RELATIVE TURÁN DENSITIES OF ORDERED GRAPHS

CHRISTIAN REIHER, VOJTĚCH RÖDL, MARCELO SALES, AND MATHIAS SCHACHT

Abstract. We introduce a modification of the Turán density of ordered graphs and investigate this graph parameter.

§1 Introduction

1.1. **Unordered graphs.** Given an (unordered) graph F and a natural number n we write $\mathrm{ex}(n, F)$ for the maximal number of edges that a graph on n vertices can have, if it is F-free, i.e., if it has no subgraphs isomorphic to F. An averaging argument shows that the sequence $n \longmapsto \mathrm{ex}(n, F)/\binom{n}{2}$ is nonincreasing and, therefore, the limit

$$\pi(F) = \lim_{n \longrightarrow \infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{2}}\,, \tag{1.1}$$

known as the *Turán density of* F, exists. Results of Erdős, Simonovits, and Stone [6,7] yield an exact formula for this graph parameter. Specifically, if F has at least one edge, then

$$\pi(F) = 1 - \frac{1}{\chi(F) - 1}\,, \tag{1.2}$$

where $\chi(F)$ denotes the chromatic number of F. With the exception of the bipartite case, this gives a fairly complete picture.

One popular direction of further study replaces the ambient host graph K_n, in which the extremal F-free graphs are thought of as living and whose number of edges appears in the denominator of (1.1), by other graphs. For instance, beginning with the work of Kostočka [8] people have been investigating Turán problems in hypercubes (see also [1,3]). Trying to optimise over the host graph, however, is less interesting than it might appear at first. By averaging over all permutations of $V(G)$ one can show that

every graph G has an F-free subgraph G' with at least $\pi(F)e(G)$ edges. (1.3)

Moreover, for every fixed graph F this statement becomes false if we replace $\pi(F)$ by any larger constant (as can be seen by taking $G = K_n$ and letting n tend to infinity).

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1.2. **Ordered graphs.** From now on all graphs we consider will be *ordered*, that is they will be equipped with a distinguished linear ordering of their vertex sets. Accordingly, when we say that a graph G contains another graph F as a subgraph, this also means that F appears 'correctly ordered' in G. Bearing this in mind, one can define extremal numbers $\vec{\mathrm{ex}}(n, F)$ and Turán densities $\vec{\pi}(F)$ as in the unordered case. Research on these quantities was initiated by Pach and Tardos [9], who found the appropriate adaptation of (1.2) to ordered graphs, namely

$$\vec{\pi}(F) = 1 - \frac{1}{\chi_<(F) - 1}\,. \tag{1.4}$$

Here $\chi_<(F)$ denotes the so-called *interval chromatic number* of F, defined to be the least number of colours required to colour $V(F)$ properly, but with the additional constraint that every colour class needs to be an interval.

A notable example, where χ and $\chi_<$ differ significantly, is the ascending path P_k with k edges, defined by $V(P_k) = [k+1]$ and $E(P_k) = \{\{i, i+1\} : i \in [k]\}$. It is plain that P_k is bipartite and has Turán density zero in the unordered sense; but due to $\chi_<(P_k) = k+1$ we have

$$\vec{\pi}(P_k) = \frac{k-1}{k}\,.$$

The averaging argument (1.3) does not extend to ordered graphs, because the class of (extremal) F-free graphs is, in general, not closed under permutations of vertices. This leads us to a new interesting graph parameter.

Definition 1.1. Given an ordered graph F we define $\varrho(F)$, called the *relative Turán density of F*, to be the largest real number $\varsigma \in [0,1]$ with the following property: Every graph G has an F-free subgraph G' with $e(G') \geqslant \varsigma e(G)$.

It follows from the definition that

$$\varrho(F) \leqslant \vec{\pi}(F) \tag{1.5}$$

for every ordered graph F and equality holds for the ordered cliques K_r on $r \geqslant 2$ vertices. In the other direction we observe for any ordered graph F

$$\varrho(F) \geqslant \frac{\ell(F) - 1}{2\ell(F)}\,, \tag{1.6}$$

where $\ell(F)$ is number of edges of a longest monotone path in F. Given an ordered graph G we consider all maps $\varphi \colon V(G) \longrightarrow [\ell(F)]$. For each of them we let G_φ be the subgraph of G having all edges xy with $x < y$ and $\varphi(x) < \varphi(y)$. Since there are no strictly monotone sequences of length $\ell(F)+1$ in $[\ell(F)]$, all these graphs G_φ are F-free. Moreover, on average G_φ has

$$\frac{\ell(F)-1}{2\ell(F)}e(G)$$

edges and thus there exists some map φ such that $G_\varphi \subseteq G$ exemplifies the lower bound (1.6).

