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Fourier Analysis - Exercise sheet 2

In Lectures 2 and 3 we discussed an abstract way to approximate functions in certain functions spaces — by means of *approximate identities* and most prominently Theorem 3.7. In the following we start investigating the connection to Fourier series and also highlight how this approach provides direct proofs for previously seen results (see Ex. 2.2).

Ex 2.0:

Let $(a_n)_{n\in\mathbb{N}}$ be a converging sequence in a Banach space. Show that the sequence of arithmetic means $(b_n)_{n\in\mathbb{N}}$, $b_n = \frac{1}{n}\sum_{k=1}^{n} a_k$, converges to the same limit. Also show that the converse fails in general.

Ex 2.1: *(Fejér kernel and Dirichlet kernel)*

Recall the definition of the *Dirichlet kernel* $(D_n)_{n\in\mathbb{N}_0} \subset C(\mathbb{T})$, $D_n(t) = \frac{1}{2\pi}\sum_{k=-n}^{n} \mathrm{e}^{ikt}$. Further define the *Fejér kernel* $(F_n)_{n\in\mathbb{N}_0}$ by

$$F_n = \frac{1}{n+1}\sum_{k=0}^{n} D_k.$$

(a) Show that $D_n(t) = \frac{1}{2\pi}\frac{\sin\left((n+\frac{1}{2})t\right)}{\sin t/2}$ for all $t \in \mathbb{T}$ and $n \in \mathbb{N}_0$.

(b) Show that $F_n(t) = \frac{1}{2\pi}\sum_{k=-n}^{n}(1-\frac{|k|}{n+1})\mathrm{e}^{ikt} = \frac{1}{2\pi(n+1)}\left(\frac{\sin\left((n+1)\frac{t}{2}\right)}{\sin t/2}\right)^2$

(c) Use (b) to conclude that $(F_n)_n$ is an approximate identity with the additional properties

$$F_n(t) \geq 0, \qquad F_n(t) = F_n(-t), \qquad \lim_{n\to\infty}\sup_{s\in[\delta,2\pi-\delta]} |F_n(s)| = 0$$

for $n \in \mathbb{N}_0$, $t \in \mathbb{T}$, $\delta \in (0,\pi)$.

(d) Let $f \in L^p(\mathbb{T})$, $p \in [1,\infty)$. Show that $\sum_{k=-n}^{n} \hat{f}(k)(1-\frac{|k|}{n+1})\mathrm{e}^{ik\cdot}$ converges to f in $L^p(\mathbb{T})$.

Ex 2.2:

Reprove the following results which we have already encountered in the lecture.

(a) Prove Weierstrass's Theorem — the set of trigonometric polynomials $\mathrm{Trig}(\mathbb{T})$ lies dense in $C(\mathbb{T})$ — by a constructive argument (given $f \in C(\mathbb{T})$ explicitly construct a sequence in $\mathrm{Trig}(\mathbb{T})$ that converge to f) *Hint: approximate identity.*

(b) Prove that $C(\mathbb{T})$ lies dense in $L^1(\mathbb{T})$ with the methods of Section 3
⚠*Be aware of a circular argument*

(c) Prove that the Fourier coefficients of an $L^1(\mathbb{T})$ function are unique, i.e. $T : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is injective by using the Fejér kernel from Exercise 2.1.

Ex 2.3:

(a) Let $f, g \in L^1(\mathbb{T})$ and $h \in L^\infty(\mathbb{T})$. Show that $\int_{\mathbb{T}}(f*g)(s)h(s)\,ds = \int_{\mathbb{T}} f(s)(g*h)(s)\,ds$.

