldots, y'_{t-1} \in Y$ such that $\{x_i, y'_i, u\}$ is a blue edge of H for every $i \in [t-1]$. Therefore, if K is red, then it can be extended into a red copy of $\widetilde{K}^{(3)}_t$ including v'; if, on the other hand, K is blue, then it can be extended into a blue copy of $\widetilde{K}^{(3)}_t$ including u. In either case, a contradiction to the assumption that ψ admits no monochromatic copies of $\widetilde{K}^{(3)}_t$ is reached.

It remains to consider the complementary case where blue is a majority colour in $E_H(A_u, Y, A_v)$. The argument in this case parallels that seen in the previous one with the sole cardinal difference being that instead of finding a monochromatic copy of $\widetilde{K}^{(3)}_{t-1}$ in a subset of $A_u \subseteq X$, such a copy is found in a subset of $A_v \subseteq Z$. An argument for this case is provided for completeness. If blue is a majority colour seen along $E_H(A_u, Y, A_v)$, then there exists a vertex $u' \in A_u \subseteq X$ satisfying

$$\left| E\left(L^{(b)}_H(u')\right) \cap (Y \times A_v)\right| \overset{(3.11)}{\geqslant} \left(\frac{d_3^2 d_2^6 \eta}{130 v(t/2)}\right) |Y||Z| \geqslant \left(\frac{d_3^2 d_2^6 \eta}{130 v(t/2)}\right) |Y||A_v|.$$

Consequently, the set

$$A_{v,u'} = \left\{ z \in A_v : \deg_{L^{(b)}_H(u')}(z) \geqslant t \right\} \subseteq A_v \subseteq Z$$

satisfies

$$\begin{aligned} |A_{v,u'}| &\geqslant \left(\frac{d_3^2 d_2^6 \eta}{260 v(t/2)}\right) |A_v| \\ &\overset{(3.10)}{\geqslant} \left(\frac{d_3^2 d_2^6 \eta}{260 v(t/2)}\right) \cdot \left(\frac{\eta}{4 v(t/2)}\right) |Z| \\ &\geqslant \left(\frac{d_3^2 d_2^6 \eta^2}{1100 v(t/2)^2}\right) \cdot \left\lfloor \frac{n}{\tilde{t}} \right\rfloor \\ &\overset{(3.7)}{\geqslant} \mu n, \end{aligned}$$

where the first inequality holds by Observation 3.3. Then, $\Gamma[A_{v,u'}] \longrightarrow (\widetilde{K}^{(3)}_{t-1})_2$ owing to R being $(\widetilde{K}^{(3)}_{t-1}, \mu)$-Ramsey, by Property (C.3). A monochromatic copy of $\widetilde{K}^{(3)}_{t-1}$ appearing in $\Gamma[A_{v,u'}]$ can be either extended into a red copy of $\widetilde{K}^{(3)}_t$ including the vertex v or into a blue such copy including u'. In either case, a contradiction to the assumption that ψ admits no monochromatic copy of $\widetilde{K}^{(3)}_t$ is reached. □

The following counterpart of Claim 3.6 holds as well.

Claim 3.7. *If red is a majority colour of ψ in H, then $e\left(L_H^{(b)}(v)\right) \leqslant \frac{\eta}{2v(t/2)} \cdot |Y||Z|$ holds for every $v \in X$.*

Proceeding with the proof of Theorem 1.1, assume first that blue is a majority colour of ψ in H. By Property (C.1), either there is a red copy of $\widetilde{K}_t^{(3)}$ (within X) or there is a blue copy of $\widetilde{K}_{t/2}^{(3)}$ within X not supported on $\mathcal{F}$. If the former occurs, then the proof concludes. Assume then that $K \subseteq \Gamma[X]$ is a blue copy of $\widetilde{K}_{t/2}^{(3)}$ such that $V(K) \notin \mathcal{F}$, and write $L_H(K,P) = \bigcap_{x \in V(K)} L_H(x,P)$ to denote the joint link graph of the members of $V(K)$ supported on P. Then,

$$e(L_H(K,P)) \geqslant \left(d_3^{v(t/2)} - \varepsilon\right) d_2^{2v(t/2)+1}|Y||Z|,$$

holds by (3.8). Remove $E(L_H^{(r)}(x))$ from $E(L_H(K,P))$ for every $x \in V(K)$; that is, remove any edge in $L_H(K,P)$ that together with a vertex of K gives rise to a red edge of H with respect to ψ. By Claim 3.6, at most

$$\sum_{x \in V(K)} e\left(L_H^{(r)}(x)\right) \leqslant v(t/2) \cdot \frac{\eta}{2v(t/2)}|Y||Z| = \frac{\eta}{2}|Y||Z|$$

edges are thus discarded from $L_H(K,P)$, leaving at least

$$\begin{aligned}\left[\left(d_3^{v(t/2)} - \varepsilon\right) d_2^{2v(t/2)+1} - \frac{\eta}{2}\right]|Y||Z| &\overset{(3.2)}{\geqslant} \left(\frac{d_3^{v(t/2)} d_2^{2v(t/2)+1}}{2} - \frac{\eta}{2}\right)|Y||Z| \\ &\overset{(3.6)}{=} \left(\eta - \frac{\eta}{2}\right)|Y||Z| \\ &= \frac{\eta}{2}|Y||Z|\end{aligned}$$

edges in the *residual* joint link graph of K, denoted $L'_H(K,P)$. It follows by Lemma 3.4 and Observation 3.5 that $L'_H(K,P)$ contains at least

$$\xi_{3.5}(\zeta_{3.4}(\eta/2))\frac{2n}{T_{1.4}} \overset{(3.7)}{\geqslant} \mu n$$

vertex-disjoint copies of the bipartite graph $K_{1,t/2}$. Let $S \subseteq V(L'_H(K,P))$ consist of the centre-vertices of all said copies of $K_{1,t/2}$. Property (C.4) coupled with $|S| \geqslant \mu n$ collectively assert that $\Gamma[S] \longrightarrow (\widetilde{K}_t^{(3)}, \widetilde{K}_{t/2}^{(3)})$. If the first alternative occurs, then there is a red copy of $\widetilde{K}_t^{(3)}$ and thus the proof concludes. Suppose then that the second alternative takes place so that a blue copy K' of $\widetilde{K}_{t/2}^{(3)}$ arises in $\Gamma[S]$. Let $u_1, \ldots, u_{t/2}$ denote the branch-vertices of K' and let $x_1, \ldots, x_{t/2}$ denote the branch-vertices of K. It follows by the definitions of $L'_H(K,P)$ and S that there are $t^2/4$ distinct vertices $\{w_{ij} : i,j \in [t/2]\} \subseteq V(L'_H(K,P)) \smallsetminus \{u_1, \ldots, u_{t/2}, x_1, \ldots, x_{t/2}\}$ such that $\{u_i, x_j, w_{ij}\}$ forms a blue edge of H for every $i,j \in [t/2]$. We conclude that Γ admits a copy of $\widetilde{K}_t^{(3)}$ which is blue under ψ.

Next, assume that red is a majority colour seen for ψ in H. Replacing the appeals to Claim 3.6, Properties (C.1) and (C.4) in the argument above with appeals to Claim 3.7 and

Properties (C.2), and (C.5), respectively, leads to the rise of a monochromatic copy of $\widetilde{K}_t^{(3)}$ in Γ under ψ in this case as well. □

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