A result on shift graphs due to Arman, Rödl, and Sales [2] implies $\varrho(P_2)=\frac{1}{4}$, which shows that the lower bound (1.6) is optimal in this case. Our main result generalises this to longer monotone paths and will be proved in §3.

Theorem 1.2. *We have $\varrho(P_k)=\frac{k-1}{2k}$ for every $k \geqslant 2$.*

We shall also show that, like many other variants of the Turán density, ϱ is invariant under taking blow-ups. Given an ordered graph F and a positive integer t we shall write $F(t)$ for the ordered graph obtained from F by replacing each vertex x by an interval I_x of size t and every edge xy by all t^2 edges from I_x to I_y. We also require that for all vertices $x<y$ of F, all vertices in I_x precede all vertices in I_y with respect to the ordering of $F(t)$. A standard supersaturation argument carried out in §2 yields the following.

Proposition 1.3. *For all ordered graphs F and integers $t \geqslant 1$ we have $\varrho(F(t))=\varrho(F)$.*

§2 Blow-ups

Proof of Proposition 1.3. The estimate $\varrho(F(t)) \geqslant \varrho(F)$ being clear, we shall show that

$$\varrho(F(t)) \leqslant \varrho(F)+2\varepsilon$$

holds for every given $\varepsilon>0$. To this end we take a graph G which has no F-free subgraph G' of size $e(G') \geqslant (\varrho(F)+\varepsilon)e(G)$. Suppose for the sake of notational simplicity that $V(G)=[m]$ holds for some natural number m.

We contend that for a sufficiently large integer n (relative to m, $v(F)$, t, and ε) the blow-up $H=G(n)$ exemplifies $\varrho(F(t)) \leqslant \varrho(F)+2\varepsilon$. In other words, we shall prove that every subgraph H' of H of size $e(H') \geqslant (\varrho(F)+2\varepsilon)e(H)$ contains a copy of $F(t)$.

Let $I_1,\dots,I_m$ be the vertex classes of H. By a *transversal* we shall mean an m-element subset of $V(H)$ intersecting each of these classes exactly once. Denoting the set of all transversals by $\mathfrak{T}$ we have $|\mathfrak{T}|=n^m$ and

$$\sum_{T\in\mathfrak{T}} e_{H'}(T)=n^{m-2}e(H') \geqslant (\varrho(F)+2\varepsilon)n^{m-2}e(H)=(\varrho(F)+2\varepsilon)n^m e(G)\,.$$

Consequently, the subset

$$\mathfrak{T}_\star=\{T\in\mathfrak{T}\colon e_{H'}(T) \geqslant (\varrho(F)+\varepsilon)e(G)$$

of 'rich' transversals satisfies

$$(\varrho(F) + 2\varepsilon)e(G)n^m \leqslant (\varrho(F) + \varepsilon)e(G)n^m + e(G)|\mathfrak{T}_\star|\,,$$

whence $|\mathfrak{T}_\star| \geqslant \varepsilon n^m$.

By our choice of G, each rich transversal $T \in \mathfrak{T}_\star$ contains a copy of F which *crosses* the partition $\{I_1, \dots, I_m\}$, i.e., whose vertex set intersects each class I_i at most once. Conversely, every crossing copy of F in H' belongs to at most $n^{m-v(F)}$ transversals. For these reasons, there are at least $\varepsilon n^{v(F)}$ crossing copies of F in H'. Setting $f = v(F)$ this yields f indices $m(1) < \dots < m(f)$ in $[m]$ such that the f-partite subgraph of H' induced by $V_{m(1)}, \dots, V_{m(f)}$ contains at least $\varepsilon n^f/\binom{m}{f}$ crossing copies of F. So the f-partite f-uniform hypergraph with these vertex classes whose edges correspond to the crossing copies of F has positive density. By a result of Erdős [4], a sufficiently large choice of n guarantees that this hypergraph contains a complete f-partite hypergraph with vertex classes of size t. Consequently, H' has indeed a subgraph isomorphic to $F(t)$. □