(b) Use (a) and Ex. 2.1 to show that for any $h \in L^\infty(\mathbb{T})$ there exists a sequence $(h_n)_{n\in\mathbb{N}} \subset C(\mathbb{T})$ which converges to h in weak* sense, i.e.

$$\int_{\mathbb{T}} f(s)\overline{h_n(s)}\,ds \overset{n\to\infty}{\longrightarrow} \int_{\mathbb{T}} f(s)\overline{h(s)}\,ds \qquad \forall f \in L^1(\mathbb{T}).$$

Ex 2.4: *(Minkowski's inequality)*

(a) Prove *Minkowski's inequality* for homogeneous Banach spaces X on $\mathbb{T}$:

For $f \in X$ and $g \in L^1(\mathbb{T})$ it follows that $f * g \in X$ and $\|f * g\|_X \leq \|g\|_{L^1(\mathbb{T})}\|f\|_X$. [1]

This can be rephrased as

$$\mathcal{M}_g : \begin{cases} f & \mapsto & g * f \\ X & \to & X \end{cases}$$

is a bounded linear operator from X to X with norm less or equal $\|g\|_{L^1(\mathbb{T})}$.

(b) Show Minkowski's inequality for $X = L^\infty(\mathbb{T})$.

(c) Show that Minkowski's inequality is sharp[2] for the cases

$$X = L^1(\mathbb{T}), X = L^\infty(\mathbb{T}), X = C(\mathbb{T})$$

by showing that $\|\mathcal{M}_g\|_{X \to X} = \|g\|_{L^1(\mathbb{T})}$.

(d) For $X = L^2(\mathbb{T})$ show that $\|\mathcal{M}_g\|_{X \to X} = 2\pi\|\hat{g}\|_{\ell^\infty(\mathbb{Z})}$ and again conclude that Minkowski's inequality is sharp.

Hints: (a): Consider first $g \in C(\mathbb{T})$ and inspect (the proof) of Theorem 3.7.
(b): To show the case $X = L^1$ "imagine" that there exists $e \in X \setminus \{0\}$ such $e * g = g$. For $X = L^\infty(\mathbb{T})$ find a suitable f and for $X = C(\mathbb{T})$ use Ex. 2.1.

Ex 2.5: (*Convergence of Fourier series in $L^1(\mathbb{T})$, $C(\mathbb{T})$ and pointwise*)

(a) Show that the following statement is wrong for general functions $f \in L^1(\mathbb{T})$.

The partial sums

$$S_N f = \sum_{k=-N}^{N} \hat{f}(k)\mathrm{e}^{ikt}$$

of the Fourier series converge to f in the $L^1(\mathbb{T})$-norm (as $N \to \infty$).

Can we find a "large" (say "dense") subspace D of $L^1(\mathbb{T})$ such that the above statement holds indeed true for all $f \in D$?

(b) Show that the following statement is wrong for general functions $f \in C(\mathbb{T})$.

The partial sums of the Fourier series converge to f pointwise.

(c) Conclude that the Dirichlet kernel $(D_n)_{n \in \mathbb{N}_0}$ is not an approximate identity.

Hint: (a): you may freely use that $\sup_{n\in\mathbb{N}} \|D_n\|_1 = \infty$ and Exercise 2.4(c).
(b): similar as in (a), also note that $\mathcal{M}_g \tau_s = \tau_s \mathcal{M}_g$, where $\tau_s f = f(\cdot - s)$.

Ex 2.6*: (*Approximate identity*)
Show that there exists no element e in $L^1(\mathbb{T})$ such that $e * f = f$ for all $f \in L^1(\mathbb{T})$.

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[1] Note that this proves the special case $p = r$ of *Young's inequality for convolutions*,

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}, \qquad f \in L^p(\mathbb{T}), g \in L^q(\mathbb{T}), \tfrac{1}{p} + \tfrac{1}{q} = 1 + \tfrac{1}{r}.$$

[2] Here "sharp" means that the inequality in the above statement (for fixed X) can not be replaced by $\|f * g\|_X \leq c\|g\|_{L^1}\|f\|_X$ for any $c < 1$ (with c not depending on f and g!)