§3 Paths

Throughout this section, which is devoted to the proof of Theorem 1.2, we fix an integer $k \geqslant 2$. We note that the lower bound follows from the general inequality (1.6). The corresponding upper bound requires the construction of appropriate graphs G. Before introducing those, we shall discuss a quadratic inequality, which we require later. We write

$$\Delta_k = \{(\alpha_1, \dots, \alpha_k) \in [0,1]^k \colon \alpha_1 + \dots + \alpha_k = 1\}$$

for the $(k-1)$-dimensional standard simplex. We designate elements of $\mathbb{R}^k$ by lowercase greek letters and the coordinates of any $\xi \in \mathbb{R}^k$ will be denoted by $\xi_1, \dots, \xi_k$. For every nonnegative integer d the function $h_d \colon \Delta_k \longrightarrow \mathbb{R}$ is defined by

$$h_d(\alpha) = (d+2)(1 - \|\alpha\|^2) + k\sum_{r=1}^{d} \frac{1}{r}\,,$$

where $\|\cdot\|$ refers to the Euclidean standard norm.

Lemma 3.1. *If $\alpha, \beta, \gamma \in \Delta_k$ satisfy $2\alpha = \beta + \gamma$ and $d \geqslant 1$, then*

$$h_{d-1}(\beta) + h_{d-1}(\gamma) + 4\sum_{1 \leqslant i < j \leqslant k} \beta_i \gamma_j \leqslant 2h_d(\alpha)\,.$$

Proof. Set $\eta = \beta - \alpha = \alpha - \gamma$. The parallelogram law tells us $\|\beta\|^2 + \|\gamma\|^2 = 2(\|\alpha\|^2 + \|\eta\|^2)$ and thus we only need to show

$$2\sum_{1 \leqslant i < j \leqslant k} \beta_i \gamma_j \leqslant (1 - \|\alpha\|^2) + (d+1)\|\eta\|^2 + \frac{k}{d}\,.$$

The left side evaluates to

$$2 \sum_{1 \leqslant i < j \leqslant k} (\alpha_i + \eta_i)(\alpha_j - \eta_j) = 2 \sum_{1 \leqslant i < j \leqslant k} (\alpha_i \alpha_j - \eta_i \eta_j) + 2 \sum_{1 \leqslant i \leqslant k} \lambda_i \eta_i \,, \tag{3.1}$$

where $\lambda_i = \sum_{j>i} \alpha_j - \sum_{j<i} \alpha_j$ satisfies $|\lambda_i| \leqslant 1$ due to $\alpha \in \Delta_k$. Because of $\sum_i \alpha_i = 1$ and $\sum_i \eta_i = 0$ the double sum on the right side of (3.1) simplifies to $1 - \|\alpha\|^2 + \|\eta\|^2$. Therefore, it remains to prove

$$2 \sum_{i=1}^{k} \lambda_i \eta_i \leqslant d\|\eta\|^2 + \frac{k}{d} \,.$$

But this is clearly implied by

$$0 \leqslant d \sum_{i=1}^{k} (\eta_i - \lambda_i/d)^2 \leqslant d\|\eta\|^2 - 2 \sum_{i=1}^{k} \lambda_i \eta_i + d^{-1} \sum_{i=1}^{k} \lambda_i^2 \,. \qquad \square$$

Arman, Rödl, and Sales describe in [2, Definition 5] a class of graphs $G_\varepsilon(n, d)$ that we shall need as well. If a real number $\varepsilon \in (0, 1]$ and a nonnegative integer d are given, such graphs $G_\varepsilon(n, d)$ exist for all sufficiently large multiples n of 2^d. The construction proceeds by recursion on d. To begin with, for every $n \in \mathbb{N}$ we let $G_\varepsilon(n, 0)$ be the empty graph on $[n]$ without any edges. Now suppose that all graphs of the form $G_\varepsilon(n, d - 1)$ have already been defined and let n be a sufficiently large integer divisible by 2^d. Fix a quasirandom bipartite graph $B = B_\varepsilon(n, d)$ with vertex classes $[n/2]$, $(n/2, n]$ and density 2^{1-d}. More precisely, the demands on this bipartite graph are $e(B) = n^2/2^{d+1}$ and

$$e_B(X, Y) = \frac{|X||Y|}{2^{d-1}} \pm \frac{\varepsilon n^2}{k 2^{d+2}}$$

for all subsets $X \subseteq [n/2]$ and $Y \subseteq (n/2, n]$. (It is well known that such bipartite graphs exist for all sufficiently large numbers n divisible by 2^{d+1}; we refer to the appendix of [2] for the standard probabilistic proof.) Having chosen $B_\varepsilon(n, d)$ we define $G_\varepsilon(n, d)$ such that

- its subgraphs induced by $[n/2]$ and $(n/2, n]$ are isomorphic to $G_\varepsilon(n/2, d - 1)$
- and its bipartite subgraph between $[n/2]$ and $(n/2, n]$ is isomorphic to $B_\varepsilon(n, d)$.

For instance, $G_\varepsilon(n, 1) = B_\varepsilon(n, 1)$ is the complete bipartite graph with vertex classes $[n/2]$ and $(n/2, n]$. An easy induction on d discloses

$$e(G_\varepsilon(n, d)) = \frac{dn^2}{2^{d+1}} \,, \tag{3.2}$$

whenever the graph $G_\varepsilon(n, d)$ is defined (cf. [2, Eq. (4)]).

Lemma 3.2. *Given $\varepsilon > 0$ suppose that n and d are such that the graph $G_\varepsilon(n, d)$ exists. If $[n] = V_1 \cup \ldots \cup V_k$ is a partition, and $\alpha_i = |V_i|/n$ for every $i \in [k]$, then the number of edges xy of $G_\varepsilon(n, d)$ with $x < y$ and $x \in V_i$, $y \in V_j$ for some $i < j$ is at most $(h_d(\alpha) + d\varepsilon)n^2/2^{d+2}$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$.*

Proof. We argue by induction on d. The base case, $d = 0$, is clear, because $G_\varepsilon(n, 0)$ has no edges at all. Now consider the induction step from $d-1$ to d and let $|V_i \cap [1, n/2]| = \beta_i n/2$ as well as $|V_i \cap (n/2, n]| = \gamma_i n/2$ for every $i \in [k]$. Clearly the vectors $\beta = (\beta_1, \dots, \beta_k)$ and $\gamma = (\gamma_1, \dots, \gamma_k)$ are in Δ_k and satisfy $2\alpha = \beta + \gamma$. There are three kinds of edges to consider:

(*a*) those with $x, y \in [1, n/2]$,
(*b*) those with $x, y \in (n/2, n]$,
(*c*) and those with $1 \leqslant x \leqslant n/2 < y \leqslant n$.

By the induction hypothesis there are at most

$$\frac{(h_{d-1}(\beta) + (d-1)\varepsilon)(n/2)^2}{2^{d+1}} \quad \text{and} \quad \frac{(h_{d-1}(\gamma) + (d-1)\varepsilon)(n/2)^2}{2^{d+1}}$$

edges of types (*a*) and (*b*), respectively. Moreover, by quasirandomness, there are at most

$$\sum_{i=1}^{k} \Big(\frac{\beta_i \sum_{j>i} \gamma_j}{2^{d-1}} \Big(\frac{n}{2}\Big)^2 + \frac{\varepsilon n^2}{k 2^{d+2}} \Big) = \frac{\big(4 \sum_{i<j} \beta_i \gamma_j + 2\varepsilon\big) n^2}{2^{d+3}}$$

edges of type (*c*). Altogether, the number Ω of edges under consideration satisfies

$$\frac{\Omega}{n^2/2^{d+3}} \leqslant h_{d-1}(\beta) + h_{d-1}(\gamma) + 4 \sum_{i<j} \beta_i \gamma_j + 2d\varepsilon \,,$$

and by Lemma 3.1 the right side is at most $2h_d(\alpha) + 2d\varepsilon$. □

Now the upper bound $\varrho(P_k) \leqslant \frac{k-1}{2k}$ we still seek to establish is a straightforward consequence of the following result.

Lemma 3.3. *For every $\varepsilon > 0$ there are positive integers d and n such that $G = G_\varepsilon(n, d)$ is defined and every P_k-free subgraph G' of G satisfies $e(G') \leqslant (\frac{k-1}{2k} + \varepsilon) e(G)$.*

Proof. Since $\sum_{r=1}^{d} 1/r = \log d + O(1) = o(d)$, we can choose d so large that

$$2 + k \sum_{r=1}^{d} \frac{1}{r} \leqslant \varepsilon d \,. \tag{3.3}$$

Let n be an arbitrary number for which the graph $G = G_\varepsilon(n, d)$ is defined and consider any P_k-free subgraph G' of G.

For each vertex $x \in [n]$ let $f(x)$ be the largest positive integer such that G' contains an ascending path of length $f(x) - 1$ ending in x. By our assumption on G', this function only attains values in $[k]$. Moreover, if xy with $x < y$ is an edge in G', then $f(x) < f(y)$. Thus, setting $\alpha_i = |f^{-1}(i)|/n$ for every $i \in [k]$ and $\alpha = (\alpha_1, \dots, \alpha_k)$, the previous lemma and (3.2) yield

$$\frac{e(G')}{e(G)} \leqslant \frac{h_d(\alpha) + d\varepsilon}{2d} \,.$$

Due to $\|\alpha\|^2 \geqslant 1/k$ we also have

$$h_d(\alpha) \leqslant \frac{d(k-1)}{k} + 2 + k\sum_{r=1}^{d}\frac{1}{r} \overset{(3.3)}{\leqslant} d\left(\frac{k-1}{k} + \varepsilon\right),$$

which leads indeed to $e(G') \leqslant (\frac{k-1}{2k} + \varepsilon)e(G)$. □

This completes the proof of Theorem 1.2.

§4 Concluding remarks

The case $k = 3$ of Theorem 1.2 yields a positive answer to [2, Problem 9]. Here we discuss a few further problems for future research.

Let C_ℓ denote the *ordered cycle* with vertex set $V(C_\ell) = [\ell]$ and edge set defined by $E(C_\ell) = \{\{i, i+1\}: i \in [\ell-1]\} \cup \{\{1, \ell\}\}$. Since C_ℓ contains a copy of the monotone path $P_{\ell-1}$, Theorem 1.2 yields $\varrho(C_\ell) \geqslant \frac{\ell-2}{2\ell-2}$ leading to the following problem.

Problem 4.1. *Determine $\varrho(C_\ell)$ for every fixed $\ell \geqslant 4$*

The obvious inequality (1.5) suggests the next question.

Problem 4.2. *Characterise the class $\{F: \varrho(F) = \vec{\pi}(F)\}$.*

For instance, all ordered cliques K_r are in this class. Are there any other such graphs? Similarly, when $F = P_k$ is a path, then Theorem 1.2 yields $\varrho(F) = \frac{1}{2}\vec{\pi}(F)$; we may thus ask for a characterisation of the class $\{F: \varrho(F) = \frac{1}{2}\vec{\pi}(F)\}$. Are there any graphs F satisfying

$$\frac{1}{2}\vec{\pi}(F) < \varrho(F) < \vec{\pi}(F)\,?$$

It would also be interesting to know whether there is any stability result accompanying Theorem 1.2.

Problem 4.3. *Given $k \geqslant 2$ and $\varepsilon > 0$, describe the structure of all graphs G with the property that every subgraph G' satisfying $e(G') \geqslant (\frac{k-1}{2k} + \varepsilon)e(G)$ contains a copy of P_k.*

In particular, one may ask whether such graphs need to have any resemblance to $G_\eta(n, d)$ for some small $\eta = \eta(k, \varepsilon)$. Perhaps one should also assume here that G be dense, i.e., that $e(G) \geqslant \varepsilon v(G)^2$.

Finally, the definition of $\varrho(\cdot)$ generalises straightforwardly to hypergraphs. A special case studied by Erdős, Hajnal, and Szemerédi [5] in the context of independent sets in shift graphs concerns the ascending r-uniform path $P_2^{(r)}$ of length 2, i.e., the hypergraph on $[r+1]$ with edges $[r]$ and $\{2, \ldots, r+1\}$. In the current notation, they showed

$$\varrho(P_2^{(r)}) \geqslant \begin{cases} \frac{1}{2} - \frac{1}{r} & \text{if } r \text{ is even} \\ \frac{1}{2} - \frac{1}{2r} & \text{if } r \text{ is odd.} \end{cases}$$

For $r = 4$ the quantitative improvement $\varrho(P_2^{(4)}) \geqslant \frac{3}{8}$ was obtained in [2] (see the footnote on page 9).

Problem 4.4. *Determine $\varrho(P_2^{(r)})$ for all $r \geqslant 3$. In particular, is $\varrho(P_2^{(3)}) = \frac{1}{3}$ true?*

Of course, one may also ask the same question for longer paths.

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Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany
Email address: `christian.reiher@uni-hamburg.de`

Department of Mathematics, Emory University, Atlanta, USA
Email address: `vrodl@emory.edu`

Department of Mathematics, University of California, Irvine, CA, USA
Email address: `mtsales@uci.edu`

Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany
Email address: `schacht@math.uni-hamburg.de`